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The Cauchy Problem for Weakly Hyperbolic Equations (II); Infinite Degenerate Case

Hitoshi URYU

Waseda University

Introduction

In this paper we shall deal with well-posedness of the Cauchy problem for some weakly hyperbolic operators with involutive and non-involutive multiple characteristics. For the second order equations, Oleinik [4] obtained a sufficient condition for well-posedness. Menikoff [1] extended Oleinik's results to the equations of higher order, and Ohya [3] improved and simplified Menifoff's proof. In a previous paper [5], we considered weakly hyperbolic operators whose characteristic roots come across one another with finite order at $t=0$.

Recently Nishitani [2] has proved well-posedness of the Cauchy problem for a hyperbolic operator with characteristic roots coming into double at $t=0$ in contact with each other of infinite order. In this article we shall treat the case where the characteristic roots may have $m(\geqq 2)$ multiplicities at $t=0$ in contact with one another of infinite order.

Now our concern is the following operator P which is a partial differential operator of the form:

$$P=D_t^m+\sum_{\substack{|\alpha|+j\leqq m\\ j\leqq m-1}} a_{\alpha,j}(x, t)D_x^\alpha D_t^j$$

where each coefficient $a_{\alpha,j}(x, t)$ belongs to $\mathscr{B}((0, T)\times R^n)$. Let $\lambda_j(x, t, \xi)$, $j=1, \cdots, m$, be the characteristic roots of P. If all the $\lambda_j(x, t, \xi)$ are real valued functions in $\mathscr{B}((0, T), S^1)$ for some $T>0$, P is said to be a weakly hyperbolic operator. We consider only such operators throughout the paper.

We say that a weakly hyperbolic operator P has involutive characteristic roots if for any $i, j(1\leqq i, j\leqq m)$, there exist pseudo-differential operators $A_{i,j}$, $B_{i,j}$ and $C_{i,j}\in\mathscr{B}((0, T), S^0)$ such that

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$$(0.1)\qquad [\partial_i, \partial_j]=A_{i,j}\partial_i+B_{i,j}\partial_j+C_{i,j}$$

where $[A, B]=AB-BA$ is the commutator, and ∂_j is a pseudo-differential operator

$$\partial_j=D_t-\lambda_j(x, t, D_x)$$

where $D_t=-i\partial_t$ and $D_x=-i\partial_x$.

In the following we denote by (u, v) the L^2-scalar product with respect to x, by $\|u(\cdot, t)\|_s$ the norm in Sobolev space H^s and

$$\|u(\cdot, t)\|_{k,s}^2=\sum_{j=0}^{k}\|D_t^j u(\cdot, t)\|_{s+k-j}^2 .$$

§ 1. Involutive case.

Let P be a weakly hyperbolic operator with involutive characteristic roots defined in introduction. Following to Zeman [6] we consider the modules $W_k(0\leqq k\leqq m-1)$ over the ring of pseudo-differential operators in x of order zero. Let $\Pi_m=\partial_1\partial_2\cdots\partial_m$. Let W_{m-1} be the module generated by the monomial operators $\Pi_m/\partial_i=\partial_1\partial_2\cdots\check{\partial}_i\cdots\partial_m$ of order $m-1$ and let W_{m-2} be the module generated by the operators $\Pi_m/\partial_i\partial_j(i\neq j)$ of order $m-2$ and so on.

Zeman [6] proved the following theorem in the case of multiplicity m.

THEOREM. *Let P be a weakly hyperbolic operator with involutive characteristic roots satisfying the condition:*

$$(1.1)\qquad P=\Pi_m+\sum_{j=1}^{m} a_{\alpha,j}\omega_{m-j}^{\alpha}$$

where $a_{\alpha,j}\in\mathscr{B}((0, T), S^0)$ and $\omega_{m-j}^{\alpha}\in W_{m-j}$. Then the Cauchy problem for P is H^∞-well-posed, where $H^\infty=\bigcap_s H^s$.

In this section we shall consider the application of this theorem.

THEOREM 1.1. *Let P be a weakly hyperbolic operator with the property:*

> *The characteristic roots $\lambda_j(x, t, \xi)$ are of the form $\lambda_j(x, t, \xi)=\sigma(x)\tilde{\lambda}_j(x, t, \xi)$, where $\tilde{\lambda}_j\in\mathscr{B}((0, T), S^1)$, $\sigma(x)\in\mathscr{B}(R^n)$ and $\tilde{\lambda}_i\neq\tilde{\lambda}_j$ when $i\neq j$.*

Then the Cauchy problem for P is well-posed if the lower order terms $P_{m-j}(j=1, 2, \cdots, m-1)$ satisfy

(1.2) $$P_{m-j}(x, t, \xi, \sigma(x)\lambda_i(x, t, \xi)) = \sigma(x)^{m-j}K_{i,j}(x, t, \xi)$$

with $K_{i,j}(x, t, \xi) \in \mathscr{B}((0, T), S^{m-j})$.

REMARK. The condition (1.2) implies that if we write

$$P_{m-j}(x, t, \xi, \tau) = \sum_{i=0}^{m-j} a_{i,j}(x, t, \xi)\tau^{m-j-i}$$

then we have

$$a_{i,j}(x, t, \xi) = \sigma(x)^i \tilde{a}_{i,j}(x, t, \xi)$$

where $\tilde{a}_{i,j}$ belong to $\mathscr{B}((0, T), S^i)$.

The theorem will be proved by showing that P satisfies the conditions of the Zeman theorem. To begin with the lemma which shows P is a weakly hyperbolic operator with involutive characteristic roots.

