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A Characterization of Homogeneous Self-dual Cones

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Introduction

A *convex cone* V in the n-dimensional real number space $\boldsymbol{R}^n$ is a non-empty open subset of $\boldsymbol{R}^n$ satisfying the following conditions:

(1) If $x \in V$ and $\lambda \in \boldsymbol{R}$ with $\lambda > 0$, then $\lambda x \in V$.

(2) If $x, y \in V$, then $x+y \in V$.

(3) V contains no full straight line.

We denote by $G(V)$ the group of all linear automorphisms of V, that is,

$$G(V)=\{A \in GL(n);\ AV=V\} .$$

If the group $G(V)$ acts transitively on V, then V is called *homogeneous*. Let $\langle\ ,\ \rangle$ be an inner product in $\boldsymbol{R}^n$. Then the *dual cone* V^* of V with respect to the inner product $\langle\ ,\ \rangle$ is defined by

$$V^*=\{y \in \boldsymbol{R}^n;\ \langle x, y\rangle>0 \text{ for every } x \in \bar{V}-(0)\} ,$$

where $\bar{V}$ is the topological closure of V in $\boldsymbol{R}^n$. A cone V is called *self-dual* if the dual cone V^* of V with respect to a suitable inner product coincides with V. The *characteristic function* φ_V of V is defined on V by

$$\varphi_V(x)=\int_{V^*} \exp -\langle x, y\rangle dy ,$$

where dy is a canonical Euclidean measure on $\boldsymbol{R}^n$. The characteristic function of a homogeneous convex cone V is determined uniquely up to a constant factor by the following property:

$$\varphi_V(Ax)=\varphi_V(x)/|\det A|$$

for every $x \in V$, $A \in G(V)$. Let us take a system of linear coordinates $(x_1, x_2, \cdots, x_n)$ of $\boldsymbol{R}^n$. Then using the characteristic function we can define a $G(V)$-invariant Riemannian metric g_V on V by

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$$g_V=\sum_{i,j}\frac{\partial^2\log\varphi_V}{\partial x_i\partial x_j}dx_i dx_j .$$

This Riemannian metric g_V is called the *canonical metric* of V (cf. Vinberg [6]).

The theory of homogeneous convex cones has been developed mainly by M. Koecher, E. B. Vinberg, O. S. Rothaus and others (cf. e.g., [2], [6], [5]). It is known that every homogeneous self-dual cone is a Riemannian symmetric space with respect to the canonical metric (cf. [2], [4]). In 1965, Y. Matsushima raised the question whether every Riemannian symmetric homogeneous convex cone is self-dual or not. The purpose of the present paper is to give an affirmative answer to the above Matsushima's problem. Our main result is stated as follows:

THEOREM.*) *If a homogeneous convex cone V is Riemannian symmetric with respect to the canonical metric, then V is self-dual.*

In proving the above theorem, the notion of T-algebras due to Vinberg [6] plays an important role. In § 1, we will recall a fundamental correspondence between homogeneous convex cones and T-algebras. In § 2, by making use of the theory of invariant connections due to Nomizu [3] and the results in § 1, we will calculate the Riemannian connection of the canonical metric (Lemma 2.2) and some of the covariant derivatives of the curvature tensor. And we will give necessary conditions for a homogeneous convex cone to be Riemannian symmetric (Lemmas 2.3 and 2.6). Using the results obtained in § 2, we will prove in § 3 the main theorem (Theorem 3.2) and give an application (Corollary 3.3).

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§ 1. Homogeneous convex cones and T-algebras.

In this section we will recall some of fundamental results on homogeneous convex cones due to Vinberg [6].

1.1. It is known that there exists a natural bijection between the set of all isomorphism classes of homogeneous convex cones and the set of all isomorphism classes of T-algebras. For later use, we will review this bijection.

