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Characterizations of the Ranges of Wave Operators for Symmetric Systems and an Application

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§1. Introduction.

The spectral and scattering theory for partial differential operators has been investigated extensively (see [1], [8], [10], [11], [13], [14], and references there). But certain ellipticity has been assumed to ensure the completeness of wave operators. Such an assumption excludes, for example, the equation of magnetgasdynamics which has a characteristic root like ξ_1 (see [3, p. 298]). The aim in this paper is to remove such an ellipticity assumption.

In [14] the author established the existence and completeness of the wave operators for "symmetric systems", which are rather general ones including symmetric hyperbolic systems and Schrödinger equations. The existence of the wave operators was established under the condition that perturbations are short range; no additional condition was assumed. In establishing the completeness, however, certain ellipticity was further assumed in order to apply the compactness argument. Without such an ellipticity assumption we shall show in this paper that the wave operators are complete in a weak sense. It is shown, for example, that the ranges of the wave operators for symmetric hyperbolic systems with characteristic roots of constant multiplicity are scattering subspaces. (For an abstract scattering theory which is not based on the subspace of absolute continuity, see [18].) In order to establish it we apply the method recently invented by Enss [5], which does not essentially rely on the compactness argument and is more direct than the method of Kato-Kuroda. The crucial tool which makes this application possible is the L_2-boundedness theorem of Calderon-Vaillancourt [4].

Now, we explain notations in order to state the results. We write $D_j=-i\partial/\partial x_j$, $x=(x_1, \cdots, x_n)\in \boldsymbol{R}^n$, $\langle x\rangle=(1+|x|^2)^{1/2}$, $\langle D\rangle^2=1+D_1^2+\cdots+D_n^2$, $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, where $\alpha=(\alpha_1, \cdots, \alpha_n)$ is a multi-index. For G_1 and G_2

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in $\boldsymbol{R}^n$, we write $G_1 \subset\subset G_2$ if the closure of G_1 is compact and is contained in the interior of G_2. We denote by $\mathscr{B}(\boldsymbol{R}^n)$ or $\mathscr{B}$ the space of C^∞-functions on $\boldsymbol{R}^n$, all of whose derivatives are bounded. No confusion will arise from using the same symbol for the space of matrix-valued functions, all of whose elements belong to $\mathscr{B}$. $L_2(\boldsymbol{R}^n)$ denotes the usual L_2-space on $\boldsymbol{R}^n$. For a positive integer m, we denote by $\boldsymbol{H}_0$ the Hilbert space $[L_2(\boldsymbol{R}^n)]^m$ with natural inner product. For $f \in \boldsymbol{H}_0$, the Fourier transform of f is denoted by $\hat{f}$. For an $m\times m$ matrix A, the norm of A is denoted by $|A|$. For a densely defined closed linear operator T between Banach spaces, $D(T)$, $R(T)$, $\sigma_p(T)$, and $\rho(T)$ denote the domain, range, point spectrum, and resolvent set of T, respectively. For Banach spaces X and Y, $\boldsymbol{B}(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y.

Consider the symmetric system

$$i\,\frac{\partial u}{\partial t}=L(x, D)u\equiv M(x)\Big[P(D)+\sum_{j=1}^{K} q_j(x)Q_j(D)\Big]u\,, \tag{1.1}$$

where $u(x, t)$ is a $\boldsymbol{C}^m$-valued function. We assume the following conditions (A.I) and (A.II).

(A.I) (i) $M(x)$ is an $m\times m$ matrix-valued measurable function on $\boldsymbol{R}^n$ such that $CI \leqq M(x) \leqq C^{-1}I$ for some positive constant C, where I is the unit matrix; (ii) $P(\xi)$ is a polynomial with $m\times m$ matrix-coefficients, and is Hermitian symmetric for each $\xi \in \boldsymbol{R}^n$; (iii) the differential operator $\sum_{j=1}^{K} q_j(x)Q_j(D)$ is formally self-adjoint.

(A.II) There exists a constant $s>1$ such that

$$\Big(|M(x)-I|+\sum_{j=1}^{K}|q_j(x)|\Big)\langle x\rangle^s \in L_\infty(\boldsymbol{R}^n)\,.$$

Let $\boldsymbol{H}$ be a Hilbert space with the inner product

$$(f, g)_{\boldsymbol{H}}=\int_{\boldsymbol{R}^n} M(x)^{-1}f(x)\overline{g(x)}dx,\ f, g \in [L_2(\boldsymbol{R}^n)]^m\,.$$

By virtue of (A.I.i) $\boldsymbol{H}$ is, as a vector space, equal to $\boldsymbol{H}_0$. The identification operator J form $\boldsymbol{H}_0$ to $\boldsymbol{H}$ is well-defined: $(Jf)(x)=f(x)$. We denote by $H^\infty(\boldsymbol{R}^n)$ the space of $\boldsymbol{C}^m$-valued C^∞-functions on $\boldsymbol{R}^n$, all of whose derivatives are L_2-functions. Let H be a self-adjoint extension of $L(x, D)|_{H^\infty(\boldsymbol{R}^n)}$ in $\boldsymbol{H}$, and let H_0 be the natural self-adjoint realization in $\boldsymbol{H}_0$ of the differential operator $P(D)$. Denote by $E_{0,ac}$ the projection of $\boldsymbol{H}_0$ onto the subspace of absolute continuity for H_0. Then it was shown in [14] that the wave operators

(1.2) $$W_{\pm}=s-\lim_{t\to\pm\infty} e^{itH}Je^{-itH_0}E_{0,ac}$$

exist and are isometric on $E_{0,ac}\boldsymbol{H}_0$.

Now, let us state our main results. Let $\boldsymbol{H}_c$ be the subspace of continuity with respect to H. Put

(1.3) $$\mathcal{N}_{\pm}=\{f\in\boldsymbol{H}_c;\ \lim_{\substack{j\to\infty\\ T\to\infty}}\sup_{\pm t\geq T}\|(1-J\psi_j(D)E_{0,ac}J^{-1})e^{-itH}f\|=0 \text{ for all } \psi_j \text{ with } \lim_{j\to\infty}\psi_j(\xi)=1 \text{ and } \sup_j\|\psi_j\|_{L_\infty(\boldsymbol{R}^n)}<\infty\}\ .$$

Then the first result is as follows.

Theorem 1. Under (A.I) and (A.II) one has

(1.4) $$R(W_{\pm})=\mathcal{N}_{\pm}\ .$$

It will be shown that $\mathcal{N}_{\pm}$ are closed subspaces, reduce H, and are contained in the scattering subspaces $\mathcal{M}_{\pm}$ for H. Here $\mathcal{M}_{\pm}$ are defined by

(1.5) $$\mathcal{M}_{\pm}=\left\{f\in\boldsymbol{H};\ \lim_{T\to\pm\infty}\frac{1}{T}\int_0^T\|\chi_R e^{-itH}f\|dt=0 \text{ for each } R>0\right\}\ ,$$

where χ_R is the characteristic function of the set $\{x\in\boldsymbol{R}^n;\ |x|<R\}$. (Different but similar definitions of scattering subspaces were given in [2], [16], and [18].)

