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A Local Isotopy Lemma

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Introduction

First of all, let us recall Thom's first isotopy lemma: *Let $\pi: E \to B$ be a proper differentiable and stratified map. Then for each stratum Z of B, the restricted map $\pi: \pi^{-1}(Z) \to Z$ is a locally trivial fibration* (see R. Thom [5], J. Mather [3]).

This lemma is very powerful in the topological studies of analytic sets (see T. Fukuda [1]), of Landau singularities, of Feynman integrals (see F. Pham [4]) and so on. However we meet many situations where the mapping $\pi: E \to B$ is not proper, and we can not apply the lemma to the situations.

For example, consider the function $I(z)$ $(z=(z_1, \cdots, z_4) \in C^4)$ defined by the integral

$$I(z)=\int_0^{z_1} (z_2-2z_3\tau+z_4\tau^2)^{-1} d\tau .$$

It is the solution of the Cauchy problem:

$$\left\{\left(\frac{\partial}{\partial z_1}\right)^2+2z_3\left(\frac{\partial}{\partial z_1}\right)\left(\frac{\partial}{\partial z_2}\right)+z_4\left(\frac{\partial}{\partial z_1}\right)\left(\frac{\partial}{\partial z_3}\right)\right\} I(z)=0$$

with initial data

$$I(0, z_2, z_3, z_4)=0 , \qquad \frac{\partial}{\partial z_1} I(0, z_2, z_3, z_4)=\frac{1}{z_2} .$$

Obviously $I(z)$ is holomorphic as long as the integral path with initial point $\tau=0$ and with terminal point $\tau=z_1$ can be continuously deformed, escaping the singularities of the integrand, $z_2-2z_3\tau+z_4\tau^2=0$. However,

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since the canonical projection $\pi: C_\tau \times C_z^4 \to C_z^4$ is not proper, we cannot apply Thom's isotopy lemma directly.

The purpose of the present note is to give a local and relative version of the lemma which works on the cases where π's are not necessarily proper. One important application of it to the Cauchy problem with singular data in the complex domain will appear in T. Kobayashi [2]. In the above example, applying the local isotopy lemma, we can show that $I(z)$ is holomorphically continued along any path in $C^4\backslash(\Sigma_0 \cup \Sigma_\infty)$, where Σ_0 and Σ_∞ are given by

$$\Sigma_0: z_2(z_2 - 2z_3z_1 + z_4z_1^2)(z_2z_4 - z_3^2) = 0 ,$$
$$\Sigma_\infty: z_4 = 0 ,$$

(see [2]).

§1. The results.

Let $\pi: E \to B$ be a continuous map, $A_1, \cdots, A_k$ subsets of E and Z a subset of B. We say that the restricted map $\pi: \pi^{-1}(Z) \to Z$ is *a locally trivial fibration relative to* $A_1, \cdots, A_k$ if for any point z of Z there exists an open neighborhood U of z in Z and a relative homeomorphism $H: (\pi^{-1}(U), A_1 \cap \pi^{-1}(U), \cdots, A_k \cap \pi^{-1}(U)) \to (\pi^{-1}(z) \times U, (\pi^{-1}(z) \cap A_1) \times U, \cdots, (\pi^{-1}(z) \cap A_k) \times U)$ such that the following diagram commutes

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "H"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowleft"] & \pi^{-1}(z) \times U \arrow[dl, "\pi'"] \\
& U &
\end{tikzcd}$$

where π' is the canonical projection.

From now on, consider the following situation. Let $E = T \times W$ with paracompact connected manifolds T and W, $\pi: E \to W$ the canonical projection and $S_1, \cdots, S_k$ subsets of E. We assume that there exist Whitney stratifications $\mathscr{S}_1 = \mathscr{S}_1(E)$ and $\mathscr{S}_2 = \mathscr{S}_2(W)$ of at most a countable number of strata such that

i) *π is a stratified map with respect to $\mathscr{S}_1$ and $\mathscr{S}_2$,*

ii) *$S_1, \cdots, S_k$ are stratified subsets of E.*

REMARK. For example, when $E = C^m \times C^n$ and $S_1, \cdots, S_k$ are complex algebraic subsets of $C^m \times C^n$, there exist stratifications $\mathscr{S}_1$ and $\mathscr{S}_2$ satisfying i), ii) and

iii) *strata of* $\mathscr{S}_1$ *and* $\mathscr{S}_2$ *are constructible and of a finite number* (see [1]).

Let $\rho_0: T \to \boldsymbol{R}$ be a smooth positive-valued and proper function and set $\overline{B(r)} = \{t \in T; \rho_0(t) \leqq r\}$ (which is compact because ρ_0 is proper), $B(r) = \{t \in T; \rho_0(t) < r\}$ and $S(r) = \overline{B(r)} \backslash B(r)$ for a positive number $r > \min \rho_0(t)$.

REMARK. A manifold which is countable at infinity, hence T, admits a smooth positive-valued and proper function.