We begin with some definitions on T-algebras. Let r be a positive

*) H. Shima obtained the same result independently [8].

integer and $\mathfrak{A}$ a finite dimensional algebra over the field $\boldsymbol{R}$ of real numbers. Then $\mathfrak{A}$ is called a *matrix algebra of rank* r if $\mathfrak{A}$ is bigraded with subspaces $\mathfrak{A}_{ij}$ $(1\leqq i,\ j\leqq r)$ such that

$$(1.1)\qquad \mathfrak{A}_{ij}\mathfrak{A}_{kl}\subset\delta_{jk}\mathfrak{A}_{il}\quad\text{and}\quad \mathfrak{A}_{ii}\neq(0)\qquad(1\leqq i,\ j,\ k,\ l\leqq r)\ .$$

An *involution* of a matrix algebra $\mathfrak{A}$ is an involutive anti-automorphism $*$ of $\mathfrak{A}$ such that $\mathfrak{A}_{ij}^*=\mathfrak{A}_{ji}$ $(1\leqq i,\ j\leqq r)$. We will employ the following notations:

$$[ab]=ab-ba\ ,\qquad [abc]=a(bc)-(ab)c\quad\text{for every}\quad a,\ b,\ c\in\mathfrak{A}\ .$$

$$n_{ij}=\dim\mathfrak{A}_{ij}\ ,\qquad n_i=1+\frac{1}{2}\sum_{k<i}n_{ki}+\frac{1}{2}\sum_{i<k}n_{ik}\ .$$

a_{ij} denotes an arbitrary element of $\mathfrak{A}_{ij}$; also we write $a=(a_{ij})$, where a_{ij} is the $\mathfrak{A}_{ij}$-component of an element $a\in\mathfrak{A}$.

DEFINITION. A matrix algebra $\mathfrak{A}=\sum_{1\leqq i,j\leqq r}\mathfrak{A}_{ij}$ of rank r with an involution $*$ is called a *T-algebra* if the following axioms (T.1)-(T.7) are satisfied:

(T.1) Each subalgebra $\mathfrak{A}_{ii}$ $(1\leqq i\leqq r)$ is isomorphic to the algebra $\boldsymbol{R}$ of real numbers under an isomorphism ρ.

(T.2) $a_{ii}a_{ij}=\rho(a_{ii})a_{ij}$, $a_{ij}a_{jj}=\rho(a_{jj})a_{ij}$ $(1\leqq i,\ j\leqq r)$.

(T.3) $\mathrm{Sp}\,[ab]=0$ for every $a,\ b\in\mathfrak{A}$, where Sp is the *trace* of an element $a=(a_{ij})$ defined by $\mathrm{Sp}\,a=\sum_{1\leqq i\leqq r}n_i\rho(a_{ii})$.

(T.4) $\mathrm{Sp}\,[abc]=0$ for every $a,\ b,\ c\in\mathfrak{A}$.

(T.5) $\mathrm{Sp}\,aa^*>0$ for every $a\neq0\in\mathfrak{A}$.

(T.6) $[abc]=0$ for every $a,\ b,\ c\in\sum_{1\leqq i\leqq j\leqq r}\mathfrak{A}_{ij}$.

(T.7) $[abb^*]=0$ for every $a,\ b\in\sum_{1\leqq i\leqq j\leqq r}\mathfrak{A}_{ij}$.

In what follows, we will identify an element of $\mathfrak{A}_{ii}$ with a real number under the identification ρ. We put

$$T(\mathfrak{A})=\{t=(t_{ij});\ t_{ii}>0\ \text{for}\ 1\leqq i\leqq r,\ t_{ij}=0\ \text{for}\ 1\leqq j<i\leqq r\}\ ,$$
$$V(\mathfrak{A})=\{tt^*;\ t\in T(\mathfrak{A})\}\quad\text{and}\quad X(\mathfrak{A})=\{a\in\mathfrak{A};\ a^*=a\}\ .$$

Then the following results are known in Vinberg [6]: *For any T-algebra $\mathfrak{A}$, the set $V(\mathfrak{A})$ is a homogeneous convex cone in the real vector space $X(\mathfrak{A})$. The set $T(\mathfrak{A})$ is a connected Lie group which acts linearly and simply transitively on $V(\mathfrak{A})$. Conversely, every homogeneous convex cone is realized in this form and this correspondence induces naturally a bijection between the set of all isomorphism classes of homogeneous convex cones and the set of all isomorphism classes of T-algebras.*