LEMMA 1.2. *For any* i, j *there exist* $A_{i,j}, B_{i,j}, C_{i,j} \in \mathscr{B}((0, T), S^0)$ *such that*

(1.3) $$[\partial_i, \partial_j] = A_{i,j}\partial_i + B_{i,j}\partial_j + C_{i,j} .$$

PROOF. Let $\sigma_0([\partial_i, \partial_j])$ be the principal symbol of $[\partial_i, \partial_j]$. Then, by the formula of product of pseudo-differential operators, we obtain

$$\begin{aligned}\sigma_0([\partial_i, \partial_j]) &= \sum_{\alpha=0}^{n} \{D_{\xi_\alpha}(\xi_0 - \sigma(x)\tilde{\lambda}_i)\partial_{x_\alpha}(\xi_0 - \sigma(x)\tilde{\lambda}_j) \\ &\quad - D_{\xi_\alpha}(\xi_0 - \sigma(x)\tilde{\lambda}_j)\partial_{x_\alpha}(\xi_0 - \sigma(x)\tilde{\lambda}_i)\} \\ &= \sigma(x)D_{i,j}(x, t, \xi)\end{aligned}$$

where $D_{i,j} \in \mathscr{B}((0, T), S^1)$. Here we used the notation

$$x_0 = t \quad \text{and} \quad \xi_0 = \tau .$$

If we define functions $A_{i,j}$ and $B_{i,j}$ for $i \neq j$ by $A_{i,j} = D_{i,j}(x, t, \xi)/(\tilde{\lambda}_j - \tilde{\lambda}_i)$, $B_{i,j} = D_{i,j}(x, t, \xi)/(\tilde{\lambda}_i - \tilde{\lambda}_j)$ respectively, then $A_{i,j}, B_{i,j} \in \mathscr{B}((0, T), S^0)$ and the equality:

$$A_{i,j}(x, t, \xi)(\xi_0 - \sigma(x)\tilde{\lambda}_i) + B_{i,j}(x, t, \xi)(\xi_0 - \sigma(x)\tilde{\lambda}_j) = \sigma(x)D_{i,j}(x, t, \xi)$$

holds. Hence we have $[\partial_i, \partial_j] = A_{i,j}\partial_i + B_{i,j}\partial_j + C_{i,j}$ for some $C_{i,j} \in \mathscr{B}((0, T), S^0)$. Q.E.D.

The following elementary lemma is helpful to show that P can be represented in the form of (1.1).

Lemma 1.3. *Let $\Pi_s=\partial_{i_1}\cdots\partial_{i_s}$ where i_j are integers and $1\leqq i_1\leqq\cdots\leqq i_s\leqq m$. Then $\sigma(\Pi_s)$, the symbol of Π_s, can be written in the form;*

$$\sigma(\Pi_s)=\prod_{\alpha=1}^{s}(\tau-\sigma(x)\tilde{\lambda}_j)+R_{s-1}+\cdots+R_0 \tag{1.4}$$

where $R_{s-j}(x, t, \xi, \tau)=\sum_{\beta=0}^{s-j}b_{\beta,j}(x, t, \xi)\sigma(x)^{\beta}\tau^{s-j-\beta}$ for some $b_{\beta,j}\in\mathscr{B}((0, T), S^{\beta})$ $(j=1, \cdots, s)$.

Proof. Let us prove by induction on s. When $s=1$, (1.4) is trivial. Suppose it holds for s. Since $\Pi_{s+1}=\Pi_s\partial_{i_{s+1}}$, the symbol of Π_{s+1} is given by $\sigma(\Pi_{s+1})=\sigma(\Pi_s)(\xi_0-\sigma(x)\tilde{\lambda}_{j_{s+1}})+\sum_{\alpha\neq 0}D_{\xi}^{\alpha}\sigma(\Pi_s)\partial_x^{\alpha}(\xi_0-\sigma(x)\tilde{\lambda}_{i_{s+1}})$. From the assumption of induction, $\sigma(\Pi_s)$ in the above equality can be replaced by the right hand side of (1.4). Arranging the terms suitably we have (1.4) for $s+1$. Q.E.D.

Now we shall show (1.2) implies (1.1). From (1.2) and Lemma 1.3 with $s=m$, we have

$$\sigma(P-\Pi_m)=\sum_{j=1}^{m}\sum_{i=0}^{m-j}\tilde{c}_{i,j}(x, t, \xi)\sigma(x)^i\tau^{m-j-i}$$

for some $\tilde{c}_{i,j}\in\mathscr{B}((0, T), S^i)$. Pick up the homogeneous part of degree $m-1$ in the right hand side, and denote by

$$\tilde{P}_{m-1}(x, t, \xi, \tau)=\sum_{i=0}^{m-1}\tilde{c}_{i,1}(x, t, \xi)\sigma(x)^i\tau^{m-1-i}\,.$$

To show (1.1) we must determine $A_j\in\mathscr{B}((0, T), S^0)$ so that

$$\tilde{P}_{m-1}(x, t, \xi, \tau)=\sum_{j=1}^{m}A_j(x, t, \xi)\prod_{i\neq j}(\tau-\sigma(x)\tilde{\lambda}_i)\,. \tag{1.5}$$

Let us note $\tilde{P}_{m-1}(x, t, \xi, \sigma(x)\tilde{\lambda}_j)=\sigma(x)^{m-1}\tilde{K}_j(x, t, \xi)$ for some $\tilde{K}_j(x, t, \xi)\in\mathscr{B}((0, T), S^{m-1})$ by the assumption. Define $A_j\in\mathscr{B}((0, T), S^0)$ by

$$A_j(x, t, \xi)=\Big[\prod_{i\neq j}(\tilde{\lambda}_j-\tilde{\lambda}_i)\Big]^{-1}\tilde{K}_j(x, t, \xi)\,.$$

Then Lemma 1.3 for $s=m-1$ yields

$$\sigma\Big(P-\Pi_m-\sum_{j=1}^{m}A_j\prod_{i\neq j}\partial_i\Big)=\sum_{j=2}^{m}\sum_{i=0}^{m-j}d_{i,j}(x, t, \xi)\sigma(x)^i\tau^{m-i-j}$$

where $d_{i,j}\in\mathscr{B}((0, T), S^i)$.
Repeating these steps we attain to the representation (1.1). Q.E.D.

Therefore the proof of Theorem 1.1 is completed by the help of Zeman's theorem.

§ 2. Non-involutive case.

