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The Microlocal Structure of Weighted Homogeneous Polynomials Associated with Coxeter Systems, II

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Introduction

This paper is a continuation of the previous paper [9]. We retain the notation used there.

Let (W, S) be a finite Coxeter system. If the rank of W is l, there exist l-number of algebraically independent W-invariant polynomials $x_1, \cdots, x_l$ which freely generate the W-invariant ring. Let $f_W(x_1, \cdots, x_l)$ be the square of an anti-invariant of W. Then there exist vector fields $X_1, \cdots, X_l$ such that they form a free basis of the Lie algebra of vector fields thereby the set $\{x \in C^l; f_W(x)=0\}$ being left invariant. In particular, we have $X_i f_W = c_i(x) f_W$ with a certain polynomial $c_i(x)$ $(i=1, \cdots, l)$. We studied in [9] the microlocal structure of the $\mathscr{D}_{C^l}$-Module

$$\mathscr{N}_\alpha' = \mathscr{D}_{C^l} \Big/ \sum_{i=1}^{l} \mathscr{D}_{C^l}(X_i - \alpha c_i(x)) \quad (\alpha \in C) .$$

The main purpose of this paper is to determine the holonomy diagram of the system $\mathscr{N}_\alpha'$ for an irreducible finite Coxeter system (except of types E_7 and E_8), which gives enough information for the system $\mathscr{N}_\alpha'$.

In connection with the study, we shall also obtain rather computational results concerning the basic invariants $x_1, \cdots, x_l$, which are complementary to our study but may be useful to the theory of logarithmic poles developed by Professor K. Saito (cf. [4]). Since we have only intermediate results when W is of type E_7 or E_8, we treat in this paper the Coxeter systems of types A_l, B_l, D_l, E_6, F_4, G_2, H_3, H_4, $I_2(p)$.

§ 1. The holonomy diagram.

(1.1) We begin with explaining the holonomy diagram of a holonomic system. The holonomy diagram gives enough information for the holonomic system. A typical application will be seen in [3].

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For the notation, see [9]. Let X be an l-dimensional complex manifold and let T^*X be the cotangent bundle over X. Let, further, π denote the projection of T^*X to X. A holonomic system $\mathscr{L}$ is then a coherent $\mathscr{D}_X$-Module with $\operatorname{codim}_{T^*X} \check{SS}(\mathscr{L})=l$. An irreducible component of $\check{SS}(\mathscr{L})$ is called a holonomic variety.

The holonomy diagram of $\mathscr{L}$ is determined as follows. To each holonomic variety Λ corresponds a circle inscribing Λ. The diagram $\textcircled{\Lambda}-\textcircled{\Lambda'}$ means that two holonomic varieties Λ and Λ' intersect in a one codimensional analytic subset. If $\Lambda_1, \cdots, \Lambda_k$ are holonomic varieties such that $\operatorname{codim}_\Lambda (\Lambda \cap \Lambda_1 \cap \cdots \cap \Lambda_k)=1$, then Λ and Λ_j $(j=1, \cdots, k)$ are joined by a branching segment starting from Λ and branching off at a point in the k directions of Λ_j $(j=1, \cdots, k)$. If $\Lambda'_1, \cdots, \Lambda'_m$ are another holonomic varieties such that $\operatorname{codim}_\Lambda (\Lambda \cap \Lambda'_1 \cap \cdots \cap \Lambda'_m)=1$ and $\Lambda \cap \Lambda_1 \cap \cdots \cap \Lambda_k \neq \Lambda \cap \Lambda'_1 \cap \cdots \cap \Lambda'_m$, then the corresponding branching segments should have different starting points. In this way, we can depict a diagram consisting of circles and segments. This is the holonomy diagram of $\mathscr{L}$. If $\mathscr{L}$ is generated by an unknown function u and Λ is a simple holonomic variety of $\check{SS}(\mathscr{L})$, the order of u on Λ is sometimes denoted beside Λ.

(1.2) We now restrict our attention to the study of the system $\mathscr{N}'_\alpha$ corresponding to a Coxeter system given in [9]. We always assume that a Coxeter system is irreducible. The general case of a reducible Coxeter system is reduced to that case. (cf. the first step of [9, Th. 4.1]).

(1.3) We refer the reader to [1], [9] for notation and results on Coxeter systems. Let E be an l-dimensional Euclidean space with an orthonormal basis $\{e_i\}$ and E^* its dual with the dual basis $\{\xi_i\}$. Let W be a finite irreducible Coxeter group acting on E as a standard representation. Let R be the W-invariant subalgebra of the symmetric algebra $S(E^*)$ of E^*. There exist algebraically independent homogeneous elements $x_1, \cdots, x_l$ of $S(E^*)$ such that $R=\boldsymbol{R}[x_1, \cdots, x_l]$. Let $D(\xi)$ be the product of linear functions defining the hyperplanes of reflections in W. Then, since $D(\xi)^2$ is contained in R, we denote it by $f_W(x_1, \cdots, x_l)$.

Let X be the quotient space of the complexification of E by the action of W whose coordinate ring is $\boldsymbol{C}\otimes R$. We use the notation T^*X and π as in (1.1).

(1.4) From the definition, the polynomials

$$m_{ij}(x)=\frac{1}{2}\sum_{k=1}^{l}\frac{\partial x_i}{\partial \xi_k}\frac{\partial x_j}{\partial \xi_k} \quad (i, j=1, \cdots, l)$$

belong to R and the vector fields

$$X_i = \sum_{j=1}^{l} m_{ij}(x) \frac{\partial}{\partial x_j}$$

leave the set $f_W^{-1}(0)$ invariant. Hence we have

$$X_i f_W = c_i(x) f_W(x)$$

with certain polynomials $c_i(x) \in R$. It should be here noted that $X_1, \cdots, X_l$ form a free basis of the Lie algebra $\mathscr{G}_W$ of vector fields thereby $f_W^{-1}(0)$ being left invariant. It follows from this fact that

$$[X_i, X_j] = \sum_{k=1}^{l} c_{ij}^k(x) X_k$$

with $c_{ij}^k(x) \in R$. Putting a matrix

$$M(W; x_1, \cdots, x_l) = (m_{ij}(x))_{i,j=1,\cdots,l}\ ,$$

we have

$$\det M(W; x_1, \cdots, x_l) = c f_W(x)$$

with a non-zero constant c.

(1.5) We shall discuss in the following sections the coherent $\mathscr{D}_X$-Module

$$\mathscr{N}_\alpha' = \mathscr{D}_X \Big/ \sum_{i=1}^{l} \mathscr{D}_X (X_i - \alpha c_i(x))$$

corresponding to each Coxeter system of types A_l, B_l, D_l, E_6, F_4, H_3, H_4, $I_2(p)$. For convenience we denote by u the generator of $\mathscr{N}_\alpha'$ which is the class of $1 \in \mathscr{D}_X$. In the sequel we always assume that α is not a sum of a root of the b-function of $f_W(x)$ and a positive integer.

(1.6) We now explain a procedure to determine the holonomy diagram of $\mathscr{N}_\alpha'$. There is a one to one correspondence between the totality of holonomic varieties and the set of conjugate classes of S-subgroups (cf. [9, Th. 4.1(1)]). Hence we first determine the latter. For a holonomic variety Λ, the order of u on Λ and the codimension of $\pi(\Lambda)$ in X are determined by [9, Th. 4.1(2)]. We next examine the intersections of holonomic varieties. For this purpose, it is sufficient to investigate whether two holonomic varieties intersect in an analytic subset of codimension one or not. We have already obtained a partial result in [9, Th. 4.1(3)]. But it may occur that there are holonomic varities Λ and Λ' with $\operatorname{codim}_\Lambda (\Lambda \cap \Lambda') = 1$ not satisfying that condition. One will find this phenomenon in cases D_l $(l \geqq 4)$, F_4 and E_6. In this way, we obtain the holonomy diagram of $\mathscr{N}_\alpha'$.

(1.7) In the following sections, we treat each type of Coxeter systems separately, and describe (1)–(10).

(1) Definition of the Coxeter group W.
(2) The W-invariant ring.
(3) The square of a fundamental anti-invariant of W.
(4) The matrix $M(W; x_1, \cdots, x_l)$.
(5) Vector fields $X_1, \cdots, X_l$.
(6) Commutation relations of $X_1, \cdots, X_l$ and the equation $X_i f_W = c_i(x) f_W$.
(7) Conjugate classes of S-subgroups.
(8) Intersections of holonomic varieties.
(9) Properties of holonomic varieties.
(10) The holonomy diagram.

§ 2. The holonomy diagram of $\mathcal{N}_\alpha'$ for a Coxeter system of the classical type.

(2.1) Type A_l.

(2.1.1) The Coxeter group $W(A_l)$ of type A_l is isomorphic to the $(l+1)$-st symmetric group $\mathfrak{S}_{l+1}$. Let E' be an $(l+1)$-dimensional Euclidean space and let E'^* be the dual space of it with an orthonormal basis $\{\zeta_1, \cdots, \zeta_{l+1}\}$. Let E be the subspace of E' defined by $\zeta_1 + \cdots + \zeta_{l+1} = 0$. Then $W(A_l)$ acts on E by permutation of indices of $(\zeta_1, \cdots, \zeta_{l+1})$. The dual space E^* of E is isomorphic to $E'^*/\boldsymbol{R}(\zeta_1 + \cdots + \zeta_{l+1})$.

(2.1.2) Let $S(E^*)$ and $S(E'^*)$ denote the symmetric algebras of E^* and E'^*, respectively. Let $p_i(\zeta_1, \cdots, \zeta_{l+1})$ $(1 \leqq i \leqq l+1)$ be the i-th elementary symmetric polynomial of $\zeta_l, \cdots, \zeta_{l+1}$. Define l elements $x_2, \cdots, x_{l+1}$ of $S(E^*) = S(E'^*)/p_1 S(E'^*)$ by

$$x_i = p_i \mod p_1 S(E'^*) \quad (2 \leqq i \leqq l+1) .$$

Then we have $R = \boldsymbol{R}[x_2, \cdots, x_{l+1}]$.

(2.1.3) A fundamental anti-invariant of $W(A_l)$ is

$$D = \prod_{i<j} (\zeta_i - \zeta_j) \mod p_1 S(E'^*)$$

and the square

$$f_{A_l}(x_2, \cdots, x_{l+1}) = D^2$$

is the discriminant $\Delta(x_2, \cdots, x_{l+1})$ of the polynomial of the $(l+1)$-th degree

$$P(u) = u^{l+1} + x_2 u^{l-1} + x_3 u^{l-2} + \cdots + x_{l+1}$$
$$= \prod_{i=1}^{l+1} (u + \zeta_i) \mod p_1 S(E'^*) .$$

(2.1.4) We define polynomials $m_{ij}(x)$ $(2\leqq i,\ j\leqq l+1)$ by the formula

$$\sum_{i,j=2}^{l+1} m_{ij}(x)u^{l+1-i}v^{l+1-j}=\frac{1}{u-v}(P'(u)P(v)-P(u)P'(v))+\frac{1}{l+1}P'(u)P'(v)$$

(cf. [5, Prop. (2.4.2)]) and a matrix $M(A_l;\, x_2, \cdots, x_{l+1})=(m_{ij}(x))_{i,j=2,\cdots,l+1}$. Then we have

$$\det(M(A_l;\, x_2, \cdots, x_{l+1}))=(-1)^{l(l-1)/2}f_{A_l}(x_2, \cdots, x_{l+1})$$

(cf. [5]).

(2.1.5) We define differential operators

$$X_i=\sum_{j=2}^{l+1} m_{i+2,j}(x)\frac{\partial}{\partial x_j} \quad (i=0, 1, \cdots, l-1)\ .$$

Then the following hold.

$$\begin{aligned}(2.1.5.1)\quad [X_i, X_j]=(j-i)\Big\{X_{i+j}-x_{i+j}X_0+\sum_{k=2}^{\min(i,j)}(x_kX_{i+j-k}-x_{i+j+1-k}X_{k-1})\Big\}\\ -\frac{(i+1)(l-j)}{l+1}x_{i+1}X_{j-1}+\frac{(j+1)(l-i)}{l+1}x_{j+1}X_{i-1}\ .\end{aligned}$$

$$(2.1.5.2)\qquad X_if_{A_l}=(l+1-i)(l-i)x_if_{A_l}\ .$$

We shall prove (2.1.5.1) and (2.1.5.2) in (2.4).

(2.1.6) Any S-subgroup of $W(A_l)$ is isomorphic to $W(A_{n_1-1})\times\cdots\times W(A_{n_k-1})$ with a partition $(n_1, \cdots, n_k)$ of $l+1$ and any two S-subgroups corresponding to the same partition are conjugate in $W(A_l)$. Therefore there is a one to one correspondence between $\mathscr{C}$ (cf. [9]) and the set of partitions $(\{n_1, \cdots, n_k\};\ \sum_{i=1}^k n_i=l+1,\ n_i\geqq 1\}$. We denote by $\Lambda_{(n_1,\cdots,n_k)}$ the holonomic variety corresponding to $W(A_{n_1-1})\times\cdots\times W(A_{n_k-1})$.

(2.1.7) **Proposition.** *Given an irreducible component $\Lambda=\Lambda_{(n_1,\cdots,n_k)}$ of $\check{SS}(\mathscr{N}_\alpha')$. Then the following hold.*

(1) $\operatorname{codim}_X \pi(\Lambda)=\sum_{i=1}^k(n_i-1)$.

(2) $\operatorname{ord}_\Lambda u=-\sum_{i=1}^k(n_i(n_i-1)/2)(\alpha+1/2)$.

(3) *We use the notation $(n_1, \cdots, n_k)_{ij}$ instead of $(n_1, \cdots, n_{i-1}, j, n_i-j, n_{i+1}, \cdots, n_k)$. Then the holonomic varieties $\Lambda_{(n_1,\cdots,n_k)_{ij}}$ $(1\leqq i\leqq k,\ 1\leqq j\leqq[n_i/2])$ have one codimensional intersection with Λ. Furthermore*

$$\Lambda\cap\Lambda_{(n_1,\cdots,n_k)_{ij}}=\Lambda\cap\Lambda_{(n_1,\cdots,n_k)_{i'j'}}$$

if and only if $i=i'$.

The proposition follows from Theorem 4.1 in [9].

(2.1.8) In this case, we amplify (2.1.7)(3) as follows.

