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On an Analogoue to Hecke Correspondence

Koji KATAYAMA

Tsuda College

Introduction

In [2], Hecke established the one-to-one correspondence, called Hecke correspondence, between (A) and (B) below, through Mellin and inverse Mellin transformation: (k: even, $k\geqq 4$)

(A) (i) $f(z)$ is analytic on the upper-half plane H,

(ii) $f(\sigma(z))=(cz+d)^k f(z)$ for $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$ (Γ: the full modular group),

(iii) $f(z)=\sum_{n=0}^{\infty} a(n)e^{2\pi i n z}$.

(B) (i) If $a(0)=0$, then

$$\varphi(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$$

is continued to an entire function of s.

(ii) If $a(0)\neq 0$, then $\varphi(s)$ is continued to s-plane analytically except for a simple pole at $s=k$ with residue

$$\frac{(-1)^{k/2}a(0)(2\pi)^k}{\Gamma(k)}.$$

(iii) $(2\pi)^{-s}\Gamma(s)\varphi(s)=(-1)^{k/2}(2\pi)^{s-k}\Gamma(k-s)\varphi(k-s)$.

This is a vast generalization of Hamburger's Theorem on the determination of Riemann zeta-function via the functional equation.

Note that in (A), it follows that

$$\text{if}\quad a(0)=0\quad\text{then}\quad a(n)=O(n^{k/2})$$

and

$$\text{if}\quad a(0)\neq 0\quad\text{then}\quad a(n)=O(n^{k-1}).$$

Now we shall consider

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(C) (i) $a(n)$ are as above and

$$b(n)=\frac{2^k(-1)^{k/2}\Gamma(k+(1/2))a(n)}{\pi^{k+1/2}} .$$

(ii) For

$$g(z)=\sum_{n=0}^{\infty} a(n)e^{2\pi i n^{1/2} z} ,$$
$$h(z)=\sum_{n=0}^{\infty} \frac{b(n)}{(-4nz^2+1)^{k+1/2}} , \qquad z \in H ,$$

the following "inversion formula" holds:

$$g(z)=z^{-2k}h(-z^{-1}) .$$

The main purpose of the present paper is to settle the one-to-one correspondence between (B) and (C). One significance of (C) is that the Kronecker's formula for partial fraction expansion of $e^{uz}/(e^z-1)$, $0<u<1$, belongs to (C), and we note that the well-known formula for $\zeta(2n)$, $L(2n+1, \chi)$ with χ odd and $L(2n, \chi)$ with χ even can be derived from Kronecker's formula. $(n\geqq 1)$.

The correspondence of this kind but slightly different from us (at least in procedure) was already given by K. Chandrasekharan and Raghavan Narasimhan [1] in a more general context.

§1. A proof of (B) $\Rightarrow$ (C).

We use Legendre's formula for Γ-function:

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)=\frac{(2\pi)^{1/2}}{2^{s-1/2}}\Gamma(s) \tag{1.1}$$

and Barnes' contour integral for the hypergeometric function [4]:

$$\frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds = F(a, b, c; z)\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} ,$$

where $|\arg(-z)|<\pi$, all poles of $\Gamma(a+s)\Gamma(b+s)$ lie in the left side of the line $\mathrm{Re}\, s=\sigma$ and all poles of $\Gamma(-s)$ lie in the right side of the line. In particular, for $b=c$, $t=-z$ positive real and s instead of $-s$, we have

$$\Gamma(a)(1+t)^{-a}=\frac{1}{2\pi i}\int_{-\sigma-\infty i}^{-\sigma+\infty i} \Gamma(a-s)\Gamma(s)t^{-s} ds . \tag{1.2}$$

Here all poles of $\Gamma(a-s)$ lie in the right side of the line $\mathrm{Re}\, s=-\sigma$ and all poles of $\Gamma(s)$ lie in the left of the line.