The second result states that if $Q_j(D)$ is H_0-bounded and the roots of the equation $\det(\lambda I-P(\xi))=0$ have constant multiplicity (the precise conditions, (A.III)$\sim$(A.V), will be stated in § 3), then $R(W_{\pm})=\mathcal{M}_{\pm}$:

Theorem 2. Under (A.I)$\sim$(A.V) one has

(1.6) $$R(W_{\pm})=\mathcal{M}_{\pm}\ .$$

An application of this theorem is given. It states that the energy of the solution of the equation (1.1) propagates along the classical orbits corresponding to the Hamiltonian $P(\xi)$.

We note here that $\mathcal{M}_{+}=\mathcal{M}_{-}=\boldsymbol{H}_c$ if $\chi_R(H+i)^{-1}E_c$ is compact for every $R>0$, where E_c is the projection operator onto $\boldsymbol{H}_c$ (see [2], [16]). Therefore, if we assume certain ellipticity, we can conclude from Theorem 2 that

$$R(W_{+})=R(W_{-})=\boldsymbol{H}_c\ .$$

Thus Theorem 2 almost includes the results already known. However, it does not include Theorem 5.7 in [14], where no condition concerning the regularity of characteristic roots was imposed. An important open problem is to establish (1.6) without assuming the regularity of characteristic roots.

The remainder of this paper is organized as follows. In § 2 we shall show Theorem 1. Some lemmas there will be given in a more general form than is necessary for the proof of Theorem 1; they will be used also in proving Theorem 2. In § 3 we shall state Theorem 2 precisely and give its proof. There, we shall also give a theorem concerning the propagation of energy. § 4 is devoted to the study of the scattering theory for symmetric systems in an exterior domain; results analogous to those for the entire space $\boldsymbol{R}^n$ are given.

§ 2. Proof of Theorem 1.

This section is devoted to the proof of Theorem 1. To start with, we state some properties of the spaces $\mathcal{N}_\pm$ and $\mathcal{M}_\pm$ defined by (1.3) and (1.5), respectively.

PROPOSITION 2.1. (i) *$\mathcal{N}_\pm$ and $\mathcal{M}_\pm$ are closed linear subspaces of $\boldsymbol{H}_c$ and reduce H.* (ii) $\mathcal{N}_\pm \subset \mathcal{M}_\pm$.

PROOF. $\mathcal{N}_\pm$ and $\mathcal{M}_\pm$ are clearly closed linear subspaces of $\boldsymbol{H}$. They reduce H, for they are invariant under the transformation e^{-itH}. Since $\mathcal{M}_\pm$ are orthogonal to all eigenvectors of H, we obtain that $\mathcal{M}_\pm \subset \boldsymbol{H}_c$ (cf. [2], [18]). This completes the proof of (i). Now let us show (ii). For $f \in \mathcal{N}_\pm$, put $f(t) = e^{-itH}f$. Given $\varepsilon > 0$, we first choose a C_0^∞-function ψ and $N > 0$ such that

$$\sup_{\pm t \geqq N} \|(1 - J\psi(D)E_{0,ac}J^{-1})f(t)\| < \varepsilon \, . \tag{2.1}$$

Since $J\chi_R(x)\psi(D)E_{0,ac}J^{-1}$ is a compact operator, there exists an operator S of finite rank such that

$$\|J\chi_R(x)\psi(D)E_{0,ac}J^{-1} - S\| < \varepsilon \, . \tag{2.2}$$

The inequalities (2.1) and (2.2) imply

$$\|\chi_R f(t)\| \leqq 2\varepsilon + \|Sf(t)\|, \quad \pm t \geqq N \, . \tag{2.3}$$

Since $f \in \boldsymbol{H}_c$, we obtain that for each $g \in \boldsymbol{H}$

$$\lim_{T\to\pm\infty}\frac{1}{T}\int_0^T |(g,\ f(t))|dt=0$$

(see [2, Lemma 2]). Thus

$$\lim_{T\to\pm\infty}\frac{1}{T}\int_0^T \|Sf(t)\|dt=0\ .$$

This and (2.3) imply that $f\in\mathcal{M}_\pm$. q.e.d.

We first show that the ranges $R(W_\pm)$ of the wave operators $W_\pm$ are included in $\mathcal{N}_\pm$.

PROPOSITION 2.2. *Assume* (A.I) and (A.II). *Then the wave operators* $W_\pm$ *defined by* (1.2) *exist and are isometric on* $E_{0,ac}\boldsymbol{H}_0$. *Furthermore,*

$$R(W_\pm)\subset\mathcal{N}_\pm\ . \tag{2.4}$$

PROOF. Since the first half of the proposition was already shown in [14], it remains to show (2.4). Suppose that $\lim_{j\to\infty}\psi_j(\xi)=1$ and $\sup_j\|\psi_j\|_{L_\infty}<\infty$. Let $f=W_\pm g$ for some $g\in\boldsymbol{H}_0$ with $E_{0,ac}g=g$. Given $\varepsilon>0$, we can choose $T>0$ such that

$$\|e^{-itH}f-Je^{-itH_0}g\|<\varepsilon,\qquad \pm t>T\ .$$

On the other hand,

$$\begin{aligned}\limsup_{j\to\infty}\ \sup_{\pm t\geqq T}\ &\|(1-J\psi_j(D)E_{0,ac}J^{-1})Je^{-itH_0}g\|\\ &=\lim_{j\to\infty}\ \sup_{\pm t\geqq T}\ \|Je^{-itH_0}(1-\psi_j(D))g\|=0\ .\end{aligned}$$

Thus

$$\limsup_{j\to\infty}\ \sup_{\pm t\geqq T}\ \|(1-J\psi_j(D)E_{0,ac}J^{-1})e^{-itH}f\|<(1+\sup_j\|\psi_j\|_{L_\infty})\varepsilon\ .$$

This already proves (2.4), for $R(W_\pm)$ are closed subspaces of $\boldsymbol{H}_c$. q.e.d.

Next we show the reverse inclusion along the line given by Enss [5] and Simon [17]. To this end we prepare some lemmas.

LEMMA 2.3. *Let f be a real-valued C^∞-function on an open set $\Omega\subset\boldsymbol{R}^n$ such that $D^\alpha f\in L_\infty(\Omega)$ for all $|\alpha|\geqq 1$. Let G be a bounded open set such that* $\{\mathrm{grad}\,f(\xi);\ \xi\in\Omega\}\subset G$. *Then for each $k>0$ there exists a constant C such that*

$$\|(1+|x-x_0|+|t|)^k e^{-itf(D)}\phi(x)\|_{L_2(\{x\in R^n;\,(x-x_0)/t\notin G\})}\leqq C\|\langle x-x_0\rangle^k\phi\|_{L_2(R^n)} \tag{2.5}$$

for all ϕ with $\mathrm{Supp}\,\hat{\phi}\subset\Omega$, $x_0\in\boldsymbol{R}^n$, *and* $t\in\boldsymbol{R}$.