PROPOSITION. *Fix an irreducible component* $\Lambda=\Lambda_{(n_1,\cdots,n_k)}$ *of* $\check{SS}(\mathcal{N}_\alpha')$. *Let* $(n_{i1}, \cdots, n_{ip_i})$ *be a partition of* n_i *for* $1\leqq i\leqq k$. *Then we have*

$$\operatorname{codim}_\Lambda (\Lambda\cap\Lambda_{(n_{11},\cdots,n_{1p_1},n_{21},\cdots,n_{kp_k})})=\sum_{i=1}^{k}(p_i-1)\ .$$

We omit the proof since it is elementary but long.

(2.2) Type B_l.

(2.2.1) The Coxeter group $W(B_l)$ of type B_l is a semidirect product of $\mathfrak{S}_l$ by $(Z/2Z)^l$. It acts on $(\xi_1, \cdots, \xi_l)$ by permutation of indices and arbitrary sign changes.

(2.2.2) Let $p_i(t_1, \cdots, t_l)$ be the i-th elementary symmetric polynomial of $t_1, \cdots, t_l$. Put

$$x_{2i}=p_i(\xi_1^2, \cdots, \xi_l^2)\quad (1\leqq i\leqq l)\ .$$

Then we have

$$R=\boldsymbol{R}[x_2, \cdots, x_{2l}]\ .$$

(2.2.3) A fundamental anti-invariant of $W(B_l)$ is

$$D=\xi_1\cdots\xi_l\prod_{i<j}(\xi_i^2-\xi_j^2)$$

and therefore $f_{B_l}(x_2, \cdots, x_{2l})=D^2$ is the product of x_{2l} and the discriminant $\Delta(x_2, \cdots, x_{2l})$ of the polynomial of the l-th degree

$$\begin{aligned}P(u)&=u^l+x_2u^{l-1}+\cdots+x_{2l}\\&=\prod_{i=1}^{l}(u+\xi_i^2)\ .\end{aligned}$$

(2.2.4) We define polynomials $m_{2i,2j}(x)$ $(1\leqq i, j\leqq l)$ by the formula

$$\sum_{i,j=1}^{l} m_{2i,2j}(x)u^{l-i}v^{l-j}=\frac{2}{u-v}(uP'(u)P(v)-vP(u)P'(v))$$

(cf. [5, Prop. (2.2.2)]) and a matrix $M(B_l; x_2, \cdots, x_{2l})=(m_{2i,2j}(x))_{i,j=1,\cdots,l}$. Then we have

$$\det(M(B_l; x_2, \cdots, x_{2l}))=(-1)^{(l+1)(l-2)/2}2^l f_{B_l}(x)\ .$$

(2.2.5) We define differential operators

$$X_{2i}=\sum_{j=1}^{l} m_{i+1,j}(x)\frac{\partial}{\partial x_{2j}}\quad (i=0, 1, \cdots, l-1)\ .$$

Then the following hold.

$$[X_{2i}, X_{2j}]=(2j-2i)\left\{X_{2(i+j)}+\sum_{k=1}^{\min(i,j)}(x_{2k}X_{2(i+j-k)}-x_{2(i+j+1-k)}X_{2(k-1)})\right\}. \tag{2.2.5.1}$$

Here we understand $x_0=1$, $x_{2k}=0$ for $k>l$ and $X_{2k}=0$ for $k\geqq l$.

$$X_{2i}f_{B_l}=2(l-i)^2x_{2i}f_{B_l} \quad (0\leqq i\leqq l-1). \tag{2.2.5.2}$$

We shall prove (2.2.5.1) and (2.2.5.2) in (2.4).

(2.2.6) Any S-subgroup of $W(B_l)$ is isomorphic to $W(B_{n_0})\times W(A_{n_1-1})\times\cdots\times W(A_{n_k-1})$ for integers $n_0, n_1, \cdots, n_k$ with the condition

$$n_0+\sum_{i=1}^{k}n_i=l\,,\quad n_0\geqq 0\,,\quad n_i\geqq 1\quad (i=1,\cdots,k)\,. \tag{2.2.6.1}$$

Here B_{n_0} denotes the type corresponding to the subdiagram (2.2.6.2) of the Coxeter diagram of type B_l.

(2.2.6.2)  (n_0 vertices) .

The groups $W(A_1)$ and $W(B_1)$ are isomorphic but not conjugate in $W(B_l)$, because the lengths of roots corresponding to them are different. Therefore there is a one to one correspondence between $\mathscr{C}$ and the set $\{(n_0; n_1, \cdots, n_k); k\geqq 0, n_i$ subject to (2.2.6.1)$\}$. We denote by $\Lambda_{(n_0;n_1,\cdots,n_k)}$ the holonomic variety corresponding to the class of $W(B_{n_0})\times W(A_{n_1-1})\times\cdots\times W(A_{n_k-1})$.

(2.2.7) PROPOSITION. *Fix an irreducible component* $\Lambda=\Lambda_{(n_0;n_1,\cdots,n_k)}$ *of* $\check{SS}(\mathscr{N}_\alpha')$. *Then the following hold.*

(1) $\operatorname{codim}_X \pi(\Lambda)=n_0+\sum_{i=1}^k(n_i-1)$.

(2) $\operatorname{ord}_\Lambda u=-(n_0^2+\sum_{i=1}^k(n_i(n_i-1)/2))(\alpha+1/2)$.

(3) *We use the notation* $(n_0; n_1, \cdots, n_k)_{ij}$ *instead of* $(n_0; n_1, \cdots, n_{i-1}, j, n_i-j, n_{i+1}, \cdots, n_k)$ *for* $1\leqq i\leqq k$, $1\leqq j\leqq[n_i/2]$ *and* $(j; n_0-j, n_1, \cdots, n_k)$ *for* $i=0$, $0\leqq j\leqq n_0-1$. *Then the holonomic varieties* $\Lambda_{(n_0;n_1,\cdots,n_k)_{ij}}$ *have one codimensional intersection with* Λ. *Furthermore*

$$\Lambda\cap\Lambda_{(n_0;n_1,\cdots,n_k)_{ij}}=\Lambda\cap\Lambda_{(n_0;n_1,\cdots,n_k)_{i'j'}}$$

if and only if $i=i'$ (cf. [9, Th. 4.1]).

(2.2.8) As in the case of type A_l, the following statement holds.

PROPOSITION. *Fix an irreducible component* $\Lambda=\Lambda_{(n_0;n_1,\cdots,n_k)}$ *of* $\check{SS}(\mathscr{N}_\alpha')$. *Let* $(n_{i1}, \cdots, n_{ip_i})$ *be a partition of* n_i *for* $1\leqq i\leqq k$ *and let* $(n_{01}, \cdots, n_{0p_0})$ *be a partition of* n_0-n_{00} *with a non-negative integer* n_{00}. *Then we have*

(2.2.7.1) $$\operatorname{codim}_A (A \cap A_{(n_{00}; n_{01}, \cdots, n_{0p_0}, n_{11}, \cdots, n_{kp_k})}) = p_0 + \sum_{i=1}^{k} (p_i - 1) .$$

(2.3) Type D_l.

(2.3.1) The Coxeter group $W(D_l)$ of type D_l is isomorphic to a semi-direct product of $\mathfrak{S}_l$ by $(\boldsymbol{Z}/2\boldsymbol{Z})^{l-1}$. It acts on $(\xi_1, \cdots, \xi_l)$ by permutation of indices and such sign changes $(\xi_1, \cdots, \xi_l) \mapsto (\varepsilon_1\xi_1, \cdots, \varepsilon_l\xi_l)$ that $\prod_{i=1}^{l} \varepsilon_i = 1$.

(2.3.2) Put $x_{2i} = p_i(\xi_1^2, \cdots, \xi_l^2)$ $(1 \leqq i \leqq l-1)$ as in (2.2.2) and $y = \xi_1 \cdots \xi_l$. Then we have
$$R = \boldsymbol{R}[x_2, \cdots, x_{2l-2}, y] .$$

(2.3.3) A fundamental anti-invariant of $W(D_l)$ is
$$D = \prod_{i<j} (\xi_i^2 - \xi_j^2)$$
and the square of D is
$$f_{D_l}(x_2, \cdots, x_{2l-2}, y) = \Delta(x_2, \cdots, x_{2l-2}, y^2) ,$$
where $\Delta(x_2, \cdots, x_{2l-2}, y^2)$ is the discriminant of the polynomial
$$\begin{aligned} P(u) &= u^l + x_2 u^{l-1} + \cdots + x_{2l-2} u + y^2 \\ &= \prod_{i=1}^{l} (u + \xi_i^2) . \end{aligned}$$

(2.3.4) We define a matrix
$$M(D_l; x_2, \cdots, x_{2l-2}, y) = \begin{bmatrix} I_{l-1} & \\ & \dfrac{1}{2y} \end{bmatrix} M(B_l; x_2, \cdots, x_{2l-2}, y^2) \begin{bmatrix} I_{l-1} & \\ & \dfrac{1}{2y} \end{bmatrix} .$$
Then we have
$$\det (M(D_l; x_2, \cdots, x_{2l-2}, y)) = -(-1)^{l(l-1)/2} 2^{l-2} f_{D_l}(x_2, \cdots, x_{2l-2}, y)$$
(cf. [5]).

(2.3.5) We define differential operators $X_0, X_2, \cdots, X_{2l-2}, Y$ by the formula
$${}^t(X_0, X_2, \cdots, X_{2l-4}, Y) = M(D_l; x_2, \cdots, x_{2l-2}, y)\, {}^t\!\left(\frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_{2l-2}}, \frac{\partial}{\partial y}\right) .$$
Then the following hold. (For the proof, see (2.4).)

(2.3.5.1) $$\begin{cases} [X_{2i}, X_{2j}] = (2j-2i)\left\{X_{2(i+j)} + \displaystyle\sum_{k=1}^{\min(i,j)} (x_{2k} X_{2(i+j-k)} - x_{2(i+j+1-k)} X_{2(k-1)})\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 \leqq i, j \leqq l-2) , \\ [X_{2i}, Y] = (l-2-i) x_{2i} Y - (l-1-i) y X_{2(i-1)} \quad (0 \leqq i \leqq l-2) . \end{cases}$$

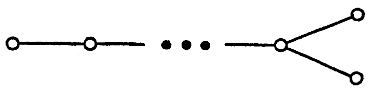
Here we understand $x_{2k}=0$ for $k>l$, $x_{2l}=y^2$, $X_{2k}=0$ for $k\geqq l$, $X_{2l-2}=2yY$.

$$\begin{cases} X_{2i}f_{D_l}=2(l-i)(l-i-1)x_{2i}f_{D_l} & (0\leqq i\leqq l-2)\ , \\ Yf_{D_l}=0\ . \end{cases} \tag{2.3.5.2}$$

(2.3.6) Any S-subgroup of $W(D_l)$ is isomorphic to $W(D_{n_0})\times W(A_{n_1-1})\times\cdots\times W(A_{n_k-1})$ with integers $n_0, n_1, \cdots, n_k$ subject to the condition

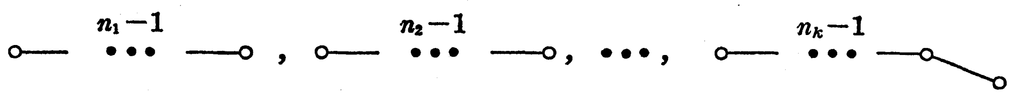
$$\sum_{i=1}^{k} n_i=l\ , \quad n_0\geqq 0,\ \neq 1\ , \quad n_i\geqq 1 \quad (i=1, \cdots, k)\ . \tag{2.3.6.1}$$

Two S-subgroups corresponding to the same symbol $(n_0; n_1, \cdots, n_k)$ with the condition (2.3.6.1) are conjugate in $W(D_l)$ except when $n_0=0$ and all n_i $(1\leqq i\leqq k)$ are even. Here D_{n_0} denotes the type corresponding to the subdiagram (2.3.6.2) of the Coxeter diagram of type D_l.

(2.3.6.2)  (n_0 vertices) .

When $n_0=0$ and n_i $(1\leqq i\leqq k)$ are even, we use the notation $(0; n_1, \cdots, n_k)$ and $(0; n_1, \cdots, n_k)_-$ respectively for the choices of subdiagrams

and



Therefore, if l is odd (resp. even), there is a one to one correspondence between $\mathscr{C}$ and the set $\{(n_0; n_1, \cdots, n_k);\ n_0, \cdots, n_k$ subject to (2.3.6.1)$\}$ (resp. the union of the set $\{(n_0; n_1, \cdots, n_k);\ n_0, \cdots, n_k$ subject to (2.3.6.1)$\}$ and the set $\{(0; n_1, \cdots, n_k)_-;\ n_i,\ i=1, \cdots, k$, are even, $\sum_{i=1}^{k} n_i=l\}$). We denote by $\Lambda_{(n_0;n_1,\cdots,n_k)}$ (or $\Lambda_{(0;n_1,\cdots,n_k)_-}$) the holonomic variety corresponding to $(n_0; n_1, \cdots, n_k)$ (or $(0; n_1, \cdots, n_k)_-$).

(2.3.7) PROPOSITION. *Fix an irreducible component* $\Lambda=\Lambda_{(n_0;n_1,\cdots,n_k)}$ *of* $\check{SS}(\mathcal{N}_\alpha')$. *Then the following hold.*

(1) $\operatorname{condim}_X \pi(\Lambda)=n_0+\sum_{i=1}^{k}(n_k-1)$.

(2) $\operatorname{ord}_\Lambda u=-(n_0(n_0-1)+\sum_{i=1}^{k} n_i(n_i-1)/2)(\alpha+1/2)$.

(3) *We use the notation* $(n_0; n_1, \cdots, n_k)_{ij}$ $(0\leqq i\leqq k)$ *instead of* $(n_0; n_1, \cdots, j, n_i-j, n_{i+1}, \cdots, n_k)$ $(1\leqq j\leqq [n_i/2])$ *for* $1\leqq i\leqq k$ *and* $(j; n_0-j, n_1, \cdots, n_k)$ $(0\leqq j\leqq n_0-1,\ j\neq 1)$ *for* $i=0$. *Then holonomic varieties*

$\Lambda_{(n_0;n_1,\cdots,n_k)ij}$, and $\Lambda_{(0;n_0-1,1,n_1,\cdots,n_k)}$ besides them when $n_0\geqq 4$, have one codimensional intersection with Λ. Furthermore

$$\Lambda\cap\Lambda_{(n_0;n_1,\cdots,n_k)ij}=\Lambda\cap\Lambda_{(n_0;n_1,\cdots,n_k)i'j'}$$

if and only if $i=i'$, and

$$\Lambda\cap\Lambda_{(0;n_0-1,1,n_1,\cdots,n_k)}=\Lambda\cap\Lambda_{(n_0;n_1,\cdots,n_k)0j}\ .$$

The statement holds if we change $\Lambda_{(0;m_1,\cdots,m_r)}$ for $\Lambda_{(0;m_1,\cdots,m_r)-}$.