Now we shall go from (B) to (C). Multiply both hands of (B), (iii) by $\Gamma((s+1)/2)$ then by (1.1), we have

$$\frac{\Gamma(s)\varphi(s/2)}{(2\pi)^s}=\frac{(-1)^{k/2}}{2^{k+1}\pi^{k+1/2}}2^s\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(k-\frac{1}{2}s\right)\varphi\left(k-\frac{1}{2}s\right).$$

For $\sigma=2k+\frac{1}{2}$, Mellin inverse transform shows

$$\frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i}(2\pi)^{-s}\Gamma(s)\varphi\left(\frac{1}{2}s\right)t^{-s}ds=\sum_{n=1}^{\infty}a(n)e^{-2\pi n^{1/2}t}, \qquad (t>0),$$

where the order of $\sum$ and $\int$ can be changed by the absolute convergence. Further by the shift integration, we have

$$\frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i}(2\pi)^{-s}\Gamma(s)\varphi\left(\frac{1}{2}s\right)t^{-s}ds$$
$$=\frac{1}{2\pi i}\int_{\sigma'-\infty i}^{\sigma'+\infty i}(2\pi)^{-s}\Gamma(s)\varphi\left(\frac{1}{2}s\right)t^{-s}ds+R$$

(with $R=\sum_{\sigma'<u<\sigma}\mathrm{Res}_{s=u}\left((2\pi)^{-s}\Gamma(s)\varphi(\frac{1}{2}s)t^{-s}\right)$)

$$=\frac{1}{2\pi i}\frac{(-1)^{k/2}}{2^{k+1}\pi^{k+1/2}}\int_{\sigma'-\infty i}^{\sigma'+\infty i}2^s\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(k-\frac{1}{2}s\right)\varphi\left(k-\frac{1}{2}s\right)t^{-s}ds$$
$$+R$$
$$=\frac{2(-1)^{k/2}}{2^{k+1}\pi^{k+1/2}}\frac{1}{2\pi i}\int_{k-\sigma'/2-\infty i}^{k-\sigma'/2+\infty i}2^{2k-2s}\Gamma\left(k+\frac{1}{2}-s\right)\Gamma(s)\varphi(s)t^{2s-2k}ds$$
$$+R$$

for some σ'. We take $k-\frac{1}{2}\sigma'>k$, i.e., $\sigma'<0$. Then the above integral is equal to

$$\frac{2^{2k+1}(-1)^{k/2}}{2^{k+1}\pi^{k+1/2}t^{2k}}\sum_{n=1}^{\infty}a(n)\frac{1}{2\pi i}\int_{k+\sigma'/2-\infty i}^{k+\sigma'/2+\infty i}\Gamma\left(k+\frac{1}{2}-s\right)\Gamma(s)(4nt^{-2})^{-s}ds$$
$$+R.$$

Further we take $-1<\sigma'$. Then all poles of $\Gamma(s)$ lie in the left of the line $\mathrm{Re}\, s=\sigma'$ and all poles of $\Gamma(k+\frac{1}{2}-s)$ lie in the right of the line. Hence we can apply (1.2) to get

the above integral $=t2^k(-1)^{k/2}\Gamma(k+\frac{1}{2})/\pi^{k+1/2}\sum_{n=1}^{\infty}(a(n)/(4n+t^2)^{k+1/2})+R$.

There remains the computation of R. Since only pole of $\varphi(\frac{1}{2}s)$ is at $s=2k$ and thus poles of $(2\pi)^{-s}\Gamma(s)\varphi(\frac{1}{2}s)$ are at $s=2k$, $s=0$, between σ'

and σ, we have

$$R=\operatorname*{Res}_{s=0}\left((2\pi)^{-s}\Gamma(s)\varphi\left(\frac{1}{2}s\right)t^{-s}\right)+\operatorname*{Res}_{s=2k}\left((2\pi)^{-s}\Gamma(s)\varphi\left(\frac{1}{2}s\right)t^{-s}\right)$$

$$=-a(0)+t^{-2k}a(0)\frac{2^k(-1)^{k/2}}{\pi^{k+1/2}}\Gamma\left(k+\frac{1}{2}\right)$$

by (B), (ii) and the information of poles of gamma-function. Summing up, we get

$$\sum_{n=1}^{\infty}a(n)e^{-2\pi n^{1/2}t}=\frac{t2^k(-1)^{k/2}\Gamma(k+\frac{1}{2})}{\pi^{k+1/2}}\sum_{n=1}^{\infty}\frac{a(n)}{(4n+t^2)^{k+1/2}} \tag{1.3}$$