1.2. Let $\mathfrak{A}$ be a T-algebra of rank r and $V(\mathfrak{A})$ the homogeneous convex cone which corresponds to $\mathfrak{A}$. Then we define the *rank* of $V(\mathfrak{A})$ by r the rank of the T-algebra $\mathfrak{A}$. We denote by $\mathfrak{t}(\mathfrak{A})$ the Lie algebra of the Lie group $T(\mathfrak{A})$. Then the bracket operation [,] in $\mathfrak{t}(\mathfrak{A})$ is given as follows:

$$[a, b]=[ab]=ab-ba \tag{1.2}$$

for every $a, b \in \mathfrak{t}(\mathfrak{A})$ (cf. the formula (20) in p. 383 of [6]). We put $e=(e_{ij})$ the unit element of $\mathfrak{A}$, that is,

$$e_{ii}=1 \text{ for } 1\leqq i\leqq r \quad\text{and}\quad e_{ij}=0 \text{ for } 1\leqq i\neq j\leqq r\ . \tag{1.3}$$

Then the tangent space of $V(\mathfrak{A})$ at the point e can be identified naturally with the ambient space $X(\mathfrak{A})$. Furthermore, there exists a natural linear isomorphism ξ from the Lie algebra $\mathfrak{t}(\mathfrak{A})$ onto the real vector space $X(\mathfrak{A})$ as follows:

$$\xi\colon t\in\mathfrak{t}(\mathfrak{A})\longmapsto t+t^*\in X(\mathfrak{A})\ . \tag{1.4}$$

By the formula (34) in p. 389 of [6], the canonical metric $g_{V(\mathfrak{A})}$ at the point e is given by

$$g_{V(\mathfrak{A})}(a, b)=\mathrm{Sp}\, ab$$

for every $a, b\in X(\mathfrak{A})$. By using $g_{V(\mathfrak{A})}$ and the isomorphism ξ, the canonical inner product can be defined on the Lie algebra $\mathfrak{t}(\mathfrak{A})$, which is denoted by $\langle\ ,\ \rangle$. Then by (1.4) we have

$$\langle a, b\rangle=\mathrm{Sp}\,(a+a^*)(b+b^*) \tag{1.5}$$

for every $a, b\in\mathfrak{t}(\mathfrak{A})$.

It is easy to see that with respect to this inner product $\langle\ ,\ \rangle$, the Lie algebra $\mathfrak{t}(\mathfrak{A})$ is an orthogonal direct sum of subspaces $\mathfrak{A}_{ij}$ ($1\leqq i\leqq j\leqq r$): $\mathfrak{t}(\mathfrak{A})=\sum_{1\leqq i\leqq j\leqq r}\mathfrak{A}_{ij}$; and the bracket operation [,] in $\mathfrak{t}(\mathfrak{A})$ satisfies the following relation:

$$[\mathfrak{A}_{ij}, \mathfrak{A}_{kl}]\subset\delta_{jk}\mathfrak{A}_{il}+\delta_{il}\mathfrak{A}_{kj}\quad(1\leqq i, j, k, l\leqq r)\ . \tag{1.6}$$

By the formula (A) in p. 393 of [6], the following equality holds:

$$\langle a_{ij}b_{jk}, a'_{ij}b'_{jk}\rangle+\langle a_{ij}b'_{jk}, a'_{ij}b_{jk}\rangle=(1/n_j)\langle a_{ij}, a'_{ij}\rangle\langle b_{jk}, b'_{jk}\rangle \tag{1.7}$$

for every $a_{ij}, a'_{ij}\in\mathfrak{A}_{ij}$, $b_{jk}, b'_{jk}\in\mathfrak{A}_{jk}$ with $1\leqq i<j<k\leqq r$. From (1.7) it follows that for $i<j<k$ with $n_{ij}n_{jk}\neq 0$, the inequality

$$\max(n_{ij}, n_{jk})\leqq n_{ik} \tag{1.8}$$

holds.

1.3. For each i with $1 \leqq i \leqq r$, we put

$$e_i = (1/2\sqrt{n_i})e_{ii} . \tag{1.9}$$

Then by (1.3) and (1.5) we have $\langle e_i, e_i \rangle = 1$.