PROOF. The proposition follows from Theorem 4.1 in [9] except the statement concerning $\Lambda_{(0;n_0-1,1,n_1,\cdots,n_k)}$. But this is derived from the following lemma.

(2.3.7.1) LEMMA. *When $n_0\geqq 4$, the holonomic varieties Λ and $\Lambda_{(0;n_0-1,1,n_1,\cdots,n_k)}$ have a one codimensional intersection, and*

$$\Lambda\cap\Lambda_{(0;n_0-1,1,n_1,\cdots,n_k)}=\Lambda\cap\Lambda_{(0;n_0,n_1,\cdots,n_k)}\ .$$

PROOF. First assume that $k=0$ and prove the lemma. In this case, Λ is the conormal bundle of the origin in X. Put $\Lambda'=\Lambda_{(0;l-1,1)}$, $l\geqq 4$. Then it suffices to show

$$\Lambda\cap\Lambda'=\Lambda\cap\Lambda_{(0;l)}\ .$$

To prove this, we put $q_{2\nu}=\sum_{i=1}^{l}\xi_i^{2\nu}$ $(\nu=1,2,\cdots,l-1)$, $y=\xi_1\cdots\xi_l$ and take $(q_2,\cdots,q_{2l-2},y)$ as a coordinate system of X instead of $(x_2,\cdots,x_{2l-2},y)$. Let us denote by Λ'^{reg} the regular part of Λ'. Then we have

$$\begin{aligned}\Lambda'^{\mathrm{reg}}=\Big\{&(q_2,\cdots,q_{2l-2},y;\zeta_2,\cdots,\zeta_{2l-2},\eta)\in T^*X\,;\\ &q_{2\nu}=(l-1)\xi_1^{2\nu}+\xi_2^{2\nu}(\nu=1,\cdots,l-1),\ y=\xi_1^{l-1}\xi_2\ (\xi_1\neq\xi_2)\,,\\ &\sum_{\nu=1}^{l-1}2\nu\zeta_{2\nu}\xi_i^{2\nu-1}+\frac{\eta y}{\xi_i}=0\ (i=1,2)\Big\}\ .\end{aligned}$$

Each element $(q_2,\cdots,q_{2l-2},y;\zeta_2,\cdots,\zeta_{2l-2},\eta)$ of Λ'^{reg} satisfies the condition

$$\begin{aligned}&\zeta_2+\sum_{\nu=2}^{l-1}\nu\zeta_{2\nu}\frac{\xi_1^{2\nu}-\xi_2^{2\nu}}{\xi_1^2-\xi_2^2}=0\,,\\ &4\zeta_4+\sum_{\nu=3}^{l-1}2\nu\zeta_{2\nu}\frac{\xi_1^{2\nu-2}-\xi_2^{2\nu-2}}{\xi_1^2-\xi_2^2}-\frac{\xi_1^{l-3}}{\xi_2}\eta=0\ .\end{aligned}$$

Taking an analytic path $\xi_1=t$, $\xi_2=at^{l-3}$ $(a\neq 0)$ and letting t tend to zero, we have that $\zeta_2=0$ and $4a\zeta_4-\eta=0$. Since we can take $a\neq 0$ arbitrary, we finally obtain

$$\Lambda \cap \Lambda' = \{q_2, \cdots, q_{2l-2}, y; \zeta_2, \cdots, \zeta_{2l-2}, \eta) \in T^*X;$$
$$q_2 = \cdots = q_{2l-2} = y = 0, \zeta_2 = 0\} .$$

Then the right-hand side of this coincides with $\Lambda \cap \Lambda_{(0;l)}$ (cf. [9, Th. 4.1]). Hence we have proved the lemma for the case $k=1$.

The general case is proved by an argument similar to the first step of the proof of Theorem 4.1 in [9]. Q.E.D.

(2.3.8) The case of type D_l is more complicated than the cases of types A_l, B_l to describe the codimension of the intersection of two irreducible components. Let $(n_0; n_1, \cdots, n_k)$ be a symbol consisting of integers with the condition (2.3.6.1).

PROPOSITION.

Case I. $n_0 \geqq 2$. *Let* $(n_{i1}, \cdots, n_{ip_i})$ *be a partition of* n_i *for* $i=1, \cdots, k$ *and let* $(n_{01}, \cdots, n_{0p_0})$ *be a partition of* $n_0 - n_{00}$ *with a non-negative integer* n_{00} *not equal to* 1. *We put* $\Lambda = \Lambda_{(n_0;n_1,\cdots,n_k)}$ *and* $\Lambda' = \Lambda_{(n_{00};n_{01},\cdots,n_{0p_0},n_{11},\cdots,n_{kp_k})}$. *Then*

$$\text{(2.3.8.1)} \qquad \operatorname{codim}_\Lambda (\Lambda \cap \Lambda') = p_0 - \varepsilon + \sum_{i=1}^{k} (p_i - 1) .$$

Here $\varepsilon = 1$ *if* $n_0, n_{00}, n_{01}, \cdots, n_{0p_0}$ *satisfy one of the conditions in* (2.3.8.2) *and* $\varepsilon = 0$ *if otherwise.*

(2.3.8.2) (i) $n_0 \geqq 4$ *and* $n_{00} = 0$.

(ii) *When* $2 \leqq p_0 \leqq [n_0/2]$: $n_{0i} = 1$ *for some* i.

When $[n_0/2] + 1 \leqq p_0$: $n_{0i} \geqq 3$ *for some* i.

In case $n_{00} = 0$, (2.3.8.1) *holds if we exchange* Λ' *for* $\Lambda_{(0;n_{01},\cdots,n_{0p_0},n_{11},\cdots,n_{kp_k})-}$.

Case II. $n_0 = 0$. *Let* $(n_{i1}, \cdots, n_{ip_i})$ $(i=1, \cdots, k)$ *be partitions as in Case* I. *We put* $\Lambda = \Lambda_{(0;n_1,\cdots,n_k)}$ *and* $\Lambda' = \Lambda_{(0;n_{11},\cdots,n_{kp_k})}$ (resp., $\Lambda = \Lambda_{(0;n_1,\cdots,n_k)-}$; *and* $\Lambda' = \Lambda_{(0;n_{11},\cdots,n_{kp_k})-}$ *when all* n_{ij} *are even, or* $\Lambda' = \Lambda_{(0;n_{11},\cdots,n_{kp_k})}$ *when one of* n_{ij} *is odd*). *Then*

$$\text{(2.3.8.3)} \qquad \operatorname{codim}_\Lambda (\Lambda \cap \Lambda') = \sum_{i=1}^{k} (p_i - 1) .$$

Since the proof is elementary but tedious, we omit it.

(2.4) Proof of (2.1.5), (2.2.5), (2.3.5).

In this paragraph, we prove (2.1.5), (2.2.5), (2.3.5). For convenience, we first prove (2.2.5), next (2.3.5) and finally (2.1.5).

(2.4.1) PROOF OF (2.2.5). We first prove (2.2.5.1) by induction on l. It is trivial for the case $l=1$. Assume (2.2.5.1) holds for B_{l-1} and prove it for B_l. The vector field X_{2i} is rewritten in the form

$$X_{2i}=\frac{1}{2}\sum_{k=1}^{l}\frac{\partial x_{2(i+1)}}{\partial\xi_k}\frac{\partial}{\partial\xi_k}\,.$$

Comparing the coefficients of $\partial/\partial\xi_m$ of both sides of (2.2.5.1), we have only to prove the following equation for each m $(1\leqq m\leqq l)$:

$$\begin{aligned}(2.4.1.1)\quad &\sum_{n=1}^{l}\left(\frac{\partial x_{2(i+1)}}{\partial\xi_n}\frac{\partial^2 x_{2(j+1)}}{\partial\xi_n\partial\xi_m}-\frac{\partial x_{2(j+1)}}{\partial\xi_n}\frac{\partial^2 x_{2(i+1)}}{\partial\xi_n\partial\xi_m}\right)\\ &=4(j-i)\left\{\sum_{k=0}^{i}\left(x_{2k}\frac{\partial x_{2(i+j+1-k)}}{\partial\xi_m}-x_{2(i+j+1-k)}\frac{\partial x_{2k}}{\partial\xi_m}\right)\right\}\quad(i<j)\,.\end{aligned}$$

We may assume $m<l$ without loss of generality. Let us denote by y_{2i} $(1\leqq i\leqq l-1)$ the i-th elementary symmetric polynomial of $\xi_1^2,\cdots,\xi_{l-1}^2$. It should be noted that $x_{2i}=\xi_l^2 y_{2(i-1)}+y_{2i}$. Here we understand $y_0=1$ and $y_{2l}=0$. For simplicity, we define

$$\begin{aligned}A(i,j)&=\sum_{n=1}^{l-1}\left(\frac{\partial y_{2(i+1)}}{\partial\xi_n}\frac{\partial^2 y_{2(j+1)}}{\partial\xi_n\partial\xi_m}-\frac{\partial y_{2(j+1)}}{\partial\xi_n}\frac{\partial^2 y_{2(i+1)}}{\partial\xi_n\partial\xi_m}\right)\\ B(i,j)&=\sum_{k=0}^{i}\left(y_{2k}\frac{\partial y_{2(i+j+1-k)}}{\partial\xi_m}-y_{2(i+j+1-k)}\frac{\partial y_{2k}}{\partial\xi_m}\right)\,.\end{aligned}$$

Then direct calculation shows that

$$\begin{aligned}(2.4.1.2)\quad &\sum_{n=1}^{l}\left(\frac{\partial x_{2(i+1)}}{\partial\xi_n}\frac{\partial^2 x_{2(j+1)}}{\partial\xi_n\partial\xi_m}-\frac{\partial x_{2(j+1)}}{\partial\xi_n}\frac{\partial^2 x_{2(i+1)}}{\partial\xi_n\partial\xi_m}\right)\\ &=A(i-1,j-1)\xi_l^4\\ &\quad+\left\{A(i-1,j)+A(i,j-1)+4\left(y_{2i}\frac{\partial y_{2j}}{\partial\xi_m}-y_{2j}\frac{\partial y_{2i}}{\partial\xi_m}\right)\right\}\xi_l^2+A(i,j)\,,\end{aligned}$$

$$\begin{aligned}(2.4.1.3)\quad &\sum_{k=0}^{i}\left(x_{2k}\frac{\partial x_{2(i+j+1-k)}}{\partial\xi_m}-x_{2(i+j+1-k)}\frac{\partial x_{2k}}{\partial\xi_m}\right)\\ &=B(i-1,j-1)\xi_l^4+(B(i-1,j)+B(i,j-1))\xi_l^2+B(i,j)\,.\end{aligned}$$

On the other hand, it follows from the induction hypothesis

$$(2.4.1.4)\quad\begin{cases}A(i-1,j-1)=4(j-i)B(i-1,j-1)\,,\\ A(i-1,j)=4(j-i+1)B(i-1,j)\,,\\ A(i,j-1)=4(j-i-1)B(i,j-1)\,,\\ A(i,j)=4(j-i)B(i,j)\,,\end{cases}$$

and the definition of $B(i,j)$ shows

$$(2.4.1.5)\qquad B(i,j-1)-B(i-1,j)=y_{2i}\frac{\partial y_{2j}}{\partial\xi_m}-y_{2j}\frac{\partial y_{2i}}{\partial\xi_m}\,.$$

In view of (2.4.1.4) and (2.4.1.5), we conclude that the right-hand side of (2.4.1.2) equals to that of (2.4.1.3). Hence (2.4.1.1) is proved.

Next we prove (2.2.5.2). For this purpose, we define $c_{2i}(x)$ by

$$X_{2i}f_{B_l}(x)=c_{2i}(x)f_{B_l}(x) .$$

and show

$$c_{2i}(x)=2(l-i)^2x_{2i}$$

by induction on l. It is trivial for the case $l=1$. Assume that (2.2.5.2) holds for B_{l-1}. We employ the notation in the preceeding argument. Let us define

$$D=\xi_1\cdots\xi_l\prod_{1\leq i<j\leq l}(\xi_i^2-\xi_j^2) ,$$

$$D'=\xi_1\cdots\xi_{l-1}\prod_{1\leq i<j\leq l-1}(\xi_i^2-\xi_j^2) .$$

Then D and D' are the fundamental anti-invariants of $W(B_l)$ and $W(B_{l-1})$ givn in (2.2.3), respectively. In particular, it follows that

$$D=\xi_l\prod_{i=1}^{l-1}(\xi_i^2-\xi_l^2)D' .$$

We define $c'_{2i}(y)=2(l-1-i)^2y_{2i}$ $(0\leq i\leq l-1)$. Then the induction hypothesis implies

$$c'_{2i}(y)=\sum_{k=1}^{l-1}\frac{\partial y_{2(i+1)}}{\partial\xi_k}\frac{\partial}{\partial\xi_k}(\log D') .$$

In view of this, we have

$$\begin{aligned}(2.4.1.6)\qquad c_{2i}(x)&=\sum_{k=1}^{l}\frac{\partial x_{2(i+1)}}{\partial\xi_k}\frac{\partial}{\partial\xi_k}(\log D)\\&=\xi_l^2c'_{2(i-1)}(y)+c'_{2i}(y)+2y_{2i}\\&\quad+4\sum_{k=1}^{l-1}(\xi_l^2y^{\hat{k}}_{2(i-1)}+y^{\hat{k}}_{2i})\frac{\xi_k^2}{\xi_k^2-\xi_l^2}-4\xi_l^2\sum_{k=1}^{l-1}\frac{y_{2i}}{\xi_k^2-\xi_l^2} .\end{aligned}$$

Here $y^{\hat{k}}_{2i}$ denotes the i-th elementary symmetric polynomial of $\xi_1^2, \cdots, \xi_{k-1}^2, \xi_{k+1}^2, \cdots, \xi_{l-1}^2$. Owing to the identities

$$y_{2i}=\xi_k^2y^{\hat{k}}_{2(i-1)}+y^{\hat{k}}_{2i}$$

$$\sum_{k=1}^{l-1}y^{\hat{k}}_{2i}=(l-1-i)y_{2i} ,$$

we have

$$\sum_{k=1}^{l-1}(\xi_l^2y^{\hat{k}}_{2(i-1)}+y^{\hat{k}}_{2i})\frac{\xi_k^2}{\xi_k^2-\xi_l^2}=\xi_l^2\left(\sum_{k=1}^{l-1}\frac{y_{2i}}{\xi_k^2-\xi_l^2}\right)+(l-1-i)y_{2i} .$$