$$+\frac{2^k(-1)^{k/2}\Gamma(k+\frac{1}{2})}{\pi^{k+1/2}}\frac{a(0)}{t^{2k}}-a(0)$$

and

$$\sum_{n=0}^{\infty}a(n)e^{-2\pi n^{1/2}t}=t^{-2k}\sum_{n=0}^{\infty}\frac{b(n)}{(4nt^{-2}+1)^{k+1/2}}, \qquad t>0,$$

which can be continued analytically to H. Changing t by $-iz$, $z\in H$, we get (C).

§2. A proof of (C) $\Rightarrow$ (B).

The method employed here is almost the same as that of (B)$\Rightarrow$(A) used by Hecke. We have to use

$$\Gamma(s)\Gamma(\beta-s)=\int_0^{\infty}\Gamma(\beta)(1+x)^{-\beta}x^{s-1}dx \tag{2.1}$$

$$\text{for}\quad 0<\operatorname{Re}s<\beta\quad(\beta>0).$$

We have, for $\operatorname{Re}s>2k$,

$$\int_0^{\infty}(g(it)-a(0))t^{s-1}dt=\sum_{n=1}^{\infty}a(n)\int_0^{\infty}e^{-2\pi n^{1/2}t}t^{s-1}dt$$

$$=\sum_{n=1}^{\infty}\frac{\Gamma(s)a(n)}{(2\pi\sqrt{n})^s}$$

$$=(2\pi)^{-s}\Gamma(s)\varphi\left(\frac{1}{2}s\right).$$

Further we have

$$\int_0^{\infty}(g(it)-a(0))t^{s-1}dt \tag{2.2}$$

$$=\int_1^{\infty}(g(it)-a(0))t^{s-1}dt+\int_0^{1}(g(it)-a(0))t^{s-1}dt$$

$$=\int_1^\infty (g(it)-a(0))t^{s-1}dt+\int_0^1 g(it)t^{s-1}dt-a(0)\int_0^1 t^{s-1}dt$$
$$=\int_1^\infty (g(it)-a(0))t^{s-1}dt+\int_0^1 t^{-2k}h(i/t)t^{s-1}dt-\frac{a(0)}{s}$$
$$=\int_1^\infty (g(it)-a(0))t^{s-1}dt+\int_1^\infty h(it)t^{2k-s-1}dt-\frac{a(0)}{s}$$
$$=\int_1^\infty (g(it)-a(0))t^{s-1}dt+\int_1^\infty (h(it)-b(0))t^{2k-s-1}dt-\frac{a(0)}{s}-\frac{b(0)}{2k-s}\ .$$

Next, we compute

$$\frac{(-1)^{k/2}}{2^{k+1}\pi^{k+1/2}}2^s\Gamma\Big(\frac{s+1}{2}\Big)\Gamma\Big(k-\frac{1}{2}-s\Big)\varphi\Big(k-\frac{1}{2}s\Big)$$

which becomes

$$\frac{(-1)^{k/2}2^{k-1}}{2^{k+1}\pi^{k+1/2}}2^{-2s}\Gamma\Big(k+\frac{1}{2}-s\Big)\Gamma(s)\varphi(s)$$

after changing s to $2k-2s$.

We have

$$\int_0^\infty (h(it)-b(0))t^{2s-1}dt$$
$$=2\sum_{n=1}^\infty \frac{b(n)}{\Gamma(k+\frac{1}{2})}\int_0^\infty \frac{\Gamma(k+\frac{1}{2})}{(4nt^2+1)^{k+1/2}}t^{2s-1}dt$$
$$=2\sum_{n=1}^\infty \frac{b(n)4^{-1-s}}{\Gamma(k+\frac{1}{2})n^s}\int_0^\infty \frac{\Gamma(k+\frac{1}{2})}{(1+u)^{k+1/2}}u^{s-1}du$$

with $u=4nt^2$ and $k<\mathrm{Re}\, s$. For $\mathrm{Re}\, s<k+\frac{1}{2}$, we can apply (2.1) to get

$$\text{the above integral}=\sum_{n=1}^\infty \frac{b(n)4^{-1-s}}{\Gamma(k+\frac{1}{2})n^s}\Gamma\Big(k+\frac{1}{2}-s\Big)\Gamma(s)\ .$$