In this section we consider the weakly hyperbolic operator P in $\Omega=R^n\times(0, T)$ with principal symbol of the form

$$P_m(x, t, \xi, \tau)=\prod_{j=1}^{m}(\tau-\sigma(t)\lambda_j(x, t, \xi)) \tag{2.1}$$

where $\lambda_j\in\mathscr{B}((0, T), S^1)$ is real valued, $\lambda_i\neq\lambda_j$ for $i\neq j$, and $\sigma(t)\in\mathscr{B}([0, T])$ is positive and strictly increasing for $t>0$. In addition, we assume that $\tau(t)\equiv\sigma(t)/\sigma'(t)$ belongs to $\mathscr{B}^m([0, T])$ and for any $N\geqq 0$ there exists a positive constant C_N such that $\sigma(t)\leqq C_N\tau(t)^N$ in $[0, T]$.

Now we state the main theorem in this section.

THEOREM 2.1. *Let P be an operator satisfying the condition* (2.1). *Then the Cauchy problem for P is well-posed if the lower order terms $P_{m-j}(x, t, \xi, \tau)$ satisfy the condition;*

$$P_{m-j}(x, t, \xi, \sigma(t)\lambda_i(x, t, \xi))=\sigma(t)^{m-j}\tau(t)^{-j}K_{i,j} \tag{2.2}$$

where $K_{i,j}(x, t, \xi)\in\mathscr{B}((0, T), S^{m-j})$.

REMARK. The condition (2.2) implies that if we write $P_{m-j}(x, t, \xi, \tau)=\sum_{i=0}^{m-j}a_{i,j}(x, t, \xi)\tau^{m-j-i}$, then $a_{i,j}=\sigma(t)^i\tau(t)^{-j}\tilde{a}_{i,j}(x, t, \xi)$ where

$$\tilde{a}_{i,j}\in\mathscr{B}((0, T), S^i)$$

for $i\geqq 1$.

We prove the Theorem by the same procedure to [5].

LEMMA 2.2. *For any i,j there exist pseudo-differential operator $A_{i,j}$, $B_{i,j}$, $C_{i,j}\in\mathscr{B}((0, T), S^0)$ such that*

$$[\partial_i, \partial_j]=\tau(t)^{-1}[A_{i,j}\partial_i+B_{i,j}\partial_j+C_{i,j}]\,. \tag{2.3}$$

PROOF. Let $\sigma_0([\partial_i, \partial_j])$ be the principal symbol of $[\partial_i, \partial_j]$. Then we obtain

$$\begin{aligned}\sigma_0([\partial_i, \partial_j])&=\sum_{\alpha=0}^{n}\{D_{\xi_\alpha}(\xi_0-\sigma(t)\lambda_i)\partial_{x_\alpha}(\xi_0-\sigma(t)\lambda_j)\\&\quad-D_{\xi_\alpha}(\xi_0-\sigma(t)\lambda_j)\partial_{x_\alpha}(\xi_0-\sigma(t)\lambda_i)\}\\&=-\sigma'(t)(\lambda_i-\lambda_j)-\sigma(t)(\partial_t\lambda_j-\partial_t\lambda_i)+\sigma(t)\{\lambda_i, \lambda_j\}\\&=\sigma(t)\tau(t)^{-1}D_{i,j}(x, t, \xi)\end{aligned}$$

where $\{\,,\,\}$ is Poisson bracket and $D_{i,j}(x, t, \xi)$ are smooth functions in $\mathscr{B}((0, T), S^1)$. On the other hand the principal symbol of $\tau(t)^{-1}[A_{i,j}\partial_i + B_{i,j}\partial_j + C_{i,j}]$ is

$$\tau(t)^{-1}[A_{i,j}(x, t, \xi)(\tau-\lambda_i)+B_{i,j}(x, t, \xi)(\tau-\lambda_j)] .$$

Put

$$A_{i,j}(x, t, \xi)=D_{i,j}(x, t, \xi)/(\lambda_j-\lambda_i) ,$$
$$B_{i,j}(x, t, \xi)=D_{i,j}(x, t, \xi)/(\lambda_i-\lambda_j) .$$

Then $A_{i,j}, B_{i,j} \in \mathscr{B}((0, T), S^0)$ and

$$[\partial_i, \partial_j]=\tau(t)^{-1}[A_{i,j}\partial_i+B_{i,j}\partial_j+C_{i,j}]$$

for some $C_{i,j} \in \mathscr{B}((0, T), S^0)$. Q.E.D.

LEMMA 2.3. *For any monomial $\omega_s^\alpha \in W_s$ there exist ∂_i and $\omega_{s+1}^\beta \in W_{s+1}$ such that*

$$\partial_i\omega_s^\alpha=\omega_{s+1}^\beta+\sum_{k=1}^{s+1}\sum_\gamma \tau(t)^{-k}c_{\gamma,k}\omega_{s+1-k}^\gamma \tag{2.4}$$

where $c_{\gamma,k} \in \mathscr{B}((0, T), S^0)$ and $\omega_{s+1-k}^\gamma \in W_{s+1-k}$.

PROOF. For any $\omega_s^\alpha=\partial_{j_1}\cdots\partial_{j_s}(j_1<j_2\cdots<j_s)$, there exists some $j \notin \{j_1, \cdots, j_s\}$ with $1\leq j\leq m$. Since $[\partial_i, \partial_j]=\tau(t)^{-1}[A_{i,j}\partial_i+B_{i,j}\partial_j+C_{i,j}]$, by Lemma 2.2, we have immediately (2.4). Q.E.D.