Substituting this equation to (2.4.1.6), we have

$$\begin{aligned}c_{2i}(x)&=\xi_i^2\cdot 2(l-1)^2y_{2(i-1)}+2(l-1-i)^2y_{2i}+(4(l-1-i)+2)y_{2i}\\&=2(l-i)^2x_{2i}\,.\end{aligned}$$

(2.4.2) PROOF OF (2.3.5). For convenience, let us denote by X'_{2i} ($i=0, 1, \cdots, l-1$) the vector fields specified in (2.2.5). Then $X_0, X_2, \cdots, X_{2(l-2)}$, Y are obtained from $X'_0, X'_2, \cdots, X'_{2(l-1)}$ in the following manner. Let

$$Z=\sum_{j=1}^{l} a_{ij}(x_2, \cdots, x_{2l})\frac{\partial}{\partial x_{2j}}$$

be a vector field on the $(x_2, \cdots, x_{2l})$-space. Then we denote by $Z|_{x_{2l}=y^2}$ the pull back of Z along the map $(x_2, \cdots, x_{2(l-1)}, y)\mapsto(x_2, \cdots, x_{2(l-1)}, y^2)$. Namely

$$Z|_{x_{2l}=y^2}=\sum_{j=1}^{l-1} a_{ij}(x_2, \cdots, x_{2(l-1)}, y^2)\frac{\partial}{\partial x_{2j}}+\frac{1}{2y}a_{il}(x_2, \cdots, x_{2(l-1)}, y^2)\frac{\partial}{\partial y}\,.$$

Under this notation, we have

$$\begin{aligned}X_{2i}&=X'_{2i}|_{x_{2l}=y^2}\quad(0\leqq i\leqq l-2)\,,\\Y&=\frac{1}{2y}(X'_{2(l-1)}|_{x_{2l}=y^2})\,.\end{aligned}$$

Due to this relation and (2.2.5), it is easy to prove that (2.3.5.1) holds at least for $i, j<l-1$. We must examine the remaining case $j=l-1$. In this case, (2.2.5.1) implies

$$[X'_{2i}, X'_{2(l-1)}]=2(l-1-i)(x_{2i}X'_{2(l-1)}+x_{2l}X'_{2(i-1)})\,.$$

It follows from this commutation relation that

$$\begin{aligned}[X_{2i}, Y]&=\left[X_{2i}, \frac{1}{2y}(X'_{2(l-1)}|_{x_{2l}=y^2})\right]\\&=X_{2i}\left(\frac{1}{2y}\right)(2yY)+\frac{1}{2y}[(X'_{2i}|_{x_{2l}=y^2}), (X'_{2(l-1)}|_{x_{2l}=y^2})]\\&=-(l-i)x_{2i}Y+(l-1-i)(2x_{2i}Y-yX_{2(i-1)})\\&=(l-2-i)x_{2i}Y-(l-1-i)yX_{2(i-1)}\,.\end{aligned}$$

Here we used the equations

$$\begin{aligned}&X_{2i}\left(\frac{1}{y}\right)=-\frac{(l-i)x_{2i}}{y}\quad(\text{cf. } (2.3.4),\ (2.3.5))\,,\\&[(X'_{2i}|_{x_{2l}=y^2}), (X'_{2(l-1)}|_{x_{2l}=y^2})]=[X'_{2i}, X'_{2(l-1)}]|_{x_{2l}=y^2}\,.\end{aligned}$$

Hence (2.3.5.1) is proved.

The identity (2.3.5.2) is derived from (2.2.5.2). Actually, due to the relation

$$f_{B_l}(x_2, \cdots, x_{2(l-1)}, y^2)=y^2 f_{D_l}(x_2, \cdots, x_{2(l-1)}, y) ,$$

it follows that

$$\begin{aligned} X_{2i} f_{D_l}(x_2, \cdots, x_{2(l-1)}, y) &= \left\{X'_{2i}\left(\frac{1}{x_{2l}} f_{B_l}(x_2, \cdots, x_{2l})\right)\right\}\Big|_{x_{2l}=y^2} \\ &= 2(l-i)(l-1-i)x_{2i} f_{D_l}(x_2, \cdots, x_{2(l-1)}, y) \end{aligned}$$

for $i=0, 1, \cdots, l-2$. A similar argument shows that

$$Y f_{D_l}(x_2, \cdots, x_{2(l-1)}, y)=0 .$$

(2.4.3) PROOF OF (2.1.5). We denote by $\tilde{X}$ the $(p_1, \cdots, p_{l+1})$-space and regard $X=E\otimes C/W(A_l)$ as the subspace of $\tilde{X}$ defined by $\{(p_1, \cdots, p_{l+1}) \in \tilde{X}; p_1=0\}$. In particular, $p_i|_X=x_i$ (cf. (2.1)). We define polynomials $\mu_{ij}(p)$ by

$$\text{(2.4.3.1)} \quad \frac{1}{4}\sum_{i,j=0}^{l+1} \mu_{ij}(p)u^{l+1-i}v^{l+1-j}=\frac{1}{u-v}(uP'(u)P(v)-vP'(v)P(u)) ,$$

where

$$\begin{aligned} \tilde{P}(u) &= u^{l+1}+p_1u^l+ \cdots +p_{l+1} \\ &= \prod_{i=1}^{l+1}(u+\zeta_i) , \end{aligned}$$

and vector fields

$$\text{(2.4.3.2)} \quad \begin{cases} Z_i=\dfrac{1}{2}\displaystyle\sum_{j=1}^{l+1}\mu_{i+1,j}(p)\frac{\partial}{\partial p_j} \quad (i=0, 1, \cdots, l) \\ Z_{-1}=\displaystyle\sum_{j=1}^{l+1}(l+2-j)p_{j-1}\frac{\partial}{\partial p_j} \quad \left(=\dfrac{1}{p_{l+1}}Z_l\right) . \end{cases}$$

Then we have

$$\text{(2.4.3.3)} \quad \begin{cases} [Z_i, Z_j]= 2(j-i)\left\{Z_{i+j}+\displaystyle\sum_{k=1}^{\min(i,j)}(p_k Z_{i+j-k}-p_{i+j+1-k}Z_{k-1})\right\} \quad (i, j\geqq 0) \\ [Z_{-1}, Z_i]=(l-i)Z_{i-1}+p_i Z_{-1} \quad (i\geqq 0) . \end{cases}$$

This follows from (2.4.3.1), (2.4.3.2), (2.2.4) and (2.2.5). We define

$$\begin{aligned} Z'_i &= Z_i-\frac{i+1}{l+1}p_{i+1}Z_{-1} \\ &= \sum_{j=1}^{l+1}\left(\frac{1}{4}\mu_{i+1,j}(p)-\frac{(i+1)(l+2-j)}{l+1}p_{i+1}p_{j-1}\right)\frac{\partial}{\partial p_j} \quad (i=0, 1, \cdots, l-1) . \end{aligned}$$

Since

$$\sum_{i,j=0}^{l+1}\left(\frac{1}{4}\mu_{ij}(p)-\frac{i(l+2-j)}{l+1}p_ip_{j-1}\right)u^{l+1-i}v^{l+1-j}$$
$$=-u\left\{\frac{1}{u-v}(P(u)P'(v)-P'(u)P(v))-\frac{1}{l+1}P'(u)P'(v)\right\}$$

and

$$X_ip_1=0\ ,$$

it follows from (2.1.4) that

$$Z_i'|_X=X_i\ .$$

On the other hand, it follows from (2.4.3.3) that

$$\text{(2.4.3.4)}\qquad [Z_i',\ Z_j']\equiv(j-i)\left\{Z_{i+j}'+\sum_{k=1}^{\min(i,j)}(p_kZ_{i+j-k}'-p_{i+j+1-k}Z_{k-1}')\right\}$$
$$-\frac{(i+1)(l-j)}{l+1}p_{i+1}Z_{j-1}'+\frac{(j+1)(l-i)}{l+1}p_{j+1}Z_{i-1}'$$
$$(\mathrm{mod}\ \mathcal{O}_{\tilde{X}}Z_{-1})$$

where $\mathcal{O}_{\tilde{X}}$ denotes the sheaf of holomorphic functions on $\tilde{X}$. Since $[Z_i',\ Z_j']p_1=0$ and $Z_{-1}p_1=l+1$, both sides of (2.4.3.4) must be equal. Restricting (2.4.3.4) on X, we obtain (2.1.5.1).

We next show (2.1.5.2). Let $\tilde{f}(p)$ be the discriminant of $\tilde{P}(u)$. Then we have $f_{B_{l+1}}(p)=p_{l+1}\tilde{f}(p)$ and $\tilde{f}|_X=f_{A_l}$. Since (2.2.5.2) implies

$$Z_if_{B_{l+1}}=(l+1-i)^2f_{B_{l+1}}\quad(i=0,\ 1,\ \cdots,\ l)\ ,$$

we have

$$Z_i'\tilde{f}=\left(Z_i-\frac{i+1}{l+1}p_{i+1}Z_{-1}\right)\left(\frac{1}{p_{l+1}}f_{B_{l+1}}\right)$$
$$=(l+1-i)(l-i)p_i\tilde{f}\ .$$

Restricting this equation on X, we obtain (2.1.5.2).

§ 3. Type F_4.

We first remark that partial results are already given in [5] and [6].

(3.1) Let $\{e_1,\ e_2,\ e_3,\ e_4\}$ be an orthonormal basis of a Euclidean space $E=\boldsymbol{R}^4$. The set of roots of type F_4 consists of $\pm e_i\pm e_j$ $(i<j)$, $\pm e_i$ $(i=1, 2, 3, 4)$ and $(\pm e_1\pm e_2\pm e_3\pm e_4)/2$. We may take $\alpha_1=e_2-e_3$, $\alpha_2=e_3-e_4$, $\alpha_3=e_4$, $\alpha_4=(e_1-e_2-e_3-e_4)/2$ as a fundamental system of roots.

Let us denote by s_i the reflection of E with respect to the root α_i ($i=1, 2, 3, 4$) and put $S=\{s_1, s_2, s_3, s_4\}$. The Coxeter group $W(F_4)$ of type F_4 is the group generated by S. It is known that $W(F_4)$ is a semi-direct product of the Coxeter group $W(D_4)$ generated by $\{s_4s_3s_2s_3s_4, s_1, s_2, s_3s_2s_3\}$ by the symmetric group of degree three generated by $\{s_3, s_4\}$.

(3.2) Let E^* denote the dual of E and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ denote the dual dasis of $\{e_1, e_2, e_3, e_4\}$. Put $q_{2\nu}=\sum_{i=1}^4 \xi_i^{2\nu}$ ($\nu=1, 2, 3, 4$). Then the polynomials

$$y_2=\frac{1}{2}q_2 \,,$$

$$y_6=3q_6-\frac{15}{4}q_2q_4+\frac{15}{16}q_2^3 \,,$$

$$y_8=9q_8-\frac{21}{2}q_2q_6-\frac{21}{4}q_4^2+\frac{63}{8}q_2^2q_4-\frac{77}{64}q_2^4 \,,$$

$$y_{12}=9q_4q_8-\frac{27}{8}q_2^2q_8-\frac{3}{2}q_6^2-\frac{33}{4}q_2q_4q_6+\frac{59}{16}q_2^3q_6-\frac{17}{4}q_4^3+\frac{63}{8}q_2^2q_4^2$$
$$-\frac{229}{64}q_2^4q_4+\frac{103}{256}q_2^6$$

form a generator system of the $W(F_4)$-invariant ring.

(3.3) We introduce an orthogonal transformation $*$ of E^* defined by

$$*\colon (\xi_1, \xi_2, \xi_3, \xi_4) \longmapsto \frac{1}{\sqrt{2}}(\xi_1+\xi_2, \xi_1-\xi_2, \xi_3+\xi_4, \xi_3-\xi_4) \,,$$

and write $f^*(\xi)$ instead of $f(*\xi)$ for a polynomial $f(\xi)$ on E.

Define

$$D'(\xi)=\prod_{i>j}(\xi_i^2-\xi_j^2)$$

and

$$D(\xi)=2^{-6}D'(\xi)D'^*(\xi) \,.$$

Then $D(\xi)$ is a fundamental anti-invariant of $W(F_4)$. Accordingly, the square of $D(\xi)$ is a polynomial of y_2, y_6, y_8, y_{12}. We denote it by $f_{F_4}(y_2, y_6, y_8, y_{12})$. Then we have

$$f_{F_4}(y)=6^{-12}\left\{4\left(y_8+y_2y_6-\frac{1}{4}y_2^4\right)^3+27\left(y_{12}+\frac{1}{2}y_2^2y_8-\frac{1}{6}y_6^2+\frac{1}{6}y_2^3y_6\right)^2\right\}$$
$$\times\left\{4\left(y_8-y_2y_6-\frac{1}{4}y_2^4\right)^3+27\left(y_{12}-\frac{1}{2}y_2^2y_8+\frac{1}{6}y_6^2+\frac{1}{6}y_2^3y_6\right)^2\right\} \,.$$

(3.4) Put

$$m_{ij}(y)=\frac{1}{2}\sum_{k=1}^{4}\frac{\partial y_i}{\partial \xi_k}\frac{\partial y_j}{\partial \xi_k}.$$

Then direct calculation shows

$$m_{22}(y)=y_2, \quad m_{26}(y)=3y_6, \quad m_{28}(y)=4y_8, \quad m_{2\,12}(y)=6y_{12},$$

$$m_{66}(y)=-15y_2y_8+\frac{3}{2}y_2^5, \quad m_{68}(y)=-18y_{12}-6y_2^3y_6,$$

$$m_{6\,12}(y)=4y_8^2-4y_2^4y_8+4y_2^2y_6^2, \quad m_{88}(y)=-7y_2^3y_8+7y_2y_6^2+\frac{1}{2}y_2^7,$$

$$m_{8\,12}(y)=9y_2^2y_6y_8-y_6^3-\frac{3}{2}y_2^6y_6,$$

$$m_{12\,12}(y)=\frac{11}{3}y_2^3y_8^2-\frac{11}{3}y_2y_6^2y_8+\frac{11}{6}y_2^5y_6^2+\frac{1}{12}y_2^{11}.$$

We introduce a matrix

$$M(F_4;\, y_2, y_6, y_8, y_{12})=(m_{ij}(y))_{i,j=2,6,8,12}.$$

Then we have

$$\det(M(F_4;\, y_2, y_6, y_8, y_{12}))=2^{16}3^{12}f_{F_4}(y_2, y_6, y_8, y_{12}).$$

(3.5) Define vector fields Y_0, Y_4, Y_6, Y_{10} by

$${}^t(Y_0, Y_4, Y_6, Y_{10})=M(F_4;\, y_2, y_6, y_8, y_{12})\,{}^t\!\left(\frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_6}, \frac{\partial}{\partial y_8}, \frac{\partial}{\partial y_{12}}\right).$$

Then the following hold.

$$(3.5.1)\quad \begin{cases} [Y_0, Y_{2i}]=iY_{2i} \quad (i=0, 2, 3, 5), \\ [Y_4, Y_6]=-3y_2^2y_6Y_0-y_2^3Y_4-3Y_{10}, \\ [Y_4, Y_{10}]=-6y_2\left(y_2^2y_8-\frac{1}{2}y_6^2\right)Y_0+3y_2^2y_6Y_4+3\left(y_8-\frac{1}{2}y_2^4\right)Y_6, \\ [Y_6, Y_{10}]=4y_2y_6\left(y_8-\frac{1}{2}y_2^4\right)Y_0+\left\{\frac{4}{3}\left(y_2^2y_8-\frac{1}{2}y_6^2\right)+\frac{2}{3}y_2^2\left(y_8-\frac{1}{2}y_2^4\right)\right\}Y_4 \\ \qquad +2y_2^2y_6Y_6. \end{cases}$$

$$(3.5.2)\quad Y_0f_{F_4}=12f_{F_4}, \quad Y_4f_{F_4}=0, \quad Y_6f_{F_4}=-6y_2^3f_{F_4}, \quad Y_{10}f_{F_4}=6y_2^2y_6f_{F_4}.$$

(3.6) There are twelve conjugate classes of S-subgroups of $W(F_4)$ (cf. Table 1). Taking into account that the root system of type F_4 consists of long roots and short ones, we use the symbol A_i^* in Table 1 to denote the root system of type A_i consisting of short roots ($i=1, 2$).