Thus

$$(2.3)\qquad \frac{2^{k-1}(-1)^{k/2}}{\pi^{k+1/2}}2^{-2s}\Gamma\Big(k+\frac{1}{2}-s\Big)\Gamma(s)\varphi(s)$$
$$=\int_0^\infty (h(it)-b(0))t^{2s-1}dt\ ,\qquad \text{for}\quad k<\mathrm{Re}\, s<k+\frac{1}{2}\ .$$

Further, we have, by shift integration as employed in §1,

$$(2.4)\qquad \int_0^\infty (h(it)-b(0))t^{2s-1}dt$$

$$=\int_1^\infty (h(it)-b(0))t^{2s-1}dt+\int_1^\infty (g(it)-a(0))t^{2k-2s-1}dt$$
$$-\frac{a(0)}{2(k-s)}-\frac{b(0)}{2s}\,.$$

Replacing s by $k-\frac{1}{2}s$, we have by (2.3), (2.4),

$$\frac{(-1)^{k/2}2^s}{2^{k+1}\pi^{k+1/2}}\Gamma\Big(\frac{s+1}{2}\Big)\Gamma\Big(k-\frac{1}{2}s\Big)\varphi\Big(k-\frac{1}{2}s\Big)$$
$$=\int_1^\infty (h(it)-b(0))t^{2k-2s-1}dt+\int_1^\infty (g(it)-a(0))t^{s-1}dt$$
$$-\frac{a(0)}{s}-\frac{b(0)}{2k-s}$$

which is valid for all s. We compare this with (2.2) and rewrite it by Legendre formula (1.1) to get (B), (iii).

§3. Going from (A) to (C).

We proved (B)$\Leftrightarrow$(C). Hence by Hecke's result, we get

$$(A)\Longleftrightarrow(B)\Longleftrightarrow(C)\,.$$

But, we can go from (A) to (C) directly and more easily. Its basis is the integral

$$\int_0^\infty e^{-a^2y-b^2/y}\frac{dy}{\sqrt{y}}=\frac{\sqrt{\pi}}{a}e^{-2ab}\,,\qquad (a>0,\ b\geqq 0)\,. \tag{3.1}$$

Now (A), (ii) and (iii) for $\sigma=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and $z=iy$, show

$$\sum_{n=0}^\infty a(n)e^{-2\pi n/y}=(-1)^{k/2}y^k\sum_{n=0}^\infty a(n)e^{-2\pi ny}\,. \tag{3.2}$$

Multiply both hands of (3.2) by $y^{-1/2}e^{-2\pi t^2y}$, $t>0$, and take the integral of both hands with respect to y from 0 to ∞. Then we have

$$a(0)\int_0^\infty e^{-2\pi t^2y}y^{-1/2}dy+\sum_{n=1}^\infty a(n)\int_0^\infty e^{-2\pi n/y-2\pi t^2y}y^{-1/2}dy$$
$$=(-1)^{k/2}a(0)\int_0^\infty e^{-2\pi t^2y}y^{k-1/2}dy$$
$$+(-1)^{k/2}\sum_{n=1}^\infty a(n)\int_0^\infty e^{-2\pi ny-2\pi t^2y}y^{k-1/2}dy$$

where the order of $\sum$ and $\int$ are already changed by the absolute convergence. From this, we get by (3.1) and Euler's integral,

$$a(0)\frac{\Gamma(\frac{1}{2})}{(2\pi t^2)^{1/2}}+2^{-1/2}t^{-1}\sum_{n=1}^{\infty}a(n)e^{-4\pi n^{1/2}t}$$
$$=(-1)^{k/2}a(0)\frac{\Gamma(k+\frac{1}{2})}{(2\pi t^2)^{k+1/2}}+(-1)^{k/2}\sum_{n=1}^{\infty}a(n)\frac{\Gamma(k+\frac{1}{2})}{(2\pi n+2\pi t^2)^{k+1/2}} .$$

Changing t to $\frac{1}{2}t$ shows that this formnla becomes (1.3) and the going (A)$\Rightarrow$(C) finishes.