LEMMA 2.4. *For any $u \in C^\infty(\Omega)$ and any real number s the following energy estimates hold.*

$$\frac{d}{dt}\|\omega_{m-j}^\alpha u\|_s^2\leq \text{const}\{\tau(t)\|\omega_{m-j+1}^\beta u\|_s^2 + \|\omega_{m-j}^\alpha u\|_s^2+\sum_{k=1}^{m-j+1}\sum_\gamma\tau(t)^{-2k+1}\|\omega_{m-j+1-k}^\gamma u\|_s^2\} . \tag{2.5}$$

PROOF. By Lemma 2.3 with $s=m-j$ we have

$$\partial_i\omega_{m-j}^\alpha u=\omega_{m-j}^\beta u+\sum_{k=1}^{m-j+1}\sum_\gamma \tau(t)^{-k}c_{\gamma,k}\omega_{m-j+1-k}^\gamma .$$

Putting

$$v=\omega_{m-j}^\alpha u$$
$$g=\omega_{m-j+1}^\beta u+\sum_{k=1}^{m-j+1}\sum_\gamma \tau(t)^{-k}c_{\gamma,k}\omega_{m-j+1-k}^\gamma u ,$$

we have a first order hyperbolic equation $\partial_t v = g$. Hence it follows

$$\begin{aligned}\frac{d}{dt}\|v\|_s^2 &= 2\,\mathrm{Re}\Big(\frac{d}{dt}v,\ v\Big)_s \\ &= 2\,\mathrm{Re}(\sqrt{-1}\lambda_i(x,t,D_x)v + \sqrt{-1}g,\ v)_s \\ &= 2\,\mathrm{Re}(\sqrt{-1}\lambda_i(x,t,D_x)v,\ v)_s + 2\,\mathrm{Re}(\sqrt{-1}\tau^{1/2}(t)g,\ \tau(t)^{-1/2}v)_s \\ &\leqq \mathrm{const}\{\|v\|_s^2 + \tau(t)\|g\|_s^2 + \tau(t)^{-1}\|v\|_s^2\}\ .\end{aligned}$$

Therefore we proved (2.5) using the inequality

$$\|g\|_s^2 \leqq \mathrm{const}\Big\{\sum_{k=1}^{m-j+1}\sum_{\gamma}\tau(t)^{-2k}\|\omega^{\gamma}_{m-j+1-k}u\|_s^2\Big\}\ ,$$

which immediately follows from the definition of g. Q.E.D.

Now we prove a basic lemma.

LEMMA 2.5. *Set* $\Phi(t) = \sum_{k=1}^{m}\sum_{\alpha}\tau(t)^{-2k}\|\omega^{\alpha}_{m-k}u\|_s^2$. *Then the inequality*

$$\frac{d}{dt}\Phi(t) \leqq \mathrm{const}\{\Phi(t) + \tau(t)^{-1}\Phi(t) + \tau(t)^{-1}\|\Pi_m u\|_s^2\} \tag{2.6}$$

holds for any $u \in C^\infty(\Omega)$.

PROOF. Since

$$\frac{d}{dt}\Phi(t) = \sum_{k=1}^{m}\Big\{-2k\tau(t)^{-2k-1}\tau'(t)\|\omega^{\alpha}_{m-k}u\|_s^2 + \tau(t)^{-2k}\frac{d}{dt}\|\omega^{\alpha}_{m-k}u\|_s^2\Big\}\ ,$$

from Lemma 2.4 it follows

$$\begin{aligned}\frac{d}{dt}\Phi(t) \leqq \mathrm{const}&\Big\{\sum_{k=1}^{m}\sum_{\beta}\tau(t)^{-2k-1}\|\omega^{\beta}_{m-k+1}u\|_s^2 + \sum_{k=1}^{m}\sum_{\alpha}\tau(t)^{-2k}\|\omega^{\alpha}_{m-k}u\|_s^2 \\ &+ \sum_{k=1}^{m}\sum_{j=1}^{m-k+1}\sum_{\gamma}\tau(t)^{-2k-2j+1}\|\omega^{\gamma}_{m-k+1-j}u\|_s^2\Big\} \\ \leqq \mathrm{const}&\Big\{\Phi(t) + \tau(t)^{-1}\Phi(t) + \tau(t)^{-1}\|\Pi_m u\|_s^2 \\ &+ \tau(t)^{-1}\sum_{k=1}^{m}\sum_{j=1}^{m-k+1}\sum_{\gamma}\tau(t)^{-2(k+j-1)}\|\omega^{\gamma}_{m-(k+j-1)}u\|_s^2\Big\}\ .\end{aligned}$$

Recalling $j+k-1 \leqq (m-k+1)+k-1 = m$, we have thus

$$\sum_{k=1}^{m}\sum_{j=1}^{m-k+1}\sum_{\gamma}\tau(t)^{-2(k+j-1)}\|\omega^{\gamma}_{m-(k+j-1)}u\|_s^2 \leqq \mathrm{const}\,\Phi(t)\ ,$$

which implies (2.6) by combining with the above inequality. Q.E.D.

Next let us state the main lemma in this section.

LEMMA 2.6. *Under the condition of Theorem* 2.1, *there exist* $c_{\alpha,k}\in\mathscr{B}((0, T), S^0)$ *and* $\omega_{m-k}^{\alpha}\in W_{m-k}$ *such that*

$$P-\Pi_m=\sum_{k=1}^{m}\sum_{\alpha}\tau(t)^{-k}c_{\alpha,k}\omega_{m-k}^{\alpha}\,. \tag{2.7}$$

The following lemma is needed for proving Lemma 2.6, as in the proof of Theorem 1.5.

LEMMA 2.7. *Let* $\Pi_s=\partial_{i_1}\cdots\partial_{i_s}$ *where each* i_j *is an integer with* $1\leqq i_1\leqq\cdots\leqq i_s\leqq m$. *Then* $\sigma(\Pi_s)$, *the symbol of* Π_s, *can be written in the form*:

$$\sigma(\Pi_s)=\prod_{\alpha=1}^{s}(\tau-\sigma(t)\lambda_{i_\alpha})+R_{s-1}+\cdots+R_0 \tag{2.8}$$

where $R_{s-j}(x, t, \xi, \tau)=\sum_{\beta=0}^{s-j}a_{\beta,j}(x, t, \xi)\tau^{s-j-\beta}$ *and* $a_{\beta,j}=\sigma(t)^{\beta}\tau(t)^{-j}\tilde{a}_{\beta,j}(x, t, \xi)$ *for* $\beta\geqq 1$ *with some* $\tilde{a}_{\beta,j}(x, t, \xi)\in\mathscr{B}((0, T), S^{\beta})$.