(3.7) For each irreducible component Λ of $\check{S}\check{S}(\mathcal{N}_\alpha')$, $\mathrm{ord}_\Lambda u$ and $\mathrm{codim}_X \pi(\Lambda)$ are determined by [9; Th. 4.1]. The holonomy diagram of $\check{S}\check{S}(\mathcal{N}_\alpha')$ (Figure 1) is obtained by the use of Theorem 4.1 in [9] and

TABLE 1
F_4

No.	Type Σ	Subset of S	$\mathrm{ord}_\Lambda u$	$\mathrm{codim}_X \pi(\Lambda)$
1	$\emptyset$	$\emptyset$	0	0
2	A_1	$\{s_1\}$	$-(\alpha+1/2)$	1
3	A_1^*	$\{s_4\}$	$-(\alpha+1/2)$	1
4	$A_1\times A_1^*$	$\{s_1, s_4\}$	$-2(\alpha+1/2)$	2
5	A_2	$\{s_1, s_2\}$	$-3(\alpha+1/2)$	2
6	A_2^*	$\{s_3, s_4\}$	$-3(\alpha+1/2)$	2
7	B_2	$\{s_2, s_3\}$	$-4(\alpha+1/2)$	2
8	$A_2\times A_1^*$	$\{s_1, s_2, s_4\}$	$-4(\alpha+1/2)$	3
9	$A_1\times A_2^*$	$\{s_1, s_3, s_4\}$	$-4(\alpha+1/2)$	3
10	B_3	$\{s_1, s_2, s_3\}$	$-9(\alpha+1/2)$	3
11	C_3	$\{s_2, s_3, s_4\}$	$-9(\alpha+1/2)$	3
12	F_4	$\{s_1, s_2, s_3, s_4\}$	$-24(\alpha+1/2)$	4

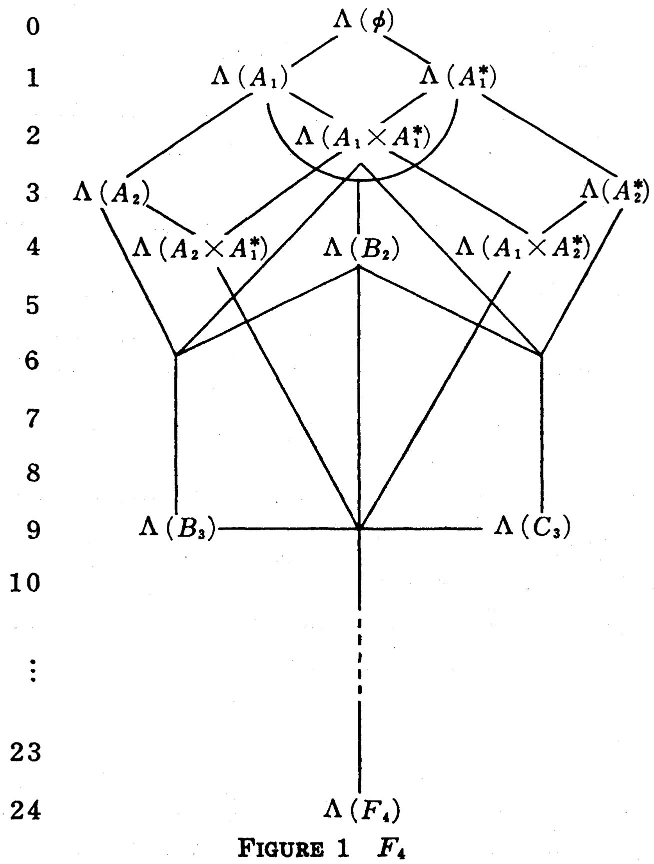


FIGURE 1 F_4

Lemma (3.7.1) below. It should be noted that any S-subgroup of a Coxeter group is also a Coxeter group. In Figure 1, $\Lambda(\Sigma)$ denotes the irreducible component of $\check{SS}(\mathcal{N}_\alpha')$ corresponding to the class of an S-subgroup G of $W(F_4)$, where G is a Coxeter group of type Σ (cf. (3.6)).

(3.7.1) LEMMA. $\Lambda(F_4)\cap\Lambda(B_2)=\Lambda(F_4)\cap\Lambda(B_3)$.

PROOF. We employ invariants x_2, x_6, x_8, x_{12} defined by

$$(3.7.1.1)\quad \begin{cases} x_2=y_2\,, \qquad x_6=-\dfrac{2}{3}y_6+5y_2^3=-2q_6+\dfrac{5}{2}q_2q_4\,, \\ x_8=-\dfrac{1}{3}y_8-\dfrac{7}{2}y_2y_6+\dfrac{133}{12}y_2^4=-3q_8+\dfrac{7}{4}q_4^2+\dfrac{7}{4}q_2^2q_4\,, \qquad x_{12}=y_{12} \end{cases}$$

as a generator system of the $W(F_4)$-invariant ring. Take $(x_2, x_6, x_8, x_{12}; \eta_2, \eta_6, \eta_8, \eta_{12})$ as a coordinate system of T^*X. Then if $(x;\eta)$ is contained in the regular part of $\Lambda(B_2)$, x_2, x_6, x_8, x_{12} are defined by (3.7.1.1) with the condition $\xi_1\xi_2\neq 0$, $\xi_1^2\neq\xi_2^2$, $\xi_3=\xi_4=0$, and $\eta_2, \eta_6, \eta_8, \eta_{12}$ satisfy the relations

$$(3.7.1.2)\quad \eta_2+(3\xi_1^4+10\xi_1^2\xi_2^2+5\xi_2^4)\eta_6+(4\xi_1^6+21\xi_1^4\xi_2^2+28\xi_1^2\xi_2^4+7\xi_2^6)\eta_8+f(\xi_1^2, \xi_2^2)\eta_{12}=0\,,$$

$$(3.7.1.3)\quad \eta_2+(3\xi_2^2+10\xi_2^2\xi_1^2+5\xi_1^2)\eta_6+(4\xi_2^6+21\xi_2^4\xi_1^2+28\xi_2^2\xi_1^4+7\xi_1^6)\eta_8+f(\xi_2^2, \xi_1^2)\eta_{12}=0\,,$$

where $f(u_1, u_2)$ is a homogeneous polynomial of degree five. Subtracting (3.7.1.2) from (3.7.1.3) and dividing the result by $\xi_2^1-\xi_2^2$, we have

$$(3.7.1.4)\qquad 2(\xi_1^2+\xi_2^2)\eta_6+(3\xi_1^4+10\xi_1^2\xi_2^2+3_2^4)\eta_8+g(\xi_1^2, \xi_2^2)\eta_{12}=0\,,$$

where $g(u_1, u_2)=(f(u_2, u_1)-f(u_1, u_2))/(u_1-u_2)$ is a polynomial of degree four. Take an analytic path $\xi_1=t+at^3$, $\xi_2=\sqrt{-1}t$, $\xi_3=\xi_4=0$ $(0<|t|<2^{-1/4}a^{-1/2})$ and let t converge to zero. Then (3.7.14) turns out to be

$$(3.7.1.5)\qquad 4a\eta_6-4\eta_8=0\,.$$

Since a can be taken arbitrary, it follows from (3.7.1.2) and (3.7.1.5) that

$$\Lambda(F_4)\cap\Lambda(B_2)=\{(x, \eta)\in T^*X;\, x=0,\, \eta_2=0\}\,.$$

The right hand side of the above equality coincides with $\Lambda(F_4)\cap\Lambda(B_3)$ by Theorem 4.1(3) in [9]. Hence the lemma.

§ 4. Type E_6.

(4.1) Let E' be an 8-dimensional Euclidean space with an orthonormal basis $\{e_1, e_2, \cdots, e_8\}$ and let E be a 6-dimensional subspace of E' spanned by $e_1, \cdots, e_5$ and $e_6'=(e_8-e_7-e_6)/\sqrt{3}$. The set of roots of type E_6 consists of

$$\pm e_i \pm e_j \quad (1 \leqq i < j \leqq 5)\,, \quad \pm\frac{1}{2}\left\{\sum_{i=1}^{5}(-1)^{\nu(i)}e_i+\sqrt{3}\,e_6'\right\} \quad \text{with} \quad \sum_{i=1}^{5}\nu(i) \quad \text{even}\,.$$

We can take

$$\alpha_1=\frac{1}{2}(e_1-e_2-e_3-e_4-e_5+\sqrt{3}\,e_6')\,, \qquad \alpha_2=e_2+e_1\,, \qquad \alpha_3=e_2-e_1\,,$$
$$\alpha_4=e_3-e_2\,, \qquad \alpha_5=e_4-e_3\,, \qquad \alpha_6=e_5-e_4\,,$$

as a fundamental system of roots.

Let us denote by s_i the reflection of E with respect to the root α_i $(i=1, \cdots, 6)$ and put $S=\{s_1, \cdots, s_6\}$. Then the Coxeter group $W(E_6)$ of type E_6 is the group generated by S. It is known that $W(E_6)$ is identified with the group of the 51840 automorphisms of the 27 lines on a general cubic surface.

We change the basis of E by the orthogonal transformation $(f_1, \cdots, f_6)=(e_1, \cdots, e_6')P$, where

$$P=\frac{1}{2\sqrt{2}}\begin{bmatrix} \sqrt{3} & -1 & 0 & 0 & -\sqrt{3} & 1 \\ -\sqrt{3} & 1 & 0 & 0 & -\sqrt{3} & 1 \\ 1/\sqrt{3} & 1 & 4/\sqrt{3} & 0 & 1/\sqrt{3} & 1 \\ 1/\sqrt{3} & 1 & -2/\sqrt{3} & 2 & 1/\sqrt{3} & 1 \\ 1/\sqrt{3} & 1 & -2/\sqrt{3} & -2 & 1/\sqrt{3} & 1 \\ -1 & -\sqrt{3} & 0 & 0 & 1 & \sqrt{3} \end{bmatrix}.$$

We shall identify $(u_1, v_1, u_2, v_2, u_3, v_3)$ with the vector

$$u_1f_1+v_1f_2+u_2f_3+v_2f_4+u_3f_5+v_3f_6\,.$$

Then α_i $(i=1, \cdots, 6)$ are represented as follows:

$$\alpha_1=(0, -\sqrt{2}, 0, 0, 0, 0)$$
$$\alpha_2=\left(0, 0, 0, 0, -\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right)$$
$$\alpha_3=\left(-\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}, 0, 0, 0, 0\right)$$
$$\alpha_4=\left(\frac{\sqrt{6}}{3}, 0, \frac{\sqrt{6}}{3}, 0, \frac{\sqrt{6}}{3}, 0\right)$$

$$\alpha_5=\left(0,\,0,\,-\frac{\sqrt{6}}{2},\,\frac{\sqrt{2}}{2},\,0,\,0\right)$$
$$\alpha_6=(0,\,0,\,0,\,-\sqrt{2},\,0,\,0)\,.$$

(4.2) The algebra of $W(E_6)$-invariant polynomials is generated by six elements of degrees 2, 5, 6, 8, 9, 12. We describe a complete set of six fundamental invariants indicated in J. S. Frame [2]. Set

$$p_i=u_i^2+v_i^2\,,\qquad q_i=\frac{1}{3}u_i^3-u_iv_i^2\,,\quad i=1,\,2,\,3\,.$$

Then the fundamental invariants x_i $(i=2, 5, 6, 8, 9, 12)$ are defined as follows: (We use the notation x_i $(i=2, \cdots, 12)$ instead of A, B, C, H, J, K in [2].)

$$\begin{aligned}
x_2&=\frac{1}{2}|p,\,1,\,1|^+\\
x_5&=|q,\,p,\,1|\\
x_6&=|p^2,\,p,\,1|^+-\frac{1}{2}|p,\,p,\,p|^++\frac{1}{2}|q^2,\,1,\,1|^+-5|q,\,q,\,1|^+\,,\\
x_8&=\frac{1}{2}|p^2,\,p^2,\,1|^+-\frac{1}{2}|p^2,\,p,\,p|^++2|q^2,\,p,\,1|^+-6|q,\,qp,\,1|^++4|q,\,q,\,p|^+\\
x_9&=|qp,\,p^2,\,1|+2|q,\,p,\,p^2|+4|q,\,q^2,\,1|\\
x_{12}&=\frac{1}{3}|p^3,\,p^3,\,1|^+-|p^3,\,p^2,\,p|^++\frac{2}{3}|p^2,\,p^2,\,p^2|^++4|q^2p,\,p^2,\,1|^++2|q^2,\,p^3,\,1|^+\\
&\quad-6|q^2,\,p^2,\,p|^+-10|qp^2,\,qp,\,1|^++4|qp^2,\,q,\,p|^++12|qp,\,qp,\,p|^+\\
&\quad-10|qp,\,q,\,p^2|^++4|q,\,q,\,p^3|^+-8|q^3,\,q,\,1|^++16|q^2,\,q^2,\,1|^+-8|q^2,\,q,\,q|^+\,.
\end{aligned}$$