PROOF. Since $e^{-itf(D)}$ commutes with translation, we may assume that $x_0=0$. Choose a C^∞-function ψ on R^n such that $\psi(x)=1$ on $R^n \backslash G$ and $\psi(x)=0$ in an open neighborhood of the closure of $\{\operatorname{grad} f(\xi);\, \xi\in\Omega\}$, and put $\psi_t(x)=\psi(x/t)$. It is sufficient to show that

$$\|\psi_t(x)(1+|x|^2+t^2)^{k/2}e^{-itf(D)}\phi(x)\|_{L_2(R^n)} \leqq C\|\langle x\rangle^k\phi(x)\|_{L_2(R^n)} \tag{2.6}$$

for all ϕ with $\hat{\phi}\in C_0^\infty(\Omega)$. We have by integration by parts

$$e^{-itf(D)}\phi(x)=(2\pi)^{-n/2}\int e^{-itf(\xi)+ix\xi}\left[\nabla\cdot\frac{i(t\nabla f-x)}{|t\nabla f-x|^2}\right]^k\hat{\phi}(\xi)d\xi\ ,$$

where $\nabla=(\partial/\partial\xi_1,\cdots,\partial/\partial\xi_n)$. Elementary calculations show that there exists a positive constant C such that

$$\inf_{\xi\in\Omega}|t\nabla f(\xi)-x|\geqq C(|t|+|x|)$$

for all t and x with $\psi_t(x)\neq 0$. Thus we have

$$\begin{aligned}(1+|x|^2+t^2)^{k/2}\psi_t(x)e^{-itf(D)}\phi(x)\\ =\sum_{|\alpha|\leqq k}\int e^{ix\xi}a_\alpha(t;x,\xi)[e^{-itf(\xi)}D^\alpha\hat{\phi}(\xi)]d\xi\ ,\end{aligned} \tag{2.7}$$

where $a_\alpha(t;x,\xi)\in\mathscr{B}(R_x^n\times R_\xi^n)$. Hence (2.6) follows from the L_2-boundedness theorem of [4].

LEMMA 2.4. *Let f, g, and h be $m\times m$ matrix-valued functions on R^n which are commutative with one another and belong to $\mathscr{B}$. Then the following statements hold.*

(i) *If $d\equiv\operatorname{dis}(\operatorname{Supp} g, \operatorname{Supp} h)$ is positive, then for each k there exists a constant C such that*

$$\|g(x)f(D)h(x)\phi(x)\|\leqq Cd^{-k}\|\phi\|,\quad \phi\in[L_2(R^n)]^m\ . \tag{2.8}$$

(ii) *There exists a constant C such that for all $R>1$*

$$\|(g(x/R)f(D)-f(D)g(x/R))\phi(x)\|\leqq CR^{-1}\|\phi\|,\quad \phi\in[L_2(R^n)]^m\ . \tag{2.9}$$

PROOF. (i) The symbol $a(x,\xi)$ of the pseudo-differential operator $g(x)f(D)h(x)$ is represented by the oscillatory integral

$$\mathrm{Os}-\iint e^{-iy\eta}g(x)f(\xi+\eta)h(x+y)dy đ\eta\ ,$$

where $đ\eta=(2\pi)^{-n}d\eta$. (See [9].) We have for any $k>n/2$

$$a(x, \xi)=\iint e^{-iy\eta}g(x)\langle D_\eta\rangle^{2k}\{\langle\eta\rangle^{-2k}f(\xi+\eta)\}\langle y\rangle^{-2k}\langle D_y\rangle^{2k}h(x+y)dyđ\eta\ .$$

Since $|y|\geqq d$ if $g(x)h(x+y)\neq 0$, we obtain that $|a(x, \xi)|\leqq Cd^{n-2k}$. Similarly, for any multi-indices α and β there is a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha\partial_\xi^\beta a(x, \xi)|\leqq C_{\alpha,\beta}d^{n-2k}\ .$$

Thus (2.8) follows from the L_2-boundedness theorem.

(ii) Denoting by $b(x, \xi)$ the symbol of the pseudo-differential operator $f(D)g(x/R)$, we have

$$b(x, \xi)=\mathrm{Os}-\iint e^{-iy\eta}f(\xi+\eta)g((x+y)/R)dyđ\eta\ .$$

Put

$$G_R(x, y)=\int_0^1 \nabla g((x+ty)/R)dt\ .$$

Since $g((x+y)/R)=g(x/R)+R^{-1}G_R(x, y)\cdot y$, we have

$$b(x, \xi)=g(x/R)f(\xi)+R^{-1}\,\mathrm{Os}-\iint e^{-iy\eta}(-i\nabla_\eta f(\xi+\eta))\cdot G_R(x, y)dyđ\eta\ .$$

Since $\{G_R\}_{R>1}$ is a bounded set in $\mathscr{B}$, this implies (2.9). q.e.d.

LEMMA 2.5 ([17, §2, Lemma 2]). *Let f be a rapidly decreasing function such that $f\geqq 0$ and $\int fdx=1$. For each lattice point $\alpha\in \boldsymbol{Z}^n$, put*

$$f_\alpha=f*\chi_\alpha\ , \tag{2.10}$$

where χ_α is the characteristic function of the unit cube centered at α. Let $g_\alpha(\alpha\in\boldsymbol{Z}^n)$ be functions such that

$$\sup_\alpha \|\langle D\rangle^{2n}g_\alpha\|_{L_2(R^n)}<\infty\ .$$

For a rapidly decreacing function h, define Th by

$$(Th)(x)=\sum_\alpha g_\alpha(D)f_\alpha(x)h(x)\ . \tag{2.11}$$

Then

$$\|Th\|_{L_2}\leqq C\|h\|_{L_2}\ .$$

In order to state a lemma which is a key to the proof of the inclusion $\mathscr{N}_\pm\subset R(W_\pm)$, we review some of the notations and propositions appearing in [14]. Decompose the characteristic polynomial $\det(\lambda I-P(\xi))$ into irreducible factors $R_j (j=1, \cdots, q)$:

$$\det(\lambda I-P(\xi))=R_1^{m_1}\cdots R_q^{m_q} . \tag{2.12}$$

Here $R_i \neq R_j$ if $i \neq j$. Put

$$R=R_1\cdots R_q \text{ and } S(\xi)=\text{the discriminant of } R(\lambda, \xi) . \tag{2.13}$$

It is easily seen that $S(\xi)\not\equiv 0$. Enumerate the roots of the equation $R(\lambda, \xi)=0$ which are not identically constants as

$$\lambda_1(\xi)\leqq\cdots\leqq\lambda_r(\xi) , \tag{2.14}$$

where $\lambda_i \not\equiv \lambda_j$ if $i\neq j$. The remaining roots $\lambda_j(\xi) (j=r+1, \cdots, k)$ are identically constants: $\lambda_j(\xi)=a_j$. We denote by Λ_e the set of all exceptional values of $P(\xi)$:

$$\Lambda_e=\{\lambda\in \boldsymbol{R};\ R(\lambda, \xi)=S(\xi)=0 \text{ for some } \xi\in \boldsymbol{R}^n\}\cup\{a_j;\ j=r+1, \cdots, k\} . \tag{2.15}$$

We denote by $F_j(\xi)$ the orthogonal projection matrix in $\boldsymbol{C}^m$ onto the eigenspace corresponding to the eigenvalue $\lambda_j(\xi)$. Note that $F_j(\xi)$ and $\lambda_j(\xi)$ are smooth in $\{\xi\in \boldsymbol{R}^n;\ S(\xi)\neq 0\}$. As for the projection $E_{0,ac}$ of H_0 onto the subspace of absolute continuity for H_0, we have

PROPOSITION 2.6 ([14, Proposition 2.1]). $E_{0,ac}=\sum_{j=1}^{r} F_j(D)$.