PROOF. We carry out the proof by induction on s. When $s=1$ (2.8) is trivial. Assume (2.8) is valid for s. Since $\Pi_{s+1}=\Pi_s\partial_{i_{s+1}}$, by the product formula for two symbols we have

$$\sigma(\Pi_{s+1})=\sigma(\Pi_s)(\xi_0-\sigma(t)\lambda_{i_{s+1}})+\sum_{\alpha\neq 0}D_{\xi}^{\alpha}\sigma(\Pi_s)\partial_x^{\alpha}(\xi_0-\sigma(t)\lambda_{i_{s+1}})\,. \tag{2.9}$$

Since $\sigma(\Pi_s)$ satisfies (2.8) by the assumption, so does $\sigma(\Pi_s)(\xi_0-\sigma(t)\lambda_{i_{s+1}})$ and since

$$\begin{aligned}&\sum_{\alpha\neq 0}D_{\xi}^{\alpha}\sigma(\Pi_s)\partial_x^{\alpha}(\xi_0-\sigma(t)\lambda_{i_{s+1}})\\&\quad=\sum_{j=0}^{s}\sum_{\beta=0}^{s-j}\sum_{\alpha\neq 0}\tilde{b}_{\beta-|\alpha|+1}(x, t, \xi)\sigma(t)^{\beta}\tau(t)^{-j}\sigma(t)\tau(t)^{-\alpha_0}\tau^{s-j-\beta-\alpha_0}\end{aligned}$$

where $\tilde{b}_i\in\mathscr{B}((0, T), S^i)$, the formula (2.8) is also valid for $s+1$, which completes the induction. Q.E.D.

PROOF OF LEMMA 2.6. From the condition (2.2) and Lemma 2.7 with $s=m$, we obtain

$$\sigma(P-\Pi_m)=\sum_{j=1}^{m}\sum_{i=0}^{m-j}c_{i,j}(x, t, \xi)\tau^{m-i-j}$$

where

$$c_{i,j}(x, t, \xi)=\sigma(t)^{i}\tau(t)^{-j}\tilde{c}_{i,j}(x, t, \xi)\ \text{for}\ i\geqq 1\ \text{with}\ \tilde{c}_{i,j}\in\mathscr{B}((0, T), S^{i})\,. \tag{2.10}$$

Let the homogeneous part of degree $m-1$ on (τ, ξ) be

$$\tilde{P}_{m-1}(x, t, \xi, \tau)=\sum_{i=0}^{m-1} c_{i,1}(x, t, \xi)\tau^{m-1-i} .$$

We want to determine $A_j(x, t, \xi) \in \mathscr{B}((0, T), S^0)$ so that

$$\tau(t)^{-1}\sum_{j=1}^{m} A_j(x, t, \xi)\prod_{i\neq j}(\tau-\sigma(t)\lambda_i)=\tilde{P}_{m-1}(x, t, \xi, \tau) . \tag{2.11}$$

From the condition (2.10) for $j=1$, we obtain

$$\tilde{P}_{m-1}(x, t, \xi, \sigma(t)\lambda_j)=\sigma(t)^{m-1}\tau(t)^{-1}K_j(x, t, \xi)$$

where each $K_j(x, t, \xi)$ is a smooth function in $\mathscr{B}((0, T), S^{m-1})$. Putting $\tau=\sigma(t)\lambda_j$ into (2.11) gives

$$\sigma(t)^{m-1}\tau(t)^{-1}A_j(x, t, \xi)\prod_{i\neq j}(\lambda_j-\lambda_i)=\sigma(t)^{m-1}\tau(t)^{-1}K_j(x, t, \xi) .$$

Then we can find

$$A_j(x, t, \xi)=\left[\prod_{i\neq j}(\lambda_j-\lambda_i)\right]^{-1}K_j(x, t, \xi) \text{ in } \mathscr{B}((0, T), S^0) .$$

Applying Lemma 2.7 with $s=m-1$ to $\sigma(\sum_{j=1}^{m} A_j \prod_{i\neq j}\partial_i)$, we obtain

$$\sigma\Big(P_{m-1}-\tau(t)^{-1}\sum_{j=1}^{m} A_j\prod_{i\neq j}\partial_i\Big)=\tau(t)^{-1}\Big\{\sum_{j=1}^{m-1}\sum_{i=0}^{m-1-j} d_{i,j}(x, t, \xi)\tau^{m-1-i-j}\Big\}$$

with $d_{i,j}(x, t, \xi)=\sigma(t)^i\tau(t)^{-1}\tilde{d}_{i,j}(x, t, \xi)$ for $i\geqq 1$ where $\tilde{d}_{i,j}\in\mathscr{B}((0, T), S^i)$. Next we pick up the homogeneous part of degree $m-2$ on (τ, ξ) in $\sum_{j=1}^{m-1}\sum_{i=0}^{m-1-j} d_{i,j}(x, t, \xi)\tau^{m-1-i-j}$, i.e., $\tilde{R}_{m-2}(x, t, \xi, \tau)=\sum_{i=0}^{m-2} d_{i,1}(x, t, \xi)\tau^{m-2-i}$. We shall represent this in a form:

$$\tau(t)^{-1}\sum_{j=2}^{m}\tilde{A}_j(x, t, \xi)\prod_{\substack{i\neq j\\ i\geqq 2}}(\tau-\sigma(t)\lambda_i) \quad \text{for } \tilde{A}_j\in\mathscr{B}((0, T), S^0) . \tag{2.12}$$