DEFINITION. Let $\rho = \rho_0 \circ \pi_T: E \to \boldsymbol{R}$ where $\pi_T: E \to T$ is the canonical projection. For a point $w \in W$ and a stratum $X \in \mathscr{S}_1$, let $D_X(w)$ denote the set of all critical values of $\rho|_{X \cap \{T \times w\}}$ ($\rho(X \cap \{T \times w\})$ are also critical values when $\dim X \cap \{T \times w\} = 0$). We set $D(w) = \bigcup_{X \in \mathscr{S}_1} D_X(w)$.

REMARK. By Sard's theorem, $D_X(w)$ is measure zero, hence so is $D(w)$.

Now we can state our results.

THEOREM. *For any stratum* $Z \in \mathscr{S}_2(W)$, *any point* z° *of* Z *and for almost every positive number* r *except those of* $D(z^\circ)$, *there is an open neighborhood* $U = U(z^\circ, r)$ *of* z° *in* Z *such that the canonical projection* $\pi: \overline{B(r)} \times U \to U$ *is a stratified map with respect to the stratifications* $\mathscr{S}_1(\overline{B(r)} \times U)$ *and* $\mathscr{S}_2(U)$ *given by*

1) $\mathscr{S}_2(U) = \{U\}$,

2) $\mathscr{S}_1(\overline{B(r)} \times U) = \{X \cap (B(r) \times U), X \cap (S(r) \times U); X \in \mathscr{S}_1(E)\}$,

and $S_1 \cap (\overline{B(r)} \times U), \cdots, S_k \cap (\overline{B(r)} \times U)$ *are stratified subsets of* $\overline{B(r)} \times U$.

COROLLARY. $\pi: \overline{B(r)} \times U \to U$ *is a locally trivial fibration relative to* $S_1 \cap (\overline{B(r)} \times U), \cdots, S_k \cap (\overline{B(r)} \times U)$.

PROOF OF COROLLARY. Since $\pi: \overline{B(r)} \times U \to U$ is a proper map, we can apply Thom's relative isotopy lemma (see [1]) to the sequence.

§2. Proof of theorem.

Set $D = \bigcup_{w \in W} \{w\} \times D(w) \subset W \times \boldsymbol{R}$.

LEMMA 1. *D is a closed subset of* $W \times \boldsymbol{R}$.

PROOF. To prove the lemma, it is enough to show that for any convergent sequence $(w_n, r_n) \in D$, the limit point $(w, r) = \lim_{n \to \infty} (w_n, r_n)$ also belongs to D. Since r_n is a critical value of $\rho|_{X \cap \{T \times w_n\}}$ for some stratum $X \in \mathscr{S}_1$, there is a critical point $c_n = (t_n, w_n) \in X$ of $\rho|_{X \cap \{T \times w_n\}}$ with

$\rho(c_n)=r_n$. By choosing a suitable subsequence, we may assume that the sequence $c_n=(t_n, w_n)$ converges to some point $c=(t, w)$, because for sufficiently large number n, c_n belongs to a compact set, say $\overline{B(2r)}\times\bar{V}$ where $\bar{V}$ is a compact neighborhood of w. Since $\rho(c)=\lim\rho(c_n)=\lim r_n=r$, it is enough to see that $c=(t, w)$ is a critical point of $\rho|_{Y\cap\{T\times w\}}$, where $Y\in\mathscr{S}_1$ is the stratum containing c. Note that c is a critical point of $\rho|_{Y\cap\{T\times w\}}$ if and only if $\ker d\rho_c\supset T_c(Y\cap\{T\times w\})$, where $d\rho_c\colon T_c(E)\to T_r(\boldsymbol{R})$ is the differential of ρ at c and $T_p(M)$ is the tangent space at a point p to a smooth manifold M. Since the stratification $\mathscr{S}_1(E)$ is locally finite, by choosing a suitable subsequence we may assume that the critical points c_n belong to the same stratum $X\in\mathscr{S}_1(E)$ for all n. There are two cases to see;

a) $X=Y$,

b) $X\succ Y$, i.e., $Y\subset(\bar{X}\backslash X)$.

Case a). Since $\lim_{n\to\infty}\ker d\rho_{c_n}\subset\ker d\rho_c$ and $\lim_{n\to\infty}T_{c_n}(X\cap\{T\times w_n\})=T_c(X\cap\{T\times w\})$, we have $\ker d\rho_c\supset T_c(X\cap\{T\times w\})$. Therefore c is a critical point of $\rho|_{X\cap\{T\times w\}}$.

Case b). Since $\lim_{n\to\infty}\ker d\rho_{c_n}\subset\ker d\rho_c$ and $\lim_{n\to\infty}T_{c_n}(X\cap\{T\times w_n\})\supset T_c(Y\cap\{T\times w\})$ from the Whitney condition (a) for the pair (X, Y), we have $\ker d\rho_c\supset T_c(Y\cap\{T\times w\})$. Hence c is a critical point of $\rho|_{Y\cap\{T\times w\}}$.

Q.E.D. of Lemma 1.