Here $p={}^t(p_1, p_2, p_3)$, $q={}^t(q_1, q_2, q_3)$, $1={}^t(1, 1, 1)$, and the notation $|u, v, w|^+$ and $|u, v, w|$ for three vectors u, v, w are permanent and determinant, that is:

$$\begin{aligned}
|u,\,v,\,w|^+&=u_1(v_2w_3+v_3w_2)+u_2(v_3w_1+v_1w_3)+u_3(v_1w_2+v_2w_1)\\
|u,\,v,\,w|&=u_1(v_2w_3-v_3w_2)+u_2(v_3w_1-v_1w_3)+u_3(v_1w_2-v_2w_1)\,.
\end{aligned}$$

(4.3) If we put

$$\left.\begin{aligned}
\phi_{im}&=u_i\cos\frac{2m\pi}{3}+v_i\sin\frac{2m\pi}{3}\,,\\
f_{im}&=-u_i\sin\frac{2m\pi}{3}+v_i\cos\frac{2m\pi}{3}\,,
\end{aligned}\right\}\quad i,\,m=1,\,2,\,3\,,$$

$$f_{kmn}=\phi_{1k}+\phi_{2m}+\phi_{3n}\,,\quad k,\,m,\,n=1,\,2,\,3\,,$$

then $f_{im}=0$ and $f_{kmn}=0$ define the reflection hyperplanes, and hence a fundamental anti-invariant of $W(E_6)$ is given by:

$$D=\Big(\prod_{i,m=1}^{3} f_{im}\Big)\Big(\prod_{k,m,n=1}^{3} f_{kmn}\Big) .$$

As usual, we define

$$D^2=f_{E_6}(x_2, x_5, x_6, x_8, x_9, x_{12}) .$$

(4.4) We define polynomials $m_{ij}(x)$ $(i, j=2, 5, 6, 8, 9, 12)$ by

$$\begin{aligned} m_{ij}(x)&=\frac{1}{2}\sum_{k=1}^{3}\left(\frac{\partial x_i}{\partial u_k}\cdot\frac{\partial x_j}{\partial u_k}+\frac{\partial x_i}{\partial v_k}\cdot\frac{\partial x_j}{\partial v_k}\right)\\ &=\frac{1}{2}\sum_{k=1}^{3}\left(4p_k\frac{\partial x_i}{\partial p_k}\cdot\frac{\partial x_j}{\partial p_k}+6q_k\frac{\partial x_i}{\partial q_k}\cdot\frac{\partial x_j}{\partial p_k}+6q_k\frac{\partial x_i}{\partial p_k}\cdot\frac{\partial x_j}{\partial q_k}+p_k^2\frac{\partial x_i}{\partial q_k}\cdot\frac{\partial x_j}{\partial q_k}\right) .\end{aligned}$$

Then we have the following.

$$\begin{aligned}
&m_{2i}(x)=ix_i \quad (i=2, 5, 6, 8, 9, 12) , \qquad m_{55}(x)=x_8 , \qquad m_{56}(x)=9x_9+3x_2^2x_5 ,\\
&m_{58}(x)=5x_2x_9+4x_5x_6 , \qquad m_{59}(x)=\frac{3}{2}x_{12} , \qquad m_{5\,12}(x)=6x_6x_9+7x_2x_5x_8+10x_5^3 ,\\
&m_{66}(x)=10x_2x_8+2x_2^2x_6+40x_5^2 , \qquad m_{68}(x)=12x_{12}+4x_2^2x_8+10x_2x_5^2 ,\\
&m_{69}(x)=5x_2^2x_9+13x_5x_8 , \qquad m_{6\,12}(x)=6x_2^2x_{12}+50x_2x_5x_9+8x_8^2+32x_5^2x_6 ,\\
&m_{88}(x)=6x_2x_{12}+28x_5x_9+4x_6x_8+6x_2^2x_5^2 ,\\
&m_{89}(x)=6x_6x_9+7x_2x_5x_8+10x_5^3 \quad (=m_{5\,12}(x)) .\\
&m_{8\,12}(x)=6x_6x_{12}+18x_9^2+26x_2^2x_5x_9+4x_2x_8^2+26x_5^2x_8+16x_2x_5^2x_6 ,\\
&m_{99}(x)=4x_2x_5x_9+x_8^2+4x_5^2x_6 ,\\
&m_{9\,12}(x)=3x_2x_5x_{12}+7x_2x_8x_9+28x_5^2x_9+8x_5x_6x_8+6x_2^2x_5^3 ,\\
&m_{12\,12}(x)=6x_2x_8x_{12}+24x_5^2x_{12}+10x_2^2x_9^2+44x_5x_8x_9+40x_2x_5x_6x_9+4x_6x_8^2+28x_2^2x_5^2x_8\\
&\qquad\qquad +16x_5^2x_6^2+40x_2x_5^4 .
\end{aligned}$$

Using $m_{ij}(x)$, we define a matrix

$$M(E_6; x_2, x_5, x_6, x_8, x_9, x_{12})=(m_{ij}(x))_{i,j=2,5,6,8,9,12} .$$

Then we have

$$\det(M(E_6; x))=2^{50}3^{-18}f_{E_6}(x) .$$

(4.5) Define vector fields $X_0, X_3, X_4, X_6, X_7, X_{10}$ by

$${}^t(X_0, X_3, X_4, X_6, X_7, X_{10})=M(E_6; x)\,{}^t\!\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{12}}\right) .$$

Then the following hold.

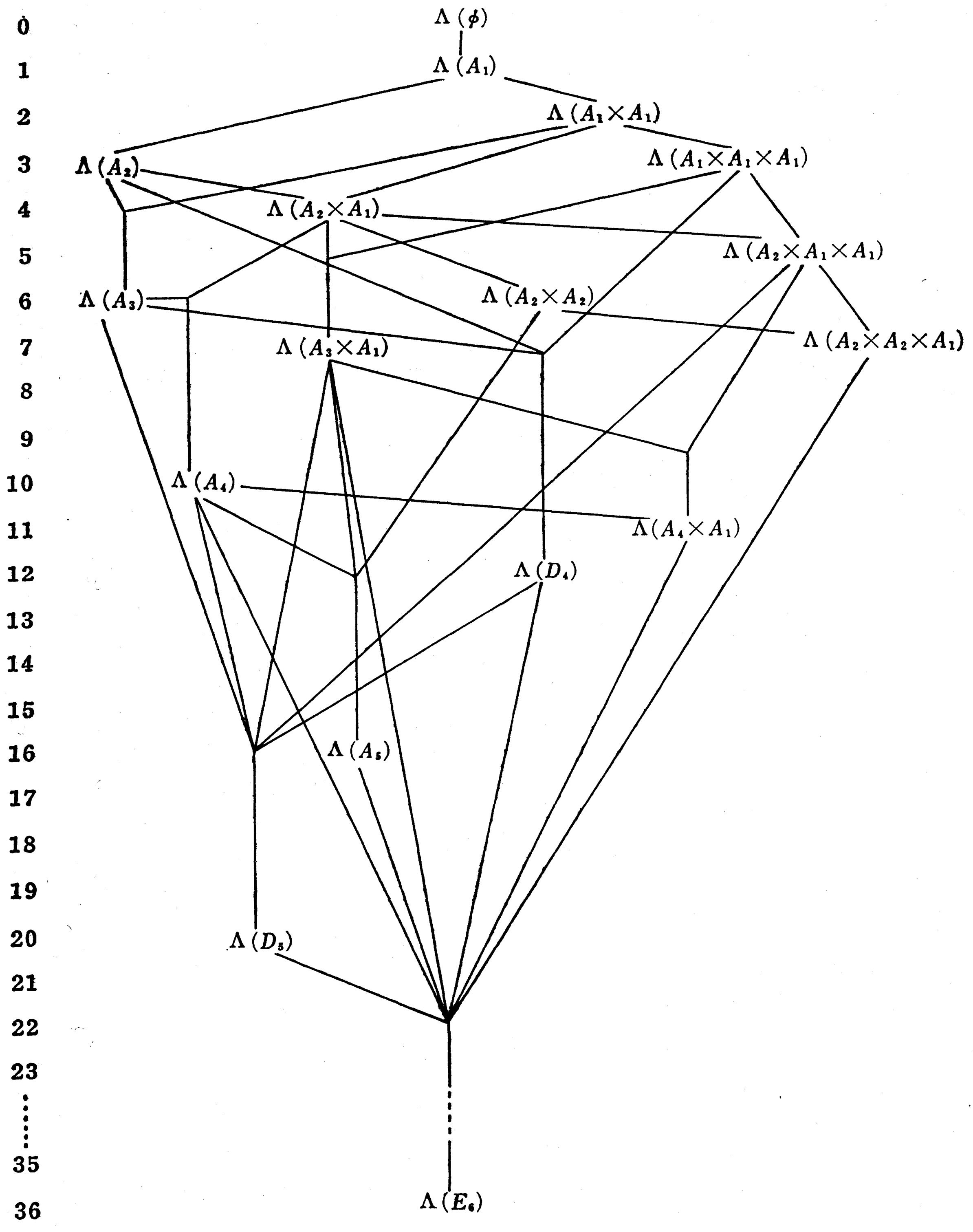
$$(4.5.1)\quad \begin{cases} [X_0, X_i]=iX_i \quad (i=0, 3, 4, 6, 7, 10)\ , \\ [X_3, X_4]=4x_2x_5X_0-x_2^2X_3+X_7\ , \\ [X_3, X_6]=3x_9X_0+2x_5X_4+x_2X_7\ , \\ [X_3, X_7]=\dfrac{1}{2}X_{10}\ , \\ [X_3, X_{10}]=5x_5x_8X_0+(3x_2x_8+14x_5^2)X_3+4x_9X_4+3x_2x_5X_6+2x_6X_7\ , \\ [X_4, X_6]=(-4x_2x_8+10x_5^2)X_0+2x_2^2X_6+2X_{10}\ , \\ [X_4, X_7]=-6x_2x_9X_0+3x_8X_3+3x_5X_6+3x_2^2X_7\ , \\ [X_4, X_{10}]=3(-x_2x_{12}+10x_5x_9)X_0+6(5x_2x_9+4x_5x_6)X_3+12x_5^2X_4+6x_8X_6 \\ \qquad +10x_2x_5X_7+4x_2^2X_{10}\ , \\ [X_6, X_7]=-3x_5x_8X_0+(x_2x_8+2x_5^2)X_3-2x_9X_4+x_2x_5X_6+2x_6X_7\ , \\ [X_6, X_{10}]=8x_2x_5x_9X_0+2(7x_2^2x_9+12x_5x_8+4x_2x_5x_6)X_3+4x_2x_5^2X_4 \\ \qquad +2(x_2x_8-x_5^2)X_6+2(4x_9+x_2^2x_5)X_7+2x_6X_{10}\ , \\ [X_7, X_{10}]=(3x_5x_{12}+3x_8x_9+4x_2x_5^3)X_0+(3x_2x_{12}+24x_5x_9+2x_2^2x_5^2)X_3 \\ \qquad +4x_5x_8X_4+3x_2x_9X_6-(x_2x_8+4x_5^2)X_7-x_2x_5X_{10}\ . \end{cases}$$

$$(4.5.2)\quad \begin{cases} X_0f_{E_6}=72f_{E_6}\ , & X_3f_{E_6}=0\ , & X_4f_{E_6}=30x_2^2f_{E_6}\ , \\ X_6f_{E_6}=24x_6f_{E_6}\ , & X_7f_{E_6}=12x_2x_5f_{E_6}\ , & X_{10}f_{E_6}=24(x_2x_8+5x_5^2)f_{E_6}\ . \end{cases}$$

TABLE 2

E_6

No.	Type Σ	Subsut of S	$\mathrm{ord}_\Lambda u$	$\mathrm{codim}_X\,\pi(\Lambda)$
1	$\emptyset$	$\emptyset$	0	0
2	A_1	$\{s_1\}$	$-(\alpha+1/2)$	1
3	$A_1\times A_1$	$\{s_1, s_6\}$	$-2(\alpha+1/2)$	2
4	A_2	$\{s_1, s_2\}$	$-3(\alpha+1/2)$	2
5	$A_1\times A_1\times A_1$	$\{s_1, s_4, s_6\}$	$-3(\alpha+1/2)$	3
6	$A_2\times A_1$	$\{s_1, s_3, s_6\}$	$-4(\alpha+1/2)$	3
7	A_3	$\{s_1, s_3, s_4\}$	$-6(\alpha+1/2)$	3
8	$A_2\times A_1\times A_1$	$\{s_1, s_3, s_2, s_6\}$	$-5(\alpha+1/2)$	4
9	$A_2\times A_2$	$\{s_1, s_3, s_5, s_6\}$	$-6(\alpha+1/2)$	4
10	$A_3\times A_1$	$\{s_1, s_3, s_4, s_6\}$	$-7(\alpha+1/2)$	4
11	A_4	$\{s_1, s_2, s_3, s_4\}$	$-10(\alpha+1/2)$	4
12	D_4	$\{s_2, s_3, s_4, s_5\}$	$-12(\alpha+1/2)$	4
13	$A_2\times A_2\times A_1$	$\{s_1, s_3, s_5, s_6, s_2\}$	$-7(\alpha+1/2)$	5
14	$A_4\times A_1$	$\{s_1, s_2, s_3, s_4, s_6\}$	$-11(\alpha+1/2)$	5
15	A_5	$\{s_1, s_3, s_4, s_5, s_6\}$	$-15\{\alpha+1/2\}$	5
16	D_5	$\{s_1, s_2, s_3, s_4, s_5\}$	$-20(\alpha+1/2)$	5
17	E_6	$\{s_1, s_2, s_3, s_4, s_5, s_6\}$	$-36(\alpha+1/2)$	6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$\Lambda(\phi)$
$\Lambda(A_1)$
$\Lambda(A_1\times A_1)$
$\Lambda(A_2)$
$\Lambda(A_1\times A_1\times A_1)$
$\Lambda(A_2\times A_1)$
$\Lambda(A_2\times A_1\times A_1)$
$\Lambda(A_3)$
$\Lambda(A_2\times A_2)$
$\Lambda(A_3\times A_1)$
$\Lambda(A_2\times A_2\times A_1)$
$\Lambda(A_4)$
$\Lambda(A_4\times A_1)$
$\Lambda(D_4)$
$\Lambda(A_5)$
$\Lambda(D_5)$
$\Lambda(E_6)$

FIGURE 2 E_6

(4.6) There are seventeen conjugate classes of S-subgroups of $W(E_6)$ (cf. Table 2).