From $d_{i,1}=\sigma(t)^i\tau(t)^{-1}\tilde{d}_{i,1}$ provided for $i\geqq 1$, it results

$$\tilde{R}_{m-2}(x, t, \xi, \sigma(t)\lambda_j)=\sigma(t)^{m-2}\tau(t)^{-1}\tilde{K}_j(x, t, \xi)$$

with $\tilde{K}_j\in\mathscr{B}((0, T), S^{m-2})$. Then (2.12) with $\tau=\sigma(t)\lambda_j$ for $j=2, 3, \cdots, m$ shows

$$\sigma(t)^{m-2}\tau(t)^{-1}\tilde{A}_j(x, t, \xi)\prod_{\substack{i\neq j\\ i\geqq 2}}(\lambda_j-\lambda_i)=\sigma(t)^{m-2}\tau(t)^{-1}\tilde{K}_j(x, t, \xi) .$$

Then we can find $\tilde{A}_j$ so that

$$\tilde{A}_j(x, t, \xi) = \Big[\prod_{\substack{i\neq j\\ i\geq 2}} (\lambda_j - \lambda_i)\Big]^{-1} \tilde{K}_j(x, t, \xi) .$$

Successive use of these steps finally makes us attain to

$$\tilde{P}_{m-1}(x, t, D_x, D_t) = \sum_{k=1}^{m} \sum_{\alpha} \tau(t)^{-k} c^1_{\alpha,k} \omega^{\alpha}_{m-k}$$

where $c^1_{\alpha,k} \in \mathscr{B}((0, T), S^0)$ and $\omega^{\alpha}_{m-k} \in W_{m-k}$.

Now we proceed to the homogeneous part of $\sigma(P-\Pi_m)$ of degree $m-2$ on (τ, ξ);

$$\tilde{P}_{m-2} = \sum_{i=0}^{m-2} c_{i,2}(x, t, \xi) \tau^{m-2-i} .$$

Here $c_{i,2}(x, t, \xi) = \sigma(t)^i \tau(t)^{-2} \tilde{c}_{i,2}(x, t, \xi)$ for $i \geq 1$ with $\tilde{c}_{i,2}(x, t, \xi) \in \mathscr{B}((0, T), S^i)$. We want to determine $B_j(x, t, \xi) \in \mathscr{B}((0, T), S^0)$ for $j=2, 3, \cdots, m$ so that

$$\tilde{P}_{m-2} = \tau(t)^{-2} \Big\{ \sum_{j=2}^{m} B_j(x, t, \xi) \prod_{\substack{i\neq j\\ i\geq 2}} (\tau - \sigma(t)\lambda_i) \Big\} . \tag{2.13}$$

From the above conditions on $c_{i,2}$ we have

$$\tilde{P}_{m-2}(x, t, \xi, \sigma(t)\lambda_j) = \sigma(t)^{m-2} \tau(t)^{-2} Q_j(x, t, \xi)$$

for $j=2, 3, \cdots, m$ where $Q_j(x, t, \xi) \in \mathscr{B}((0, T), S^{m-2})$. Let us put $\tau = \sigma(t)\lambda_j$ into (2.13). Then we have

$$\sigma(t)^{m-2} \tau(t)^{-2} \prod_{\substack{i\neq j\\ i\geq 2}} (\lambda_j - \lambda_i) B_j(x, t, \xi) = \sigma(t)^{m-2} \tau(t)^{-2} Q_j(x, t, \xi) ,$$

so that

$$B_j(x, t, \xi) = \Big[\prod_{\substack{i\neq j\\ i\geq 2}} (\lambda_j - \lambda_i)\Big]^{-1} Q_j(x, t, \xi) \text{ in } \mathscr{B}((0, T), S^0) .$$

Applying Lemma 2.7 with $s=m-2$ to $\sigma(\sum_{j=2}^{m} B_j \prod_{\substack{i\neq j\\ i\geq 2}} \partial_i)$, we obtain

$$\sigma\Big(P_{m-2} - \tau(t)^{-2} \sum_{j=2}^{m} B_j \prod_{\substack{i\neq j\\ i\geq 2}} \partial_i\Big) = \tau(t)^{-2} \Big\{ \sum_{j=2}^{m-2} \sum_{i=0}^{m-2-j} e_{i,j}(x, t, \xi) \tau^{m-2-i-j} \Big\} .$$

Here $e_{i,j} = \sigma(t)^i \tau(t)^{-j} \tilde{e}_{i,j}(x, t, \xi)$ for $i \geq 1$ with $\tilde{e}_{i,j} \in \mathscr{B}((0, T), S^i)$. In the same way as the proof for P_{m-1} we have

$$\tilde{P}_{m-2}(x, t, D_x, D_t) = \sum_{k=2}^{m} \sum_{\alpha} \tau(t)^{-k} c^2_{\alpha,k} \omega^{\alpha}_{m-k}$$

where $c^2_{\alpha,k} \in \mathscr{B}((0, T), S^0)$ and $\omega^\alpha_{m-k} \in W_{m-k}$. Repeating the same process for $\tilde{P}_{m-3}, \cdots, \tilde{P}_0$, we conclude

$$\tilde{P}_{m-j}(x, t, D_x, D_t) = \sum_{k=j}^{m} \sum_{\alpha} \tau(t)^{-k} c^j_{\alpha,k} \omega^\alpha_{m-k}$$

for $j=1, 2, \cdots, m$ where $c^j_{\alpha,k} \in \mathscr{B}((0, T), S^0)$ and $\omega^\alpha_{m-k} \in W_{m-k}$. Therefore we obtain the representation (2.7) and the proof is completed. Q.E.D.