§4. Theta-series.

As an application of (A)$\Rightarrow$(C) in the preceding section, we can derive Kronecker's formula

(4.1) $$\frac{1}{e^{2\pi iz}-1}=-\frac{1}{2}+\frac{1}{2\pi iz}+\frac{1}{2\pi i}\sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{z+n}\right)$$
for $z\notin \boldsymbol{Z}$

and

(4.2) $$\frac{e^{2\pi iuz}}{e^{2\pi iz}-1}=\frac{1}{2\pi iz}+\frac{1}{2\pi i}\sum_{n=1}^{\infty}\left(\frac{e^{-2\pi inu}}{z+n}+\frac{e^{2\pi inu}}{z-n}\right)$$
for $0<u<1,\ z\notin \boldsymbol{Z}$

from theta inversion formula

(4.3) $$\sum_{m=-\infty}^{\infty} e^{-\pi y(m+v)^2+2\pi imu}$$
$$=y^{-1/2}\sum_{m=-\infty}^{\infty} e^{-\pi y^{-1}(m-u)^2+2\pi i(m-u)v}, \qquad y>0 .$$

In fact, by (4.3) with $u=v=0$, we have

$$\sum_{m=1}^{\infty} e^{-\pi m^2 y}+\frac{1}{2}-\frac{1}{2}y^{-1/2}=y^{-1/2}\sum_{m=1}^{\infty} e^{-\pi m^2/y} .$$

Multiply both hands by $e^{-\pi yt^2}$ and integrate them with respect to y from 0 to ∞. Here note that in the exponent of e only squares of m appear. Then by (3.1) we have

$$\sum_{m=1}^{\infty} e^{-2\pi mt}=\frac{t}{\pi}\sum_{m=1}^{\infty}\frac{1}{m^2+t^2}+\frac{1}{2\pi t}-\frac{1}{2} .$$

From this follows

$$\frac{1}{e^{2\pi t}-1}=-\frac{1}{2}+\frac{1}{2\pi t}-\frac{1}{2\pi i}\sum_{m=1}^{\infty}\left(\frac{1}{ti-m}+\frac{1}{ti+m}\right)$$

which shows (4.1) by putting $t=iz$, $z\in C$, $z\notin Z$ and by the analytic continuation. A proof of (4.2) is almost the same but we need some device. We have by (4.3) with $v=0$, $0<u<1$,

$$1+\sum_{m=1}^{\infty}e^{-\pi ym^2+2\pi imu}+\sum_{m=1}^{\infty}e^{-\pi ym^2-2\pi imu}$$
$$=y^{-1/2}\left\{e^{-\pi m^2/y}+\sum_{m=1}^{\infty}e^{-\pi(m-u)^2/y}+\sum_{m=1}^{\infty}e^{-\pi(m+u)^2/y}\right\}$$

and by the same procedure as above,

$$\frac{e^{-2\pi ut}}{e^{2\pi t}-1}+\frac{e^{2\pi t}}{e^{2\pi t}-1}+e^{-2\pi ut} \tag{4.4}$$
$$=\frac{1}{\pi t}-\frac{1}{2\pi i}\sum_{m=1}^{\infty}e^{2\pi imu}\left(\frac{1}{ti+m}+\frac{1}{ti-m}\right)$$
$$-\frac{1}{2\pi i}\sum_{m=1}^{\infty}e^{2\pi imu}\left(\frac{1}{ti+m}+\frac{1}{ti-m}\right).$$

Then differentiate (4.4) with respect to u, sum up both and divide by 2. We have

$$\frac{e^{2\pi ut}}{e^{2\pi t}-1}=\frac{1}{2\pi t}-\frac{1}{2\pi i}\sum_{m=1}^{\infty}\left\{\frac{e^{2\pi imu}}{ti+m}+\frac{e^{-2\pi imu}}{ti-m}\right\}$$

which is analytically continued to (4.2) after putting $z=it$, $z\notin Z$.