We note here that the subspace of absolute continuity, subspace of continuity, and scattering subspaces for the operator H_0 are all equal.

Let Λ_c be the set of all critical values of the functions $\lambda_j(\xi)|_{\{S(\xi)\neq 0\}}$ $(j=1, \cdots, r)$ defined by

$$\Lambda_c=\bigcup_{j=1}^{r}\{\lambda_j(\xi);\ \operatorname{grad}\lambda_j(\xi)=0,\quad S(\xi)\neq 0\} . \tag{2.16}$$

We have the following proposition.

PROPOSITION 2.7 ([14, Lemmas 2.2 and 2.3]). *Λ_c is a finite set. Furthermore, there is a polynomial $T(\xi)$ such that $T(\xi)\not\equiv 0$ and*

$$\Lambda_c=\{\lambda\in \boldsymbol{R};\ R(\lambda, \xi)=T(\xi)=0 \text{ and } S(\xi)\neq 0 \text{ for some } \xi\in \boldsymbol{R}^n\} . \tag{2.17}$$

We can now define the operators $A_{j,\pm}(j=1, 2, \cdots)$ from $\boldsymbol{H}$ to $\boldsymbol{H}_0$ playing an important role in the proof of Theorem 1. Choose homogeneous functions $g_\pm$ of order zero in $C^\infty(\boldsymbol{R}^n\backslash\{0\})$ so that

$$g_+(\xi)+g_-(\xi)=1 \text{ on } \boldsymbol{R}^n\backslash\{0\}\ ,$$
$$g_\pm(\xi)=0 \text{ on } \{\mp\xi_1\geqq|\xi|/2\}\ .$$

For each $\alpha\in\boldsymbol{Z}^n$, choose a rotation R_α transferring α to $(|\alpha|, 0, \cdots, 0)$. Let Ψ be an $m\times m$ matrix-valued function on $\boldsymbol{R}^n$ belonging to $\mathscr{B}$ such that

$$V\equiv\operatorname{Supp}\Psi+\operatorname{Supp}\hat{f}\subset\{\xi\in\boldsymbol{R}^n;\ S(\xi)T(\xi)\neq 0\}\ , \tag{2.18}$$

where f is the function appearing in (2.10). Let Φ be an $m\times m$ matrix-valued C^∞-function on $\boldsymbol{R}^n$, all of whose derivatives are bounded on V. Then $A_{j,\pm}$ are defined by

$$A_{j,\pm}=\sum_{|\alpha|>j/3}\sum_{p,q=1}^{r} g_\pm(R_\alpha\nabla\lambda_p(D))F_p(D)\Phi(D)f_\alpha(x)\Psi(D)F_q(D)J^{-1}\ , \tag{2.19}$$

where $f_\alpha(x)$ is the function defined by (2.10).

The key lemma is stated as follows.

Lemma 2.8. *Let $Q(\xi)$ be an $m\times m$ matrix-valued C^∞-function, all of whose derivatives are bounded on the set V in* (2.18). *Assume that $D^\alpha\lambda_j(\xi)(j=1, \cdots, r, |\alpha|\geqq 1)$ are bounded on V and*

$$\inf\{|\nabla\lambda_j(\xi)|;\ \xi\in V,\ j=1, \cdots, r\}>0\ . \tag{2.20}$$

Then the following statements hold.

(i) *There exist positive constants δ and C such that for all t with $\pm t\geqq 0$*

$$\|\chi_{\delta(j+|t|)}(x)Q(D)e^{-itH_0}A_{j,\pm}\|_{B(H,H_0)}\leqq C(1+j+|t|)^{-2} \tag{2.21}$$

and

$$\|\chi_{\delta(j+|t|)}(x)Q(D)e^{-itH_0}J^*A^*_{j,\pm}\|_{B(H_0,H)}\leqq C(1+j+|t|)^{-2}\ . \tag{2.22}$$

(ii) *For each k there exists a positive constant C such that*

$$\|\chi_{j/5}(x)Q(D)A_{j,\pm}\|_{B(H,H_0)}\leqq C(1+j)^{-k} \tag{2.23}$$

and

$$\|\chi_{j/5}(x)Q(D)J^*A^*_{j,\pm}\|_{B(H_0,H)}\leqq C(1+j)^{-k}\ . \tag{2.24}$$

Proof. (i) We shall show only the inequality (2.21) for $t\geqq 0$, since the other ones can be proved similarly. For $\phi\in\boldsymbol{H}$ and $\alpha\in\boldsymbol{Z}^n$ with $|\alpha|>j/3$, put

$$\phi_\alpha=\sum_{p,q=1}^{r} g_+(R_\alpha\nabla\lambda_p(D))F_p(D)\Phi(D)f_\alpha(x)\Psi(D)F_q(D)J^{-1}\phi .$$

Choose positive constants a and b so that the set

$$G=\{v\in \boldsymbol{R}^n;\ a<|v|<b,\ v\cdot\alpha>-2|v||\alpha|/3\}$$

includes the set $\{\nabla\lambda_j(\xi);\ \xi\in \operatorname{Supp}\hat{\phi}_\alpha,\ j=1,\cdots,r\}$. Choose $\delta>0$ so small that for x and t with $|x|\leqq\delta(j+t)$

$$(x-\alpha)/t\notin G \text{ and } 1+|\alpha|+t\leqq 2(1+|x-\alpha|+t) .$$

Then it follows from Lemma 2.3 that for each $k>0$ there is a constant C independent of α and t such that

$$\|\chi_{\delta(j+t)}(x)Q(D)e^{-itH_0}\phi_\alpha\|_{L_2}\leqq C(1+|\alpha|+t)^{-k}\|\langle x-\alpha\rangle^k\phi_\alpha\|_{L_2} .$$

Since $\sup_\alpha\|\langle x-\alpha\rangle^k f_\alpha\|<\infty$, we have

$$\sup_\alpha\|\langle x-\alpha\rangle^k\phi_\alpha\|_{L_2}\leqq C\|\phi\|_H .$$

Hence

$$\|\chi_{\delta(j+t)}(x)Q(D)e^{-itH_0}\phi_\alpha\|_{H_0}\leqq C(1+|\alpha|+t)^{-k}\|\phi\|_H . \tag{2.25}$$

Summing up (2.25) on α we obtain (2.21).