It can be easily seen from Lemma 2.6 that

$$\begin{aligned} \|\Pi_m u\|_s^2 &= \|(\Pi_m - P)u + Pu\|_s^2 \\ &\leqq 2\{\|(\Pi_m - P)u\|_s^2 + \|Pu\|_s^2\} \\ &\leqq \text{const}\{\Phi(t) + \|Pu\|_s^2\} . \end{aligned}$$

This inequality combined with Lemma 2.5 immediately shows

LEMMA 2.8. *For any $u \in C^\infty(\Omega)$ there exists a constant c so that*

$$\frac{d}{dt}\Phi(t) \leqq c\{\Phi(t) + \tau(t)^{-1}\Phi(t) + \tau(t)^{-1}\|Pu\|_s^2\} . \tag{2.14}$$

Let us proceed to obtain the energy estimate for P. Using the inequality (2.14), we have

$$\begin{aligned} \frac{d}{dt}\{e^{-ct}\Phi(t)\} &= -ce^{-ct}\Phi(t) + e^{-ct}\frac{d}{dt}\Phi(t) \\ &\leqq ce^{-ct}\{\tau(t)^{-1}\Phi(t) + \tau(t)^{-1}\|Pu\|_s^2\} \end{aligned}$$

and then

$$\begin{aligned} \frac{d}{dt}\{\sigma(t)^{-c}e^{-ct}\Phi(t)\} &= -c\sigma(t)^{-c-1}\sigma'(t)e^{-ct}\Phi(t) + \sigma(t)^{-c}\frac{d}{dt}\{\Phi(t)e^{-ct}\} \\ &\leqq c\sigma(t)^{-c}\tau(t)^{-1}\|Pu\|_s^2 . \end{aligned} \tag{2.15}$$

Now in order to be able to integrate both sides of (2.15) from 0 to t, the value $\sigma(t)^{-c}e^{-ct}\Phi(t)|_{t=0}$ must be finite.

Hence we shall consider the following argument.

LEMMA 2.9. *Under the condition of Theorem 1.1 there exist $A_1, A_2, \cdots, A_m \in \mathscr{B}((0, T), S^0)$ and a pseudo-differential operator of order $m-1$ with respect to (x, t) L_{m-1} such that*

$$\begin{aligned} P &= \partial_1\partial_2\cdots\partial_m + \sum_{j=2}^{m} A_{j-1}\partial_j\cdots\partial_m + \tau(t)^{-(m-1)}\sigma(t)L_{m-1} \\ &= L + \tau(t)^{-(m-1)}\sigma(t)L_{m-1} . \end{aligned}$$

Since this lemma is clear we omit the proof.

LEMMA 2.10. *Let us consider a equation $Lu=f$. Then we obtain the energy inequality*:

$$\|u(t)\|_s \leqq \text{const}\left\{\sum_{j=0}^{m-1}\|D_t^j u(0)\|_{s+m-1-j}+\|f(t)\|_s\right\}. \tag{2.16}$$

PROOF. From the form of L we can reduce $Lu=f$ to the first order system with diagonal principal part. Hence it follows easily the energy inequality (2.16). Q.E.D.

Let us consider the following Cauchy problem.

$$\begin{cases} Pu=f \\ D_t^j u|_{t=0}=0 \text{ for } j=0, 1, \cdots, m-1. \end{cases} \tag{2.17}$$

Here we shall define the function $u_i(x, t)(i\geqq 0)$ analogous to Nishitani [2] successively as follows.

$$\begin{cases} u_0 \equiv 0 \\ Lu_{i+1}=f-\tau(t)^{-(m-1)}\sigma(t)L_{m-1}u_i \\ D_t^j u_{i+1}|_{t=0}=0 \text{ for } j=0, 1, \cdots, m-1 \end{cases} \quad (i\geqq 0). \tag{2.18}$$

Hence we have

LEMMA 2.11. *For any integer i, the solution $u(x, t)$ of (2.17) is decomposed*

$$\begin{aligned} u&=(u-u_{i+1})+u_{i+1} \\ &=w_{i+1}+u_{i+1} \end{aligned} \tag{2.19}$$

where w_{i+1} and u_{i+1} have the estimates

$$\|w_{i+1}\|_s\leqq \text{const}\{\tau(t)^{-(m-1)}\sigma(t)\}^{i+1}\|u\|_{(m-1)(i+1),s} \tag{2.20}$$

$$\|u_{i+1}\|_s\leqq \text{const}\|f\|_{(m-1)i,s}. \tag{2.21}$$

Furthermore let $g_{i+1}=f-Pu_{i+1}$. Then

$$\|g_{i+1}\|_s\leqq \text{const}\{\tau(t)^{-(m-1)}\sigma(t)\}^{i+1}\|f\|_{(m-1)(i+1),s}. \tag{2.22}$$

PROOF. (i) Proof of (2.20). From (2.17) and (2.18) $L(u-u_{i+1})=-\tau(t)^{-(m-1)}\sigma(t)L_{m-1}(u-u_i)$. Then it follows from Lemma 2.10

$$\|u-u_{i+1}\|_s\leqq \text{const } \tau(t)^{-(m-1)}\sigma(t)\|u-u_i\|_{m-1,s}$$

$$\begin{aligned}
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
&\leqq \text{const } \{\tau(t)^{-(m-1)}\sigma(t)\}^{i+1}\|u-u_0\|_{(m-1)(i+1),s} \\
&\leqq \text{const } \{\tau(t)^{-(m-1)}\sigma(t)\}^{i+1}\|u\|_{(m-1)(i+1),s} \,.
\end{aligned}$$

(ii) Proof of (2.21). Combining (2.18) with Lemma 2.10, we have

$$\begin{aligned}
\|u_{i+1}\|_s &\leqq \text{const } \|f-\tau(t)^{-(m-1)}\sigma(t)L_{m-1}u_i\|_s \\
&\leqq \text{const } \{\|f\|_s+\tau(t)^{-(m-1)}\sigma(t)\|u_i\|_{m-1,s}\} \\
&\quad \cdots\cdots \\
&\quad \cdots\cdots \\
&\leqq \text{const } \left\{\sum_{j=0}^{i}[\tau(t)^{-(m-1)}\sigma(t)]^j\|f\|_{(m-1)j,s}\right\} \\
&\leqq \text{const } \|f\|_{(m-1)i,s} \,.
\end{aligned}$$

(iii) Proof of (2.22). Note that

$$g_{i+1}=\tau(t)^{-(m-1)}\sigma(t)L_{m-1}(u_{i+1}-u_i) \,.$$

Then

$$\|g_{i+1}\|_s \leqq \text{const } \tau(t)^{-(m-1)}\sigma(t)\|u_{i+1}-u_i\|_{m-1,s} \,.$$

Since we can estimate $u_{i+1}-u_i$ in the similar way, the estimate (2.22) follows easily. Q.E.D.