(4.7) For each irreducible component Λ of $\check{S}\check{S}(\mathcal{N}'_\alpha)$, $\mathrm{ord}_\Lambda u$ and $\mathrm{codim}_X \pi(\Lambda)$ are determined by [9, Th. 4.1]. The holonomy diagram (Figure 2) is obtained by Theorem 4.1 in [9], Lemma (2.3.7.1) and Lemma (4.7.1) below. In Figure 2, the notation $\Lambda(\Sigma)$ has the meaning similar to the one in (3.7).

(4.7.1) LEMMA. *Let $\Sigma = D_4$, A_4 or $A_3 \times A_1$. Then*

$$\Lambda(E_6) \cap \Lambda(\Sigma) = \Lambda(E_6) \cap \Lambda(D_5) .$$

PROOF. The proof runs as same as that of Lemma (3.7.1).

(4.7.2) REMARK. We can also check that $\Lambda(A_2 \times A_1 \times A_1)$ (or $\Lambda(A_2 \times A_2)$) $\Lambda(E_6)$ intersect in a 2-codimensional analytic subset.

§ 5. Type H_3.

(5.1) Let $E = \boldsymbol{R}^3$ and let $\{e_1, e_2, e_3\}$ be an orthonomal basis of it. Define

$$\alpha_1 = e_1 , \qquad \alpha_2 = -\frac{1}{2}(ae_1 + \bar{a}e_2 + e_3) , \qquad \alpha_3 = e_3 ,$$

where we put $a = (1+\sqrt{5})/2$ and $\bar{a} = (1-\sqrt{5})/2$. We denote by s_i the reflection of E with respect to α_i $(i=1, 2, 3)$. Then we have

(5.1.1) $$s_i^2 = \mathrm{id}. \quad (i=1, 2, 3) , \qquad (s_1 s_2)^5 = (s_2 s_3)^3 = (s_1 s_3)^2 = \mathrm{id}.$$

The group generated by $S = \{s_1, s_2, s_3\}$ is the Coxeter group of type H_3. We denote this group by $W(H_3)$. The fundamental relation is given by (5.1.1).

(5.2) Put

$$p_2 = \xi_1^2 + \xi_2^2 + \xi_3^2 , \qquad p_4 = \xi_1^2\xi_2^2 + \xi_1^2\xi_3^2 + \xi_2^2\xi_3^2 , \qquad p_6 = \xi_1^2\xi_2^2\xi_3^2 ,$$
$$d = (\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)(\xi_2^2 - \xi_3^2) .$$

Then the polynomials

$$x_2 = p_2 , \qquad x_6 = -11p_6 + p_2 p_4 + \sqrt{5}\, d ,$$
$$x_{10} = 95 p_4 p_6 - 32 p_2^2 p_6 - 5 p_2 p_4^2 + 2 p_2^3 p_4 + 3 p_4(\sqrt{5}\, d)$$

form a generator system of the $W(H_3)$-invariant ring.

(5.3) Put

$$D(\xi)=2^{15}\prod_{i=1}^{3}\prod_{\nu=0}^{4}(s_1s_2)^{\nu}(\xi_i)\ .$$

Then $D(\xi)$ is a fundamental anti-invariant of $W(H_3)$ and $f_{H_3}(x_2, x_6, x_{10})=(D(\xi))^2$ is given by

$$f_{H_3}(x)=-40x_{10}^3+\left(-40x_6+\frac{16}{5}x_2^5\right)x_2^2x_{10}^2+\left(180x_6^2+48x_2^3x_6+\frac{16}{5}x_2^6\right)x_2x_6x_{10}$$
$$-\left(54x_6^3+92x_2^3x_6^2+8x_2^6x_6+\frac{16}{5}x_2^9\right)x_6^2\ .$$

(5.4) Put

$$m_{ij}(x)=\frac{1}{2}\sum_{k=1}^{3}\frac{\partial x_i}{\partial\xi_k}\frac{\partial x_j}{\partial\xi_k}\quad(i, j=2, 6, 10)\ .$$

Then straightforward calculation gives

$$m_{22}(x)=2x_2\ ,\qquad m_{26}(x)=6x_6\ ,\qquad m_{2\,10}(x)=10x_{10}\ ,$$
$$m_{66}(x)=4x_{10}+4x_2^2x_6\ ,\qquad m_{6\,10}(x)=14x_2x_6^2+4x_2^4x_6\ ,$$
$$m_{10\,10}(x)=15x_6^3+4x_2^4x_{10}+18x_2^3x_6^2\ .$$

Define a matrix

$$M(H_3;\, x_2, x_6, x_{10})=(m_{ij}(x))_{i,j=2,6,10}\ .$$

Then

$$\det M(H_3;\, x_2, x_6, x_{10})=10f_{H_3}(x_2, x_6, x_{10})\ .$$

(5.5) We define differential operators X_0, X_4, X_8 by

$$^t(X_0, X_4, X_8)=M(H_3;\, x_2, x_6, x_{10})^t\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_{10}}\right)\ .$$

Then the following hold.

(5.5.1) $[X_0, X_k]=kX_k$ $(k=0, 4, 8)$, $[X_4, X_8]=(4x_6^2+8x_2^3x_6)X_0+8x_2x_6X_4$,

(5.5.2) $X_0f_{H_3}=30f_{H_3}$, $X_4f_{H_3}=4x_2^2f_{H_3}$, $X_8f_{H_3}=(20x_2x_6+8x_2^4)f_{H_3}$.

TABLE 3

H_3

No.	Type Σ	Subset of S	$\mathrm{ord}_\Lambda u$	$\mathrm{codim}_X\,\pi(\Lambda)$
1	$\varnothing$	$\varnothing$	0	0
2	A_1	$\{s_1\}$	$-(\alpha+1/2)$	1
3	$A_1\times A_1$	$\{s_1, s_3\}$	$-2(\alpha+1/2)$	2
4	A_2	$\{s_2, s_3\}$	$-3(\alpha+1/2)$	2
5	$I_2(5)$	$\{s_1, s_2\}$	$-5(\alpha+1/2)$	2
6	H_3	$\{s_1, s_2, s_3\}$	$-15(\alpha+1/2)$	3

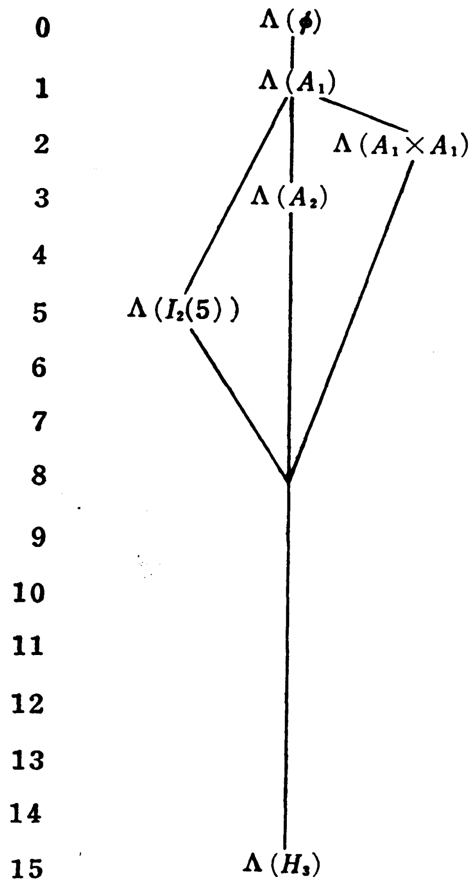

FIGURE 3 H_3

(5.6) There are six conjugate classes of S-subgroups of $W(H_3)$. They are listed in Table 3.

(5.7) The holonomy diagram is given in Figure 3.

§6. Type H_4.

(6.1) Let $E=\boldsymbol{R}^4$ and let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of it. Define

$$\alpha_1=e_1\ , \qquad \alpha_2=\frac{1}{2}(ae_1+\bar{a}e_2+e_3)\ , \qquad \alpha_3=e_3\ , \qquad \alpha_4=-\frac{1}{2}(ae_2+e_3+\bar{a}e_4)\ ,$$

where we put $a=(1+\sqrt{5})/2$ and $\bar{a}=(1-\sqrt{5})/2$. We denote by s_i the reflection of E with respect to α_i $(i=1, 2, 3, 4)$. Then we have

$$s_i^2=\text{id.}\quad (i=1, 2, 3, 4)\ ,$$
$$(s_1s_2)^5=(s_2s_3)^3=(s_3s_4)^3=(s_1s_3)^2=(s_1s_4)^2=(s_2s_4)^2=\text{id.}$$

The group generated by $S=\{s_1, s_2, s_3, s_4\}$ is the Coxeter group of type H_4. We denote it by $W(H_4)$.

(6.2) Let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the dual basis of $\{e_1, e_2, e_3, e_4\}$. Let $W(H_3)$ be the S-subgroup of $W(H_4)$ generated by $\{s_1, s_2, s_3\}$, and x_2, x_6, x_{10} the generator system of the $W(H_3)$-invariant ring specified in (5.2). Then the following polynomials $z_2, z_{12}, z_{20}, z_{30}$ of $\{\xi_0, x_2, x_6, x_{10}\}$ form a generator system of the $W(H_4)$-invariant ring (cf. [5]).

$$
\begin{aligned}
z_2 &= \xi_4^2 + x_2 \,, \\
z_{12} &= -2x_2\xi_4^{10} + 6x_2^2\xi_4^8 + (33x_6 - 14x_2^3)\xi_4^6 - (33x_2x_6 - 13x_2^4)\xi_4^4 + (11x_{10} + 2x_2^5)\xi_4^2 \\
&\quad - x_2x_{10} + \frac{3}{2}x_6^2 \,, \\
z_{20} &= 4x_2^2\xi_4^{16} - (30x_6 + 20x_2^3)\xi_4^{14} + (138x_2x_6 + 44x_2^4)\xi_4^{12} + (180x_{10} - 402x_2^2x_6 - 44x_2^5)\xi_4^{10} \\
&\quad + (-464x_2x_{10} + 294x_6^2 + 402x_2^3x_6 + 44x_2^6)\xi_4^8 + (296x_2^2x_{10} - 306x_2x_6^2 - 138x_2^4x_6 \\
&\quad - 20x_2^7)\xi_4^6 + (-114x_6x_{10} - 76x_2^3x_{10} + 168x_2^2x_6^2 + 30x_2^5x_6 + 4x_2^8)\xi_4^4 \\
&\quad + \left(4x_2^4x_{10} + \frac{57}{2}x_6^3 - 21x_2^3x_6^2\right)\xi_4^2 + x_{10}^2 - \frac{3}{2}x_2x_6^3 \,, \\
z_{30} &= \frac{32}{3}x_2^3\xi_4^{24} - (120x_2x_6 + 80x_2^4)\xi_4^{22} + \left(360x_{10} + \frac{1344}{5}x_2^5 + 672x_2^2x_6\right)\xi_4^{20} + \Big(-2880x_2x_{10} \\
&\quad + 1080x_6^2 - 1608x_2^3x_6 - \frac{1328}{3}x_2^6\Big)\xi_4^{18} + (10024x_2^2x_{10} - 5628x_2x_6^2 + 1248x_2^4x_6 + 272x_2^7)\xi_4^{16} \\
&\quad + (-7260x_6x_{10} - 16856x_2^3x_{10} + 18588x_2^2x_6^2 + 272x_2^8)\xi_4^{14} \\
&\quad + \left(23508x_2x_6x_{10} + 14216x_2^4x_{10} - 5796x_6^3 - 27396x_2^3x_6^2 - 1248x_2^6x_6 - \frac{1328}{3}x_2^9\right)\xi_4^{12} \\
&\quad + \left(3240x_{10}^2 + 7650x_2x_6^3 + 19968x_2^4x_6^2 + 1608x_2^7x_6 + \frac{1344}{5}x_2^{10}\right)\xi_4^{10} \\
&\quad + (-3232x_2x_{10}^2 - 1956x_2^2x_6^3 - 6924x_2^5x_6^2 - 672x_2^8x_6 - 80x_2^{11})\xi_4^8 \\
&\quad + \Big(1168x_2^2x_{10}^2 - 1908x_2x_6^2x_{10} - 2172x_2^4x_6x_{10} - 344x_2^7x_{10} + 2394x_6^4 + 288x_2^3x_6^3 \\
&\quad + 1332x_2^6x_6^2 + 120x_2^9x_6 + \frac{32}{3}x_2^{12}\Big)\xi_4^6 + (348x_6x_{10}^2 - 152x_2^3x_{10}^2 + 408x_2^2x_6^2x_{10} + 60x_2^5x_6x_{10} \\
&\quad + 16x_2^8x_{10} - 909x_2x_6^4 + 84x_2^4x_6^3 - 84x_2^7x_6^2)\xi_4^4 + (8x_2^4x_{10}^2 - 87x_6^3x_{10} - 42x_2^3x_6^2x_{10} + 135x_2^2x_6^4 \\
&\quad - 6x_2^5x_6^3)\xi_4^2 + \frac{4}{3}x_{10}^3 - 3x_2x_6^3x_{10} + \frac{9}{5}x_6^5 \,.
\end{aligned}
$$

(6.3) Put

$$D(\xi) = 2^{-56}\prod_{i=1}^{4}\xi_i \cdot \prod(\xi_1 \pm \xi_2 \pm \xi_3 \pm \xi_4) \cdot \prod_{\sigma \in \mathfrak{A}_4}(a\xi_{\sigma(1)} \pm \bar{a}\xi_{\sigma(2)} \pm \xi_{\sigma(3)}) \,,$$

where we take products over all combinations of signs and $\mathfrak{A}_4$ is the alternating group of degree four. Then $D(\xi)$ is a fundamental anti-invariant of $W(H_4)$ and is of degree $4+2^3+2^2\times 12=60$. We denote the square of $D(\xi)$ by $f_{H_4}(z_2, z_{12}, z_{20}, z_{30})$.