(ii) We can write

$$A_{j,\pm}=\sum_{|\alpha|>j/3} G_1(D)f_\alpha(x)G_2(D)J^{-1} ,$$

where $G_1, G_2\in\mathscr{B}$. Choosing a C^∞-function ψ such that $\psi(x)=1$ on $\{|x|\leqq 1/11\}$ and $\psi(x)=0$ on $\{|x|\geqq 1/10\}$, we put

$$f_{\alpha,1}(x)=\psi((x-\alpha)/|\alpha|)f_\alpha(x) \text{ and } f_{\alpha,2}(x)=(1-\psi((x-\alpha)/|\alpha|))f_\alpha(x) .$$

Since $\{f_{\alpha,1}\}_\alpha$ is a bounded set in $\mathscr{B}$ and

$$\operatorname{dis}(\operatorname{Supp}\chi_{j/5},\ \operatorname{Supp}f_{\alpha,1})\geqq j/30 ,$$

it follows from Lemma 2.4 that for each $k>0$ there is a constant C such that

$$\|\chi_{j/5}\sum_{|\alpha|>j/3} G_1(D)f_{\alpha,1}(x)G_2(D)J^{-1}\|_{B(H,H_0)}\leqq C(1+j)^{-k} . \tag{2.26}$$

On the other hand,

$$\|f_{\alpha,2}(x)\|_{L_\infty}\leqq C(1+|\alpha|)^{-k-n} .$$

This implies that the inequality obtained from (2.26) by replacing $f_{\alpha,1}$ with $f_{\alpha,2}$ holds. The proof of (2.23) is now complete. The inequality (2.24) can be shown similarly. q.e.d.

Completion of the proof of Theorem 1. We shall show only the inclusion $\mathcal{N}_+ \subset R(W_+)$, for the other one can be proved similarly. Let $\phi \in \mathcal{N}_+$. Given $\varepsilon > 0$, we first choose Ψ in $C_0^\infty(\{\xi \in \boldsymbol{R}^n;\ S(\xi)T(\xi) \neq 0\})$ so that

$$\|(1 - J\Psi(D)E_{0,ac}J^{-1})e^{-itH}\phi\| < \varepsilon \tag{2.27}$$

when t is sufficiently large. Since it follows from Proposition 2.1 (ii) that $\phi \in \mathcal{M}_+$, we can find a sequence $\{t_j\}_{j=1}^\infty$ of positive numbers such that $\max(j, t_{j-1}) \leqq t_j$ and

$$\lim_{j\to\infty} \|\chi_j(x)e^{-it_jH}\phi\| = 0 . \tag{2.28}$$

Choosing a rapidly decreasing function f such that $f \geqq 0$, $\int f dx = 1$, $\hat{f} \in C_0^\infty$, and

$$\operatorname{Supp} \hat{f} + \operatorname{Supp} \Psi \subset \{\xi \in \boldsymbol{R}^n;\ S(\xi)T(\xi) \neq 0\} ,$$

we define $f_\alpha(\alpha \in \boldsymbol{Z}^n)$ by (2.10). Define operators $A_{j,\pm}$ by (2.19) with $\Phi = 1$, and put

$$\phi_{j,\pm} = A_{j,\pm}e^{-it_jH}\phi \quad \text{and} \quad \phi_j = e^{-it_jH}\phi .$$

We have

$$\begin{aligned}\Psi(D)E_{0,ac}J^{-1}\phi_j &= \sum_\alpha E_{0,ac}f_\alpha(x)\Psi(D)E_{0,ac}J^{-1}\phi_j \\ &\equiv \sum_{|\alpha|>j/3} + \phi_{j,w} = \phi_{j,+} + \phi_{j,-} + \phi_{j,w} .\end{aligned}$$

Since it follows from Lemma 2.4 and (2.28) that

$$\lim_{j\to\infty} \|\chi_{j/2}(x)\Psi(D)E_{0,ac}J^{-1}\phi_j\| = 0 ,$$

we obtain

$$\lim_{j\to\infty} \|\phi_{j,w}\| = 0 . \tag{2.29}$$

We next claim that

$$\lim_{j\to\infty} \|\phi_{j,-}\| = 0 . \tag{2.30}$$

We have

$$\|A_{j,-}e^{-it_jH}-A_{j,-}Je^{-it_jH_0}J^*\|_{B(H,H_0)}=\|A_{j,-}^*-e^{-it_jH}Je^{it_jH_0}J^*A_{j,-}^*\|_{B(H_0,H)}$$
$$\leqq\|(J^*J-1)A_{j,-}^*\|+\int_0^\infty\|J(\sum_\nu q_\nu(x)Q_\nu(D)+(M(x)-I)P(D))e^{itH_0}J^*A_{j,-}^*\|dt\ .$$

Since $(J^*f)(x)=M(x)f$, we thus obtain by Lemma 2.8 and the assumption (A.II)

$$\lim_{j\to\infty}\|A_{j,-}e^{-it_jH}-A_{j,-}Je^{-it_jH_0}J^*\|_{B(H,H_0)}=0\ . \tag{2.31}$$

On the other hand, since for each R $\|\chi_R(x)e^{it_jH_0}J^*A_{j,-}^*\|\to 0$ as $j\to\infty$, we obtain that for any ψ in H_0 with compact support

$$\|A_{j,-}Je^{-it_jH_0}\psi\|\longrightarrow 0\ .$$

Thus

$$\operatorname*{s-lim}_{j\to\infty} A_{j,-}Je^{-it_jH_0}J^*=0\ . \tag{2.32}$$

Combining (2.31) and (2.32), we get the claim (2.30).

We have by Lemma 2.8

$$\|(W_+-J)\phi_{j,+}\|\leqq\int_0^\infty\|(HJ-JH_0)e^{-itH_0}\phi_{j,+}\|dt$$
$$\leqq C\int_0^\infty\{(1+j+t)^{-2}+[\delta(j+t)]^{-\epsilon}\}dt\ .$$

Thus

$$\lim_{j\to\infty}\|(W_+-J)\phi_{j,+}\|=0\ . \tag{2.33}$$

Combining (2.27), (2.29), (2.30), and (2.33), we obtain that for sufficiently large j

$$\|e^{-it_jH}\phi-W_+\phi_{j,+}\|<2\varepsilon\ . \tag{2.34}$$

Hence

$$\|\phi-W_+e^{it_jH_0}\phi_{j,+}\|<2\varepsilon\ ,$$

which implies that $\phi\in R(W_+)$. This completes the proof.

§ 3. Scattering states. Propagation of energy.

In this section we shall prove Theorem 2 and give its application. Throughout this section we use the notations (2.12)$\sim$(2.17) in § 2.

First, let us state Theorem 2 precisely; we must formulate the conditions (A.III)$\sim$(A.V). Put

$$\Lambda=\Lambda_e \cup \Lambda_c \ . \tag{3.1}$$

For an interval I and the characteristic root $\lambda_j(\xi)$ which is not identically constant, we put

$$\Omega_j(I)=\{\xi \in \boldsymbol{R}^n;\ \lambda_j(\xi) \in I\}\ . \tag{3.2}$$

The condition (A.III) is then stated as follows.