For each pair (i, j) with $i<j$ and $n_{ij} \neq 0$, we take an orthonormal basis $\{e_{ij}^{\lambda}\}_{1 \leq \lambda \leq n_{ij}}$ of the subspace $\mathfrak{A}_{ij}$. If $n_{ij}n_{jk} \neq 0$ and $1 \leqq i<j<k \leqq r$, then we define a system of linear operators $\{T_\lambda\}_{1 \leq \lambda \leq n_{ij}}$ by

$$T_\lambda : \mathfrak{A}_{jk} \longrightarrow \mathfrak{A}_{ik} , \qquad T_\lambda a_{jk} = e_{ij}^{\lambda} a_{jk} \tag{1.10}$$

for every $a_{jk} \in \mathfrak{A}_{jk}$. Then from the condition (1.7) it follows that the system $\{T_\lambda\}_{1 \leq \lambda \leq n_{ij}}$ satisfies the following conditions:

$${}^tT_\lambda T_\mu + {}^tT_\mu T_\lambda = (1/n_j)\delta_{\lambda\mu} I_{jk} , \tag{1.11}$$

where I_{jk} is the identity operator on $\mathfrak{A}_{jk}$.

LEMMA 1.1. *The bracket relations in the Lie algebra* $\mathfrak{t}(\mathfrak{A})$ *are given as follows*:

(1) *The case of* $n_{ij}n_{jk} \neq 0$ $(i<j<k)$.

$[e_i, a_{ij}] = -[a_{ij}, e_i] = (1/2\sqrt{n_i})a_{ij}$, $[e_i, a_{ik}] = -[a_{ik}, e_i] = (1/2\sqrt{n_i})a_{ik}$,

$[e_j, a_{ij}] = -[a_{ij}, e_j] = -(1/2\sqrt{n_j})a_{ij}$, $[e_j, a_{jk}] = -[a_{jk}, e_j] = (1/2\sqrt{n_j})a_{jk}$,

$[e_k, a_{ik}] = -[a_{ik}, e_k] = -(1/2\sqrt{n_k})a_{ik}$, $[e_k, a_{jk}] = -[a_{jk}, e_k] = -(1/2\sqrt{n_k})a_{jk}$,

$[e_{ij}^{\lambda}, a_{jk}] = -[a_{jk}, e_{ij}^{\lambda}] = T_\lambda a_{jk}$ *for* $1 \leqq \lambda \leqq n_{ij}$ *and all other bracket relations between elements in* $\mathfrak{A}_{\alpha\beta}$ $(\alpha, \beta \in \{i, j, k\})$ *are zero.*

(2) *The case of* $n_{ij}n_{ik} \neq 0$, $n_{jk}=0$ $(i<j<k)$ *or* $n_{jk}n_{ik} \neq 0$, $n_{ij}=0$ $(i<j<k)$.

$[e_i, a_{ij}] = -[a_{ij}, e_i] = (1/2\sqrt{n_i})a_{ij}$, $[e_i, a_{ik}] = -[a_{ik}, e_i] = (1/2\sqrt{n_i})a_{ik}$,

$[e_j, a_{ij}] = -[a_{ij}, e_j] = -(1/2\sqrt{n_j})a_{ij}$, $[e_j, a_{jk}] = -[a_{jk}, e_j] = (1/2\sqrt{n_j})a_{jk}$,

$[e_k, a_{ik}] = -[a_{ik}, e_k] = -(1/2\sqrt{n_k})a_{ik}$, $[e_k, a_{jk}] = -[a_{jk}, e_k] = -(1/2\sqrt{n_k})a_{jk}$ *and all other bracket relations between elements in* $\mathfrak{A}_{\alpha\beta}$ $(\alpha, \beta \in \{i, j, k\})$ *are zero.*

The proofs of the above lemma follow from (1.2), (1.6), (1.9) and (1.10) in a straightforward manner. So, we omit them.

§ 2. Invariant connection of the canonical metric.

Let $\mathfrak{A}$ be a T-algebra of rank r and $V(\mathfrak{A})$ the corresponding homogeneous convex cone. We will denote simply $V(\mathfrak{A})$ and $\mathfrak{t}(\mathfrak{A})$ as V and $\mathfrak{t}$, respectively. Making use of the theory of invariant connections due to

Nomizu [3], we can calculate the Riemannian connection and the curvature tensor of the canonical metric on V in terms of the Lie algebra $\mathfrak{t}$ and the inner product $\langle\ ,\ \rangle$ in (1.5).