§5. On Siegel's proof of Hamburger's theorem.

Hamburger's theorem asserts that under a certain condition of convergence, if

$$f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}, \qquad \operatorname{Re} s>1$$

and

$$g(1-s)=\sum_{n=1}^{\infty}\frac{b_n}{n^{1-s}}, \qquad \operatorname{Re} s<-\alpha<0,$$

satisfy the functional equation

$$f(s)\Gamma\left(\frac{1}{2}s\right)\pi^{-(1/2)s}=g(1-s)\Gamma\left(\frac{1}{2}(1-s)\right)\pi^{-(1/2)(1-s)},$$

then

$$f(s)=a_1\zeta(s) \qquad \text{(Riemann zeta-function)}.$$

Siegel gave an analytic proof based on the integral (4.1) in [3]. His proof goes on (B)$\Rightarrow$(A)$\Rightarrow$(C) and takes only 3 pages.

Now, we can go on (B)$\Rightarrow$(C) directly as in §1, abridge his proof at least a half page and thus get a slightly different proof of Hamburger's theorem.

Note that in Hamburger's theorem there is no assumption on poles in contrary to (A). But we can follow the same line of computation in §1, except the determination of $R=\sum_{\sigma'<u<\sigma}\mathrm{Res}_{s=u}(\Gamma(s)f(s)(2\pi)^{-s}t^{-s})$ which is nothing but Siegel's $Q(x)$ $(x=t)$. We may follow Siegel's proof after $Q(x)$.

References

[1] K. Chandrasekharan and R. Narasimhan, Heckes functional equation and arithmetical identities, Ann. of Math., vol. **74**, no. 1 (1961), 1-23.

[2] E. Hecke, Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung, Math. Ann., **112** (1936), 664-699, =Werke, 591-626.

[3] C. L. Siegel, Bemerkungen zu einem Satz von Hamburger über die Funktionalgleichung der Riemannschen Zetafunktion, Math. Ann., **86** (1922), 276-279, =Gesammelte Abhandlung Bd. I, 154-156.

[4] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, 4th ed., Cambridge Univ. Press, 1965.

Corrections to "On the values of Eisenstein series" in this Journal vol. 1, no. 1 (1978) by Koji Katayama.

p. 166 Theorem 4, (i): read $(2\pi i)^{\lambda}$ for $(2\pi i)$.
The right side of formula (3.1): read $\frac{1}{4}$ for $\frac{1}{2}$.

p. 179 Line 8: $g_4^{(0,0)}=\dfrac{1}{4!\cdot 10}$.

Line 12: $S_4=\dfrac{12}{5},\ 2^4H_4\left(\sqrt{-1};\dfrac{1}{2},\dfrac{1}{2}\right)=\dfrac{2^4\cdot 3}{5}$.

Line 15: $W_3\left(\sqrt{-1};\dfrac{1}{2},\dfrac{1}{2}\right)=\dfrac{3\pi^3\sqrt{-1}}{8}-\dfrac{3\sqrt{-1}}{10}\dfrac{\varpi_{(4)}^4}{\pi}$.

Line 17: The right side must be read as

$$-\frac{3}{64}+\frac{1}{120}+\frac{\varpi_{(4)}^4}{\pi^4}\frac{3}{640}\,.$$

p. 182 Line 6, 4 from bottom: read 3^2 for 3^3.
p. 184 Line 7, 9: read $e^{-2n\pi i\rho}-1$ for $e^{2n\pi i/\rho}-1$.
Line 7, 10, 11: read M_{6k} for M_k.
p. 185 In the formula of Theorem 11, read

$$\frac{(2^{2k-1}-1)E_{2k}-2^{2k}E_{2k}^{(1,1)}}{4k}$$

for

$$\frac{3\cdot 2^{2k-3}E_{2k}-4\cdot 2^{2k-3}E_{2k}^{(1,1)}}{k}\,.$$

Present Address:
DEPARTMENT OF MATHEMATICS
TSUDA COLLEGE
TSUDA-MACHI, KODAIRA, TOKYO 187