(A.III) (i) Λ_e is a discrete set; (ii) for each interval $I_1 \subset \boldsymbol{R}\backslash\Lambda$ and $j(j=1, \cdots, r)$, there exist positive constants a and b and an interval I_2 such that

$$I_1 \subset I_2 \subset \boldsymbol{R}\backslash\Lambda,\ \operatorname{dis}(\partial\Omega_j(I_2),\ \Omega_j(I_1))>0\ , \tag{3.3}$$

$$\{|\nabla\lambda_j(\xi)|;\ \xi \in \Omega_j(I_2)\} \subset [a, b]\ , \tag{3.4}$$

$$\inf\{|\lambda_i(\xi)-\lambda_j(\xi)|;\ \xi \in \Omega_j(I_2),\ i \neq j,\ 1 \leqq i \leqq k\} \geqq a \tag{3.5}$$

and $D^\alpha\lambda_j(\xi)(|\alpha| \geqq 2)$ are bounded on $\Omega_j(I_2)$.

Suggested by Simon [17], we formulate the conditions (A.IV) and (A.V) as follows; they are a little weaker than the condition that $Q_j(D)$ is $P(D)$-bounded.

(A.IV) There exists an $m\times m$ matrix-valued function R_1 on $\boldsymbol{R}^n$ such that $R_1^{-1} \in \mathscr{B}$, $R_1(\xi)P(\xi)=P(\xi)R_1(\xi)$, and

$$Q_j(\xi)(P(\xi)-z)^{-1}R_1^{-1}(\xi) \in \mathscr{B},\ z \in \boldsymbol{C}\backslash\boldsymbol{R},\ j=1, \cdots, K\ .$$

Put

$$X=\{f \in \boldsymbol{H};\ R_1(\xi)(P(\xi)-i)\hat{f}(\xi) \in L_2\}\ . \tag{3.6}$$

(We note that $H^\infty(\boldsymbol{R}^n) \subset X \subset JD(H_0)$.) Let H be a self-adjoint extension of $L(x, D)|_X$, and denote by E the spectral measure associated with H. Then (A.V) is stated as follows.

(A.V) There exists an $m\times m$ matrix-valued function R_2 on $\boldsymbol{R}^n$ such that $R_2^{-1} \in L_\infty$, $R_2(D)E(I)$ is a bounded operator for each $I \subset \boldsymbol{R}$, and for every $\varepsilon>0$ there is a positive constant δ such that $R_1(\xi)$ is bounded on the set $\{\xi+\eta;\ |R_2^{-1}(\xi)| \geqq \varepsilon,\ |\eta|<\delta\}$.

We can now state Theorem 2.

THEOREM 2. *Asuume* (A.I)$\sim$(A.IV). *Let H be a self-adjoint extension in $\boldsymbol{H}$ of $L(x, D)|_X$, and assume* (A.V). *Then the ranges $R(W_\pm)$ of the wave operators defined by* (1.2) *are equal to $\mathscr{M}_\pm$: $R(W_\pm)=\mathscr{M}_\pm$.*

For the proof we need a lemma.

LEMMA 3.1. *Let Φ be a continuous function on $\boldsymbol{R}$ vanishing at infinity. Let $I\subset \boldsymbol{R}$. Let g be a C^∞-function such that $g(x)=0$ on $\{|x|<1\}$ and $g(x)=1$ on $\{|x|>2\}$, and let $g_j(x)=g(x/j)(j=1, 2, \cdots)$. Then for each $\varepsilon>0$ one can choose a function Ψ in $\mathscr{B}$ such that $R_1\Psi \in \mathscr{B}$ and*

$$\|\{\Phi(H)-J\Phi(H_0)\Psi(D)J^{-1}\}g_j(x)E(I)\|_{B(H)}<\varepsilon \tag{3.7}$$

for sufficiently large j.

PROOF. We shall show the lemma along the line given in [17] (see [17, §2, Lemmas 4 and 5]). Since finite linear combinations of the functions $(t-z)^{-1}$ $(\operatorname{Im} z\neq 0)$ are dense in the Banach space of all continuous functions on $\boldsymbol{R}$ vanishing at infinity, it suffices to consider the case $\Phi(t)=(t-z)^{-1}$ with $\operatorname{Im} z\neq 0$. We have by (A.II)$\sim$(A.IV)

$$\begin{aligned}
&\|\{\Phi(H)-J\Phi(H_0)J^{-1}\}R_1^{-1}(D)g_j(x)\| \\
&\quad =\|(H-z)^{-1}J[(M(x)-I)P(D)+\sum_\nu q_\nu(x)Q_\nu(D)](P(D)-z)^{-1}R_1^{-1}(D)g_jJ^{-1}\| \\
&\quad \leqq C(j^{-\epsilon}+\|(1-g_{j/3})a_0(D)g_j\|+\sum_\nu\|(1-g_{j/3})a_\nu(D)g_j\|)\ ,
\end{aligned}$$

where a_0, $a_\nu\in\mathscr{B}$. Thus Lemma 2.4 (i) yields

$$\|\{\Phi(H)-J\Phi(H_0)J^{-1}\}R_1^{-1}(D)g_j(x)\|\leqq Cj^{-1}\ . \tag{3.8}$$

For each $\varepsilon>0$, choose Ψ in $\mathscr{B}$ so that $R_1(\xi)\Psi(\xi)\in\mathscr{B}$ and

$$\|(1-\Psi(\xi))R_2^{-1}(\xi)\|_{L_\infty}<\varepsilon\delta\ ,$$

where δ is a sufficiently small number. Then we have by (3.8)

$$\begin{aligned}
&\|\{\Phi(H)-J\Phi(H_0)\Psi(D)J^{-1}\}g_j(x)E(I)\| \\
&\qquad \leqq \|\{\Phi(H)-J\Phi(H_0)J^{-1}\}\Psi(D)g_j\|+C\|(1-\Psi(D))g_jE(I)\| \\
&\qquad \leqq C(j^{-1}\|R_1(D)\Psi(D)\|+\|[R_1(D)\Psi(D),\ g_j(x)]\|) \\
&\qquad\quad +C(\varepsilon\delta\|R_2(D)E(I)\|+\|[1-\Psi(D),\ g_j(x)]\|)\ ,
\end{aligned}$$

where $[X, Y]=XY-YX$. Hence we obtain by Lemma 2.4 (ii)

$$\|\{\Phi(H)-J\Phi(H_0)\Psi(D)J^{-1}\}g_j(x)E(I)\|\leqq C(j^{-1}+\varepsilon\delta)\ .$$

This implies (3.7).