2.1. Let α be the connection function for the canonical metric of a homogeneous convex cone V. Then α is given as follows:

$$\alpha: \mathfrak{t}\times\mathfrak{t}\longrightarrow\mathfrak{t}\,,\qquad \alpha(a, b)=U(a, b)+\frac{1}{2}[a, b]\,, \tag{2.1}$$

$$2\langle U(a, b), c\rangle=\langle [c, a], b\rangle+\langle a, [c, b]\rangle$$

for every $a, b, c\in\mathfrak{t}$ (cf. [3]).

First of all, we will calculate the symmetric part U of the connection function α in the respective cases in Lemma 1.1. From Lemma 1.1 and (2.1), we have easily the following

LEMMA 2.1. *The symmetric part U of the connection function α is given as follows*:

(1) *The case of* $n_{ij}n_{jk}\neq 0$ $(i<j<k)$.

$U(e_\lambda, e_\mu)=0$ *for* $\lambda, \mu\in\{i, j, k\}$,

$U(e_i, a_{ij})=-(1/4\sqrt{n_i})a_{ij}$, $U(e_i, a_{jk})=0$, $U(e_i, a_{ik})=-(1/4\sqrt{n_i})a_{ik}$,

$U(e_j, a_{ij})=(1/4\sqrt{n_j})a_{ij}$, $U(e_j, a_{jk})=-(1/4\sqrt{n_j})a_{jk}$, $U(e_j, a_{ik})=0$,

$U(e_k, a_{ij})=0$, $U(e_k, a_{jk})=(1/4\sqrt{n_k})a_{jk}$, $U(e_k, a_{ik})=(1/4\sqrt{n_k})a_{ik}$,

$U(a_{ij}, a'_{ij})=\langle a_{ij}, a'_{ij}\rangle((1/2\sqrt{n_i})e_i-(1/2\sqrt{n_j})e_j)$,

$U(a_{ij}, a_{jk})=0$, $U(e^\lambda_{ij}, a_{ik})=(-1/2)^tT_\lambda a_{ik}$ *for* $1\leqq\lambda\leqq n_{ij}$,

$U(a_{jk}, a'_{jk})=\langle a_{jk}, a'_{jk}\rangle((1/2\sqrt{n_j})e_j-(1/2\sqrt{n_k})e_k)$,

$U(a_{jk}, a_{ik})=(1/2)(\sum_{1\leq\lambda\leq n_{ij}}\langle T_\lambda a_{jk}, a_{ik}\rangle e^\lambda_{ij})$,

$U(a_{ik}, a'_{ik})=\langle a_{ik}, a'_{ik}\rangle((1/2\sqrt{n_i})e_i-(1/2\sqrt{n_k})e_k)$.

(2) *The case of* $n_{ij}n_{ik}\neq 0$, $n_{jk}=0$ $(i<j<k)$ *or* $n_{jk}n_{ik}\neq 0$, $n_{ij}=0$ $(i<j<k)$.

$U(e_\lambda, e_\mu)=0$ *for* $\lambda, \mu\in\{i, j, k\}$,

$U(e_i, a_{ij})=-(1/4\sqrt{n_i})a_{ij}$, $U(e_i, a_{jk})=0$, $U(e_i, a_{ik})=-(1/4\sqrt{n_i})a_{ik}$,

$U(e_j, a_{ij})=(1/4\sqrt{n_j})a_{ij}$, $U(e_j, a_{jk})=-(1/4\sqrt{n_j})a_{jk}$, $U(e_j, a_{ik})=0$,

$U(e_k, a_{ij})=0$, $U(e_k, a_{jk})=(1/4\sqrt{n_k})a_{jk}$, $U(e_k, a_{ik})=(1/4\sqrt{n_k})a_{ik}$,

$U(a_{ij}, a'_{ij})=\langle a_{ij}, a'_{ij}\rangle((1/2\sqrt{n_i})e_i-(1/2\sqrt{n_j})e_j)$, $U(a_{ij}, a_{jk})=0$,

$U(a_{ij}, a_{ik})=0$, $U(a_{jk}, a'_{jk})=\langle a_{jk}, a'_{jk}\rangle((1/2\sqrt{n_j})e_j-(1/2\sqrt{n_k})e_k)$,

$U(a_{jk}, a_{ik})=0$, $U(a_{ik}, a'_{ik})=\langle a_{ik}, a'_{ik}\rangle((1/2\sqrt{n_i})e_i-(1/2\sqrt{n_k})e_k)$.