(6.4) Put

$$m_{ij}(z)=\frac{1}{2}\sum_{k=1}^{4}\frac{\partial z_i}{\partial \xi_k}\frac{\partial z_j}{\partial \xi_k} \quad (i, j=2, 12, 20, 30) .$$

Then via direct calculation:

$$m_{2i}(z)=iz_i \quad (i=2, 12, 20, 30) , \qquad m_{12\,12}(z)=22z_2z_{20}-4z_2^5z_{12} ,$$

$$m_{12\,20}(z)=-15z_{30}-12z_2^5z_{20}+4z_2^3z_{12}^2 ,$$

$$m_{12\,30}(z)=-20z_2^5z_{30}-40z_{20}^2-8z_2^4z_{12}z_{20}-\frac{8}{3}z_2^2z_{12}^3 ,$$

$$m_{20\,20}(z)=10z_2^4z_{30}-28z_2^3z_{12}z_{20}+\frac{76}{3}z_2z_{12}^3 ,$$

$$m_{20\,30}(z)=-60z_2^3z_{12}z_{30}+24z_2^4z_{20}^2-56z_2^2z_{12}^2z_{20}-16z_{12}^4 ,$$

$$m_{30\,30}(z)=80z_2^4z_{20}z_{30}+120z_2^2z_{12}^2z_{30}+48z_2^3z_{12}z_{20}^2+\frac{464}{3}z_2z_{12}^3z_{20} .$$

We define a matrix

$$M(H_4;\, z_2, z_{12}, z_{20}, z_{30})=(m_{ij}(z))_{i,j=2,12,20,30} .$$

Then

$$\det M(H_4;\, z_2, z_{12}, z_{20}, z_{30})=f_{H_4}(z_2, z_{12}, z_{20}, z_{30}) .$$

(6.5) We introduce vector fields $X_0, X_{10}, X_{18}, X_{28}$ by

$${}^t(X_0, X_{10}, X_{18}, X_{28})=M(H_4;\, z_2, z_{12}, z_{20}, z_{30})\,{}^t\!\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_{12}}, \frac{\partial}{\partial z_{20}}, \frac{\partial}{\partial z_{30}}\right) .$$

TABLE 4

H_4

No.	Type Σ	Subset of S	$\mathrm{ord}_\Lambda u$	$\mathrm{codim}_X \pi(\Lambda)$
1	$\varnothing$	$\varnothing$	0	0
2	A_1	$\{s_1\}$	$-(\alpha+1/2)$	1
3	$A_1\times A_1$	$\{s_1, s_3\}$	$-2(\alpha+1/2)$	2
4	A_2	$\{s_2, s_3\}$	$-3(\alpha+1/2)$	2
5	$I_2(5)$	$\{s_1, s_2\}$	$-5(\alpha+1/2)$	2
6	$A_1\times A_2$	$\{s_1, s_3, s_4\}$	$-4(\alpha+1/2)$	3
7	$I_2(5)\times A_1$	$\{s_1, s_2, s_4\}$	$-6(\alpha+1/2)$	3
8	A_3	$\{s_2, s_3, s_4\}$	$-6(\alpha+1/2)$	3
9	H_3	$\{s_1, s_2, s_3\}$	$-15(\alpha+1/2)$	3
10	H_4	$\{s_1, s_2, s_3, s_4\}$	$-60(\alpha+1/2)$	4

Then the following hold.

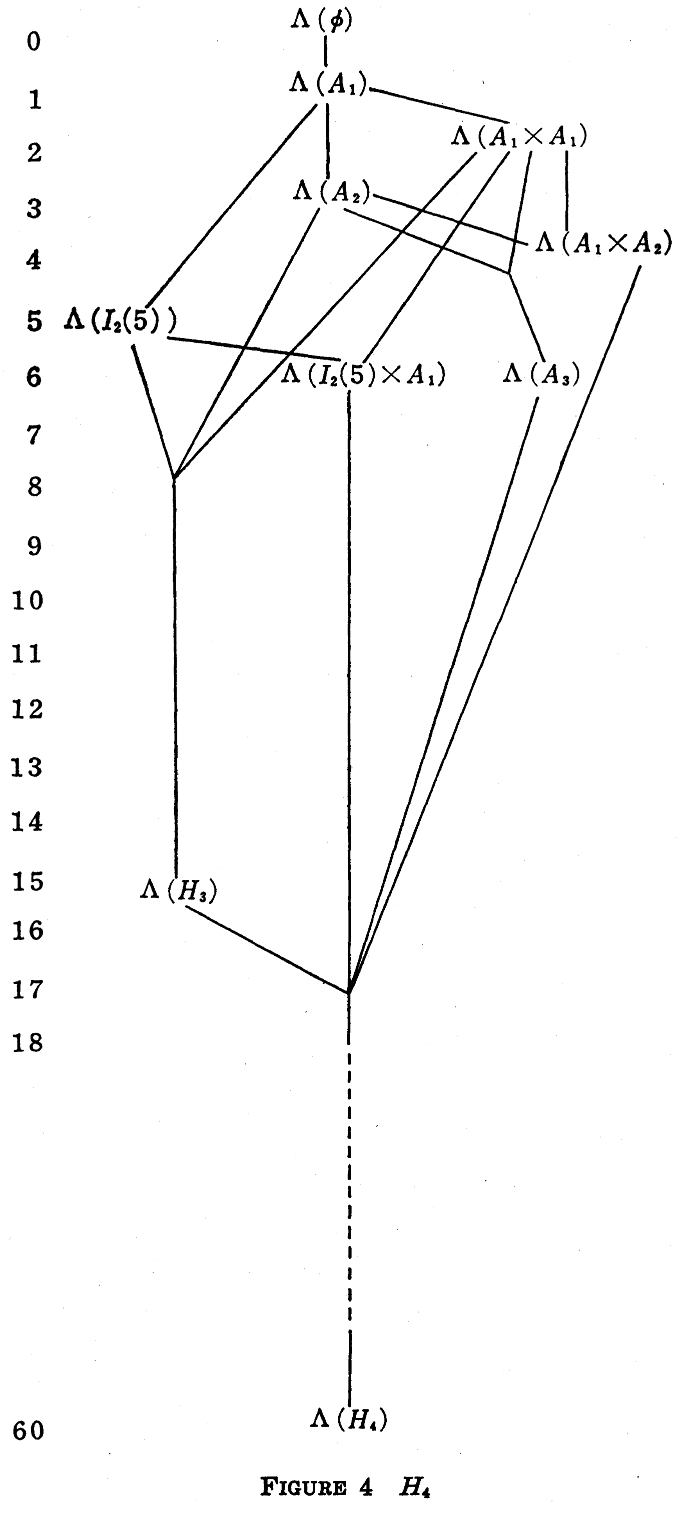

FIGURE 4 H_4

$$(6.5.1)\quad \begin{cases} [X_0, X_i]=iX_i \quad (i=0, 10, 18, 28)\ , \\ [X_{10}, X_{18}]=32z_2^2(z_2^2z_{20}-z_{12}^2)X_0+8z_2^3z_{12}X_{10}-8z_2^5X_{18}-4X_{28}\ , \\ [X_{10}, X_{28}]=(60z_2^4z_{30}+56z_2^3z_{12}z_{20}+24z_2z_{12}^3)X_0 \\ \qquad -8z_2^2(z_2^2z_{20}+z_{12}^2)X_{10}-(36z_{20}+8z_2^4z_{12})X_{18}-16z_2^5X_{28}\ , \\ [X_{18}, X_{28}]=(60z_2^2z_{12}z_{30}+40z_2^3z_{20}^2+40z_2z_{12}^2z_{20})X_0 \\ \qquad +\left(20z_2^3z_{30}+40z_2^2z_{12}z_{20}-\dfrac{40}{3}z_{12}^3\right)X_{10} \\ \qquad +(8z_2^4z_{20}-56z_2^2z_{12}^2)X_{18}-32z_2^3z_{12}X_{28}\ , \end{cases}$$

$$(6.5.2)\quad \begin{cases} X_0f_{H_4}=120f_{H_4}\ , \qquad X_{10}f_{H_4}=-60z_2^5f_{H_4}\ , \qquad X_{18}f_{H_4}=-120z_2^3z_{12}f_{H_4}\ , \\ X_{28}f_{H_4}=120z_2^2(z_2^2z_{20}+z_{12}^2)f_{H_4}\ . \end{cases}$$

(6.6) There are ten conjugate classes of S-subgroups of $W(H_4)$. They are listed in Table 4.

(6.7) The holonomy diagram is given in Figure 4.

§ 7. Type $I_2(p)$.

(7.1) Let $\{e_1, e_2\}$ be an orthonormal basis of $E=\boldsymbol{R}^2$, define $\alpha_1=e_1$, $\alpha_2=-\cos(\pi/p)\cdot e_1+\sin(\pi/p)\cdot e_2$ and denote by s_i the reflection of E with respect to α_i $(i=1, 2)$. Then we have

$$(7.1.1)\qquad s_1^2=s_2^2=(s_1s_2)^p=\mathrm{id}.$$

The Coxeter group of type $I_2(p)$ is the group generated by $S=\{s_1, s_2\}$. We denote it by $W(I_2(p))$. It is isomorphic to the dihedral group of order $2p$, and (7.1.1) is the fundamental relation of S. We remark that $W(I_2(p))$ is isomorphic to $W(A_1)\times W(A_1)$, $W(A_2)$, $W(B_2)$, $W(G_2)$ when $p=2, 3, 4, 6$, respectively.

(7.2) Define $x_2=\xi_1^2+\xi_2^2$,

$$x_p=(-2)^p\prod_{k=0}^{p-1}\left[\cos\left\{\left(k+\frac{1}{2}\right)\frac{\pi}{p}\right\}\xi_1+\sin\left\{\left(k+\frac{1}{2}\right)\frac{\pi}{p}\right\}\xi_2\right].$$

Then x_2, x_p generate the $W(I_2(p))$-invariant ring.

(7.3) A fundamental anti-invariant of $W(I_2(p))$ is

$$D(\xi)=-(-2)^p\prod_{k=0}^{p-1}\left\{\cos\left(\frac{k\pi}{p}\right)\xi_1+\sin\left(\frac{k\pi}{p}\right)\xi_2\right\}.$$

Therefore $f_{I_2(p)}=D^2$ is given by $f_{I_2(p)}=4x_2^p-x_p^2$.

(7.4) The $m_{ij}(x)=\sum_{k=1}^{2}\partial x_i/\partial\xi_k\cdot\partial x_j/\partial\xi_k$ $(i, j=2, p)$ are given by

$$m_{22}(x)=2x_2\ , \qquad m_{2p}(x)=px_p\ , \qquad m_{pp}(x)=2p^2x_2^{p-1}\ .$$

Set $M(I_2(p); x_2, x_p)=(m_{ij}(x))_{i,j=2,p}$. Then

$$\det M(I_2(p); x)=p^2 f_{I_2(p)}(x) \ .$$

(7.5) Define vector fields X_0, X_{p-2} by ${}^t(X_0, X_{p-2})=M(I_2(p); x)\cdot {}^t(\partial/\partial x_2, \partial/\partial x_p)$. Then we have

(7.5.1) $$[X_0, X_{p-2}]=(p-2)X_{p-2} \ ,$$

(7.5.2) $$X_0 f_{I_2(p)}=2pf_{I_2(p)} \ , \qquad X_{p-2}f_{I_2(p)}=0 \ .$$

(7.6) **The conjugate classes of S-subgroups are listed in Table 5.**

(7.7) **The holonomy diagram is given in Figure 5.**

(7.8) **For convenience, put $f=f_{I_2(p)}$. We define an ideal $\mathscr{J}(s)$ of $\mathscr{D}_X[s]$ by**

$$\mathscr{J}(s)=\{P(s)\in\mathscr{D}_X[s];\ P(s)f^s=0\}$$

(cf. [8]) and an ideal $\mathscr{J}(\alpha)$ of $\mathscr{D}_X$ by

$$\mathscr{J}(\alpha)=\{P\in\mathscr{D}_X;\ P=Q(\alpha) \text{ with } Q(s)\in\mathscr{J}(s)\}$$

for a complex number α. Then $\mathscr{N}_\alpha=\mathscr{D}_X/\mathscr{J}(\alpha)$ equals $\mathscr{N}_\alpha'=\mathscr{D}_X/\mathscr{G}_f(\alpha)$ and the b-function of f^s is determined. Namely, we have the following proposition.

TABLE 5
$I_2(p)$

p: odd				p: even			
Type Σ	Subset	$\mathrm{ord}_\Lambda u$	codim	Type Σ	Subset	$\mathrm{ord}_\Lambda u$	codim
$\emptyset$	$\emptyset$	0	0	$\emptyset$	$\emptyset$	0	0
A_1	$\{s_1\}$	$-(\alpha+1/2)$	1	A_1	$\{s_1\}$	$-(\alpha+1/2)$	1
$I_2(p)$	$\{s_1, s_2\}$	$-p(\alpha+1/2)$	2	A_1'	$\{s_2\}$	$-(\alpha+1/2)$	1
				$I_2(p)$	$\{s_1, s_2\}$	$-p(\alpha+1/2)$	2

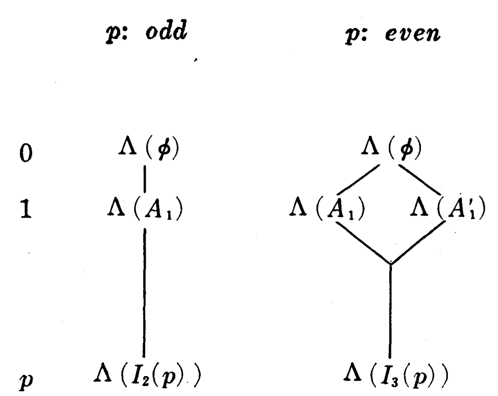

FIGURE 5 $I_2(p)$

(7.8.1) PROPOSITION.

(7.8.1.1) $$\mathcal{N}'_\alpha \cong \mathcal{N}_\alpha .$$

(7.8.1.2) $$b_f(s)=(s+1)\prod_{i=1}^{p-1}\left(s+\frac{1}{2}+\frac{i}{p}\right).$$

PROOF. First recall that

$$\mathcal{G}_f(\alpha)=\mathcal{D}_X(X_0-2p\alpha)+\mathcal{D}_X X_{p-2} .$$

Since $\partial f/\partial x_2=4px_2^{p-1}$ and $\partial f/\partial x_p=-2x_p$ form a regular sequence, it follows from § 6 in [8] that

$$\mathcal{J}(s)=\mathcal{D}_X[s](X_0-2ps)+\mathcal{D}_X[s]X_{p-2} .$$

Hence we have

$$\mathcal{G}_f(\alpha)=\mathcal{J}(\alpha)$$

for any complex number α, and this proves (7.8.1.1).

The formula (7.8.1.2) follows from Th. 3.8 in [8].

(6.8.2) Proposition (7.8.1) asserts that Conjectures in [9] hold for the Coxeter system of type $I_2(p)$.

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