PROOF OF THEOREM 2. Since $R(W_\pm)\subset\mathscr{N}_\pm\subset\mathscr{M}_\pm$, it remains to show the inclusion $\mathscr{M}_\pm \subset R(W_\pm)$. Since $\bigcup_{I\subset R\setminus\Lambda} E(I)\mathscr{M}_\pm$ are dense in $\mathscr{M}_\pm$, we have only to show that $E(I)\mathscr{M}_\pm \subset R(W_\pm)$ for each $I\subset \boldsymbol{R}\setminus\Lambda$. Let us show the inclusion $E(I)\mathscr{M}_+ \subset R(W_+)$. (We show only this inclusion, for the other one can be shown similarly.) Let $\phi\in E(I)\mathscr{M}_+$. We can find a sequence $\{t_j\}_{j=1}^\infty$ of positive numbers such that $\max(j, t_{j-1})\leqq t_j$ and $\lim_{j\to\infty}\|\chi_j(x)J^{-1}e^{-it_jH}\phi\|=0$. Choose a function Φ in $C_0^\infty(\boldsymbol{R}\setminus\Lambda)$ so that $\Phi(H)\phi=\phi$. For each $\varepsilon>0$, we can choose by Lemma 3.1 a function Ψ in $\mathscr{B}$ such that $R_1\Psi\in\mathscr{B}$ and

$$\lim_{j\to\infty}\|e^{-it_jH}\phi-J\Phi(H_0)\Psi(D)J^{-1}e^{-it_jH}\phi\|<\varepsilon .$$

By the same argument as in the proof of Theorem 1, we obtain the decomposition

$$\Phi(H_0)\Psi(D)J^{-1}e^{-it_jH}\phi=\phi_{j,+}+\phi_{j,-}+\phi_{j,w} ,$$

where

$$\lim_{j\to\infty}(\|(W_+-J)\phi_{j,+}\|+\|\phi_{j,-}\|+\|\phi_{j,w}\|)=0 .$$

(In doing so, use the equalities: $\Phi(H_0)=\Phi(H_0)E_{0,ac}$ and $e^{-itH_0}R_1^{-1}(D)=R_1^{-1}(D)e^{-itH_0}$.) Thus

$$\lim_{j\to\infty}\|\phi-W_+e^{it_jH_0}\phi_{j,+}\|<\varepsilon ,$$

which implies that $\phi\in R(W_+)$. The proof is complete.

Next, let us discuss the propagation of energy of the solution. We begin with the following theorem.

THEOREM 3.2. *Assume* (A.I) *and* (A.II). *Let H be a self-adjoint extension of $L(x, D)|_{H^\infty(R^n)}$ in $\boldsymbol{H}$. Let I be a finite or infinite interval of $\boldsymbol{R}$, and let Γ be an open set including the set*

$$\bigcup_{j=1}^{r}\{-\nabla\lambda_j(\xi);\ \lambda_j(\xi)\in I,\ S(\xi)\neq 0\} . \tag{3.9}$$

For $t\in\boldsymbol{R}$, set $\Gamma_t=\{tx;\ x\in\Gamma\}$; denote by χ_{Γ_t} the characteristic function of Γ_t. Then

$$\lim_{t\to\pm\infty}\|\chi_{\Gamma_t}e^{-itH}\phi\|_H=\|\phi\|_H,\ \ \phi\in E(I)R(W_\pm) . \tag{3.10}$$

PROOF. We put

$$\Omega=\bigcup_{1\leq j\leq r}\{\xi\in\boldsymbol{R}^n;\ \lambda_j(\xi)\in I,\ S(\xi)T(\xi)\neq 0\} .$$

We denote by E_0 the spectral measure associated with H_0. Since $E(I)W_\pm = W_\pm E_0(I)E_{0,ac}$ and $C_0^\infty(\Omega)$ is dense in $E_0(I)E_{0,ac}\boldsymbol{H}_0$, it suffices to show (3.10) for $\phi = W_\pm\psi$ with $\psi \in C_0^\infty(\Omega)$ and $E_{0,ac}\psi=\psi$. For such ψ, Lemma 2.3 gives

$$\lim_{t\to\pm\infty}\{\|(1-\chi_{\Gamma_t})e^{-itH_0}\psi\|_{H_0}+\|(J-J_1)e^{-itH_0}\psi\|_H\}=0 ,$$

where J_1 is the unitary operator from $\boldsymbol{H}_0$ to $\boldsymbol{H}$ defined by $(J_1f)(x)=M(x)^{1/2}f(x)$. Thus

$$\begin{aligned}\lim_{t\to\pm\infty}\|\chi_{\Gamma_t}e^{-itH}\phi\|_H &\geqq \lim_{t\to\pm\infty}(\|J_1\chi_{\Gamma_t}e^{-itH_0}\psi\|_H-\|e^{-itH}\phi-J_1e^{-itH_0}\psi\|_H)\\ &=\lim_{t\to\pm\infty}\|\chi_{\Gamma_t}e^{-itH_0}\psi\|_{H_0}=\|\psi\|_{H_0}=\|\phi\|_H .\end{aligned}$$

This implies (3.10). q.e.d.

The following corollary is concerned with the lower bounds at infinity for the solutions of (1.1).

COROLLARY 3.3. *Let the hypotheses of Theorem* 3.2 *be satisfied, and assume that the open set Γ includes also the origin. Assume further that $R(W_\pm)=\mathscr{M}_\pm$. Let $\phi\in\boldsymbol{H}$. If*

$$\lim_{T\to\pm\infty}\frac{1}{T}\int_0^T\|\chi_{\Gamma_t}e^{-itH}\phi\|_H^2dt=0 , \tag{3.11}$$

then $E(I)\phi=0$.

PROOF. We have

$$\begin{aligned}&\|\chi_{\Gamma_t}e^{-itH}E(I)\phi\|^2\\ &\quad\leqq\|\chi_{\Gamma_t}e^{-itH}\phi\|^2+2\,\mathrm{Re}\,((1-\chi_{\Gamma_t})e^{-itH}E(I)\phi,\ e^{-itH}E(\boldsymbol{R}\backslash I)\phi) .\end{aligned}\tag{3.12}$$

Since the assumption that $R(W_\pm)=\mathscr{M}_\pm$ and (3.11) imply that $\phi\in R(W_\pm)$, we obtain by Theorem 3.2, (3.11), and (3.12)

$$\lim_{T\to\pm\infty}\frac{1}{T}\int_0^T\|\chi_{\Gamma_t}e^{itH}E(I)\phi\|^2dt=0 .$$

Hence $E(I)\phi=0$. q.e.d.

REMARK 3.4. (i) For symmetric hyperbolic systems, we can always replace the set Γ_t in (3.11) with a bounded set by using the property of finite propagation: Let $P(D)=\sum_j A_jD_j+B$, and put

$$\sigma=\max\{\lambda;\ \det(\lambda I-\sum_j A_j\xi_j)=0,\ |\xi|=1\} .$$

Let γ be an open set including the set

$$\bigcup_{1\leq j\leq r}\{-\nabla\lambda_j(\xi);\ \lambda_j(\xi)\in I,\ S(\xi)\neq 0\}\cap\{|x|<\sigma\}\ ,$$

and put $\gamma_t=\{tx;\ x\in\gamma\}$. Then the conclusion of Corollary 3.3 holds if (3.11) is satisfied with Γ_t replaced by $\gamma_t+\{x\in \boldsymbol{R}^n;\ |x|<R\}$, where R runs over $(0, \infty)$.

(ii) Corollary 3.3 extends a part of Theorem 2.1 in [7] to the variable coefficients case. (See also [6], [12], [15].)

(iii) The assumption $R(W_\pm)=\mathscr{M}_\pm$ is satisfied also by systems which do not necessarily satisfy the condition (A.III) (see [14]).

§ 4. Exterior problems.