Now we shall investigate the Cauchy problem (2.17). Since $Pu=f$, w_{i+1} satisfies the equation $Pw_{i+1}=g_{i+1}$. Here we redefine $\Phi(t)$ replacing $u(x, t)$ by $w_{i+1}(x, t)$. From (2.15) and (2.20) we have

$$\begin{aligned}
(2.23)\quad \frac{d}{dt}\{\sigma(t)^{-c}e^{-ct}\Phi(t)\} &\leqq c\sigma(t)^{-c}\tau(t)^{-1}\|g_{i+1}\|_s^2 \\
&\leqq \text{const } \sigma(t)^{-c}\tau(t)^{-1}\{\tau(t)^{-(m-1)}\sigma(t)\}^{2(i+1)}\|f\|^2_{(m-1)(i+1),s} \\
&= \text{const } \sigma(t)^{-c+2(i+1)}\tau(t)^{-1-2(i+1)(m-1)}\|f\|^2_{(m-1)(i+1),s} \,.
\end{aligned}$$

Now we choose a positive integer i such that $i\geqq[c/2]+1$. Then $\sigma(t)^{-c}e^{-ct}\Phi(t)|_{t=0}=0$. Integrating (2.23) from 0 to t we obtain for any $\varepsilon>0$

$$\begin{aligned}
&\sigma(t)^{-c}e^{-ct}\Phi(t) \\
&\quad \leqq \text{const } \int_0^t \sigma(t)^{2(i+1)-c}\tau(t)^{-1-2(i+1)(m-1)}\|f(t)\|^2_{(m-1)(i+1),s}dt
\end{aligned}$$

$$\leqq \text{const} \int_0^t \sigma(t)^{2(i+1)-c-\varepsilon} \| f(t) \|^2_{(m-1)(i+1),s} dt$$

because $\tau(t)^{-(m-1)(i+1)} \leqq \text{const}\ \sigma(t)^{-\varepsilon}$ for any $\varepsilon > 0$. On the other hand from monotony of $\sigma(t)$ it follows

$$\sigma(t)^{-c} e^{-ct} \Phi(t) \leqq \text{const}\ \sigma(t)^{2(i+1)-c-\varepsilon} \int_0^t \| f(t) \|^2_{(m-1)(i+1),s} dt .$$

Hence

$$\Phi(t) \leqq \text{const}\ \sigma(t)^{2(i+1)-\varepsilon} \int_0^t \| f(t) \|^2_{(m-1)(i+1),s} dt . \tag{2.24}$$

From (2.21) and (2.24) we have following energy inequality.

LEMMA 2.12. *Under the condition of Theorem* 2.1 *for any real* s *and* $\varepsilon > 0$ *there exists some non-negative integer* i *only depending on* P *so that*

$$\| u(t) \|^2_s \leqq \text{const} \Big\{ \| f(t) \|^2_{(m-1)i,s} + \sigma(t)^{2(i+1)-\varepsilon} \tau(t)^{2m} \int_0^t \| f(t) \|^2_{(m-1)(i+1),s} dt \Big\} . \tag{2.25}$$

Following Nishitani [2] we shall complete the proof of Theorem 2.1. Define the δ-translation $P_\delta(x, t, D_x, D_t)$ of P by

$$P_\delta(x, t, D_x, D_t) = P(x, t+\delta, D_x, D_t) \quad (0 \leqq \delta \leqq \delta_0) .$$

Now we consider the following Cauchy problem.

$$\begin{aligned} &P_\delta(x, t, D_x, D_t) u_\delta(x, t) = f(x, t) \\ &D_t^j u_\delta(x, t)|_{t=0} = 0 \quad \text{for} \quad 0 \leqq j \leqq m-1 . \end{aligned} \tag{2.26}$$

Since P_δ is strictly hyperbolic for $\delta > 0$, (2.26) is well-posed. From (2.25) the energy inequality of (2.26) holds uniformly in δ such that

$$\| u_\delta(t) \|^2_s \leqq \text{const} \Big\{ \| f(t) \|^2_{(m-1)i,s} + \sigma(t)^{2(i+1)-\varepsilon} \tau(t)^{2m} \int_0^t \| f(t) \|^2_{(m-1)(i+1),s} dt \Big\} . \tag{2.27}$$

Then there exists a subsequence of $\{u_\delta(t)\}_{0<\delta\leqq\delta_0}$ which converges weakly in $\mathscr{B}^0((0, T), H^s)$. The limit function u is a unique solution of the Cauchy problem (2.17).

Finally we state the following theorem as a combined result of Theorem 1.1 and Theorem 2.1.

THEOREM 2.13. *Let P be a weakly hyperbolic operator with the property*:

The characteristic roots $\lambda_j(x, t, \xi)(1 \leqq j \leqq m)$ *are of the form* $\lambda_j(x, t, \xi) = \phi(x)\sigma(t)\tilde{\lambda}_j(x, t, \xi)$ *where* $\tilde{\lambda}_j \in \mathscr{B}((0, T), S^1)$ *are real valued,* $\tilde{\lambda}_i \neq \tilde{\lambda}_j$ *when* $i \neq j$ *and* $\phi(x)$ *is real valued smooth function in* $\mathscr{B}(\boldsymbol{R}^n)$. *In addition* $\sigma(t)$ *is the same function in Theorem* 2.1.

Then the Cauchy problem for P is well-posed if the lower order terms $P_{m-j}(x, t, \xi, \tau)(j=1, \cdots, m-1)$ *satisfy*

$$P_{m-j}(x, t, \xi, \lambda_i(x, t, \xi)) = \phi(x)^{m-j}\sigma(t)^{m-j}\tau(t)^{-j}K_{i,j}(x, t, \xi) \tag{2.28}$$

where $K_{i,j}(x, t, \xi) \in \mathscr{B}((0, T), S^{m-j})$.

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Present Address:
DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES AND ENGINEERINGS
WASEDA UNIVERSITY
NISHIOKUBO, SHINJUKU-KU, TOKYO 160