Let Z be a stratum of $\mathscr{S}_2$, z° a point of Z and $r\notin D(z^\circ)$. Then from Lemma 1, there are an open neighborhood $U=U(z^\circ, r)$ of z° in Z and a positive number $\varepsilon>0$ such that $U\times(r-\varepsilon, r+\varepsilon)\cap D=\emptyset$. Set $\mathscr{S}_2(U)=\{U\}$ and $\mathscr{S}_1(\overline{B(r)}\times U)=\{X\cap(B(r)\times U),\ X\cap(S(r)\times U);\ X\in\mathscr{S}_1(E)\}$. We will prove the theorem dividing it into two lemmas:

LEMMA 2. 1). *$\mathscr{S}_2(U)$ is a Whitney stratification.*

2). *$\mathscr{S}_1(\overline{B(r)}\times U)$ is a Whitney stratification of $\overline{B(r)}\times U$ such that $S_i\cap(\overline{B(r)}\times U)$, $i=1, \cdots, k$, are stratified subsets of $\overline{B(r)}\times U$.*

LEMMA 3. *$\pi\colon\overline{B(r)}\times U\to U$ is a stratified map with respect to $\mathscr{S}_1(\overline{B(r)}\times U)$ and $\mathscr{S}_2(U)$.*

PROOF OF LEMMA 2. 1). Trivial.

2). *Elements of $\mathscr{S}_1(\overline{B(r)}\times U)$ are smooth manifolds*: Each element of $\mathscr{S}_1(\overline{B(r)}\times U)$ is of the form

$$X_B=X\cap(B(r)\times U)\ ,$$

which is obviously a manifold, or of the form

$$X_S = X \cap (S(r) \times U) ,$$

which is also a manifold because X and $S(r) \times U$ intersect transversally. (Remember that r is not a critical value of $\rho|_{X \cap \{T \times z\}}$ for any $z \in U = U(z^\circ, r)$.)

$\mathscr{S}_1(\overline{B(r)} \times U)$ *satisfies the frontier condition*: Since $Y_B \cap (\bar{X}_S \backslash X_S) = \varnothing$ for any strata X, $Y \in \mathscr{S}_1(E)$, it is enough to see the following three cases:

i) $Y_B \cap (\bar{X}_B \backslash X_B) \neq \varnothing$,

ii) $Y_S \cap (\bar{X}_B \backslash X_B) \neq \varnothing$,

iii) $Y_S \cap (\bar{X}_S \backslash X_S) \neq \varnothing$.

In any case, it is easily follows from the frontier condition of $\mathscr{S}_1(E)$ that $Y_\cdot \subset (\bar{X}_* \backslash X_*)$. Incidentally we can see that a pair $(\tilde{X}, \tilde{Y})$ of $\mathscr{S}_1(\overline{B(r)} \times U)$ is $\tilde{X} \succ \tilde{Y}$ only if $(\tilde{X}, \tilde{Y})$ is one of the following:

a) $\tilde{X} = X_B$, $\tilde{Y} = X_S$ for the same $X \in \mathscr{S}_1(E)$.

b) $\tilde{X} = X_B$, $\tilde{Y} = Y_B$ with $X \succ Y$.

c) $\tilde{X} = X_B$, $\tilde{Y} = Y_S$ with $X \succ Y$.

d) $\tilde{X} = X_S$, $\tilde{Y} = Y_S$ with $X \succ Y$.

Every pair $(\tilde{X}, \tilde{Y})$ *of elements of* $\mathscr{S}_1(\overline{B(r)} \times U)$ *satisfies the Whitney conditions* (a) *and* (b): There are four cases a), b), c) and d) above to check. In any case, one can see easily that $(\tilde{X}, \tilde{Y})$ satisfies the Whitney conditions.

$S_i \cap (\overline{B(r)} \times U)$, $i = 1, \cdots, k$, *are stratified subsets*: Trivial.

Q.E.D. of Lemma 2.

PROOF OF LEMMA 3. It is enough to prove that $\pi|_{\tilde{X}}: \tilde{X} \to U$ is a submersion for each stratum $\tilde{X} \in \mathscr{S}_1(\overline{B(r)} \times U)$.

If $\tilde{X}$ is of the form $\tilde{X} = X \cap (B(r) \times U)$, then $\pi|_{\tilde{X}}: \tilde{X} \to U$ is a submersion, for $\pi|_X: X \to Z$ is a submersion.

If X is of the form $\tilde{X} = X \cap (S(r) \times U)$, to see that $\pi|_{\tilde{X}}: \tilde{X} \to U$ is a submersion it is enough to see that $\pi \times \rho: X \to Z \times \boldsymbol{R}$, defined by $\pi \times \rho((t, w)) = (w, \rho(t, w)) = (w, \rho_0(t))$, is submersive. But this is obvious from the form of $\pi \times \rho$ and from the fact that $\pi|_X: X \to Z$ is a submersion and r is not a critical value of the function $\rho|_{X \cap \{T \times z\}}$. Q.E.D. of Lemma 3.

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