In this section we shall study the scattering theory for symmetric systems in an exterior domain, and give results analogous to those for symmetric systems in $\boldsymbol{R}^n$. Since all theorems to be formulated in this section can be shown along the line described in §§ 2 and 3, we omit the proof.

Let G be a domain in $\boldsymbol{R}^n$ such that $\boldsymbol{R}^n\backslash G$ is compact. We consider the differential operator

$$L(x, D)=M(x)\Big[P(D)+\sum_{j=1}^{K}q_j(x)Q_j(D)\Big] \tag{4.1}$$

in G under the following conditions (A.I)′ and (A.II)′.

(A.I)′ (i) $M(x)$ is an $m\times m$ matrix-valued measurable function on G such that $CI\leqq M(x)\leqq C^{-1}I$ for some positive constant C; (ii) $P^*(\xi)=P(\xi)$; (iii) the differential operator $\sum_j q_j(x)Q_j(D)$ is formally self-adjoint.

(A.II)′ There exists a constant $s>1$ such that

$$\Big(|M(x)-I|+\sum_{j=1}^{K}|q_j(x)|\Big)\langle x\rangle^s\in L_\infty(G)\ .$$

Let $\boldsymbol{H}_0$ be the Hilbert space $[L_2(\boldsymbol{R}^n)]^m$ with natural inner product. Let $\boldsymbol{H}$ be the Hilbert space $[L_2(G)]^m$ with the inner product

$$(f, g)_{\boldsymbol{H}}=\int_G M(x)^{-1}f(x)\bar{g}(x)dx\ .$$

Let J_1 be the identification operator from $\boldsymbol{H}_0$ to $\boldsymbol{H}$ given by the restrictions of functions to G. Let J_2 be the operator from $\boldsymbol{H}$ to $\boldsymbol{H}_0$ defined by

$$(J_2f)(x)=\begin{cases} f(x), & x\in G \\ 0, & x\notin G\,. \end{cases}$$

Let ρ be a C^∞-function on $\boldsymbol{R}^n$ such that $\rho(x)=0$ for x with $\operatorname{dis}(x, \boldsymbol{R}^n\backslash G)\leqq 1$ and $\rho(x)=1$ on $\{x;\ \operatorname{dis}(x, \boldsymbol{R}^n\backslash G)\geqq 2\}$. Let H be a self-adjoint extension of $L(x, D)$ restricted to $\{\rho(x)f(x);\ f(x)\in H^\infty(\boldsymbol{R}^n)\}$. Put

$$\mathscr{N}_\pm=\{f\in \boldsymbol{H}_e;\ \lim_{\substack{j\to\infty\\ T\to\infty}}\sup_{\pm t\geqq T}\|(1-J_1\psi_j(D)E_{0,ac}J_2)e^{-itH}f\|=0 \quad \text{if } \lim_{j\to\infty}\psi_j(\xi)=1 \text{ and } \sup_j\|\psi_j\|_{L_\infty}<\infty\}\,. \tag{4.2}$$

Then we have the following analogue to Theorem 1.

THEOREM 4.1. *The wave operators*

$$W_\pm=s\text{-}\lim_{t\to\pm\infty} e^{itH}J_1e^{-itH_0}E_{0,ac}$$

exist and are isometric on $E_{0,ac}\boldsymbol{H}_0$. *Furthermore,*

$$R(W_\pm)=\mathscr{N}_\pm\,.$$

Now, let us state the analogue to Theorem 2. We require the following modifications of (A.IV) and (A.V).

(A.IV)′ There exists an $m\times m$ matrix-valued function R_1 on $\boldsymbol{R}^n$ such that $R_1^{-1}\in\mathscr{B}$, $R_1(\xi)P(\xi)=P(\xi)R_1(\xi)$, and

$$Q_j^{(\alpha)}(\xi)(P(\xi)-z)^{-1}R_1^{-1}(\xi)\in\mathscr{B},\ z\in\boldsymbol{C}\backslash\boldsymbol{R},\ j=1,\cdots,K,\ |\alpha|\geqq 0\,.$$

Let H be a self-adjoint extension of $L(x, D)|_X$, where

$$X=\{J_1\rho(x)(P(D)-i)^{-1}R_1^{-1}(D)g;\ g\in\boldsymbol{H}_0\}\,.$$

We denote by E the spectral measure associated with H.

(A.V)′ There exists an $m\times m$ matrix-valued function R_2 on $\boldsymbol{R}^n$ such that $R_2^{-1}\in L_\infty$, $R_2(D)J_2\rho(x)E(I)$ is a bounded operator for each $I\subset\boldsymbol{R}$, and for every $\varepsilon>0$ there is a positive constant δ such that $R_1(\xi)$ is bounded on $\{\xi+\eta;\ |R_2^{-1}(\xi)|\geqq\varepsilon,\ |\eta|<\delta\}$.

The scattering subspaces $\mathscr{M}_\pm$ for H are defined with an obvious modification; the analogue to Proposition 2.1 holds. The analogue to Theorem 2 is stated as follows.

THEOREM 4.2. *Assume* (A.I)′, (A.II)′, (A.III), *and* (A.IV)′. *Let H be a self-adjoint extension of* $L(x, D)|_X$ *in* $\boldsymbol{H}$, *and assume* (A.V)′. *Then* $R(W_\pm)=\mathscr{M}_\pm$.

The following theorem is essentially known (cf. [10], [14], [17]).

THEOREM 4.3. *Let the hypotheses of Theorem* 4.2 *be satisfied. Assume that* $\chi_R E(I)$ *is compact for each* $R>0$ *and* $I\subset \boldsymbol{R}\backslash \Lambda$, *where* Λ *is the set defined by* (3.1). *Then the following statements hold.*

(i) $R(W_+)=R(W_-)=\boldsymbol{H}_c=\boldsymbol{H}_{ac}=\mathscr{M}_+=\mathscr{M}_-$, *where* $\boldsymbol{H}_{ac}$ *is the subspace of absolute continuity for* H.

(ii) *The only possible limit points for the point spectrum of* H *are in* Λ. *Each eigenvalue not in* Λ *has finite multiplicity.*

The following theorem and corollary are analogues to Theorem 3.2 and Corollary 3.3.

THEOREM 4.4. *Let the hypotheses of Theorem* 4.1 *be satisfied. Let an interval* $I\subset \boldsymbol{R}$ *and open sets* Γ *and* Γ_t *have the same property as in Theorem* 3.2. *Then*

$$\lim_{t\to\pm\infty} \|\chi_{\Gamma_t} e^{-itH}\phi\|_{\boldsymbol{H}} = \|\phi\|_{\boldsymbol{H}}, \quad \phi\in E(I)R(W_\pm) .$$

COROLLARY 4.5. *Let the hypotheses of Theorem* 4.4 *be satisfied, and assume that the open set* Γ *includes also the origin. Assume further that* $R(W_\pm)=\mathscr{M}_\pm$. *Let* $\phi\in \boldsymbol{H}$. *If*

$$\lim_{T\to\pm\infty} \frac{1}{T}\int_0^T \|\chi_{\Gamma_t} e^{-itH}\phi\|_{\boldsymbol{H}}^2 dt = 0 ,$$

then $E(I)\phi=0$.

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