Directly from the formula (2.1), Lemmas 1.1 and 2.1 we have the following

LEMMA 2.2. *The connection function α of the canonical metric is given as follows*:

(1) *The case of* $n_{ij}n_{jk}\neq 0$ $(i<j<k)$.

$\alpha(e_\lambda, a_{mn})=0$ *for* $\lambda, m, n\in\{i, j, k\}$,

$\alpha(a_{ij}, e_i)=-(1/2\sqrt{n_i})a_{ij}$, $\alpha(a_{ij}, e_j)=(1/2\sqrt{n_j})a_{ij}$, $\alpha(a_{ij}, e_k)=0$,

$\alpha(a_{ij}, a'_{ij})=\langle a_{ij}, a'_{ij}\rangle((1/2\sqrt{n_i})e_i-(1/2\sqrt{n_j})e_j)$,

$\alpha(e^\lambda_{ij}, a_{jk})=(1/2)T_\lambda a_{jk}$ *for* $1\leqq\lambda\leqq n_{ij}$, $\alpha(e^\lambda_{ij}, a_{ik})=(-1/2)^t T_\lambda a_{ik}$ *for* $1\leqq\lambda\leqq n_{ij}$,

$\alpha(a_{jk}, e_i)=0$, $\alpha(a_{jk}, e_j)=-(1/2\sqrt{n_j})a_{jk}$, $\alpha(a_{jk}, e_k)=(1/2\sqrt{n_k})a_{jk}$,

$\alpha(a_{jk}, e^\lambda_{ij})=(-1/2)T_\lambda a_{jk}$ *for* $1\leqq\lambda\leqq n_{ij}$, $\alpha(a_{jk}, a_{ik})=(1/2)\sum_{1\leq\lambda\leq n_{ij}}\langle T_\lambda a_{jk}, a_{ik}\rangle e^\lambda_{ij}$,

$\alpha(a_{jk}, a'_{jk})=\langle a_{jk}, a'_{jk}\rangle((1/2\sqrt{n_j})e_j-(1/2\sqrt{n_k})e_k)$,

$\alpha(a_{ik}, e_i)=-(1/2\sqrt{n_i})a_{ik}$, $\alpha(a_{ik}, e_j)=0$, $\alpha(a_{ik}, e_k)=(1/2\sqrt{n_k})a_{ik}$,

$\alpha(a_{ik}, e^\lambda_{ij})=(-1/2)^t T_\lambda a_{ik}$ *for* $1\leqq\lambda\leqq n_{ij}$, $\alpha(a_{ik}, a_{jk})=(1/2)\sum_{1\leq\lambda\leq n_{ij}}\langle T_\lambda a_{jk}, a_{ik}\rangle e^\lambda_{ij}$,

$\alpha(a_{ik}, a'_{ik})=\langle a_{ik}, a'_{ik}\rangle((1/2\sqrt{n_i})e_i-(1/2\sqrt{n_k})e_k)$.

(2) *The case of* $n_{ij}n_{ik}\neq 0$, $n_{jk}=0$ $(i<j<k)$ *or* $n_{jk}n_{ik}\neq 0$, $n_{ij}=0$ $(i<j<k)$.

$\alpha(e_\lambda, a_{mn})=0$ *for* $\lambda, m, n\in\{i, j, k\}$,

$\alpha(a_{ij}, e_i)=-(1/2\sqrt{n_i})a_{ij}$, $\alpha(a_{ij}, e_j)=(1/2\sqrt{n_j})a_{ij}$, $\alpha(a_{ij}, e_k)=0$,

$\alpha(a_{ij}, a'_{ij})=\langle a_{ij}, a'_{ij}\rangle((1/2\sqrt{n_i})e_i-(1/2\sqrt{n_j})e_j)$,

$\alpha(a_{ij}, a_{jk})=0$, $\alpha(a_{ij}, a_{ik})=0$, $\alpha(a_{jk}, e_i)=0$, $\alpha(a_{jk}, e_j)=-(1/2\sqrt{n_j})a_{jk}$,

$\alpha(a_{jk}, e_k)=(1/2\sqrt{n_k})a_{jk}$, $\alpha(a_{jk}, a_{ij})=0$,

$\alpha(a_{jk}, a'_{jk})=\langle a_{jk}, a'_{jk}\rangle((1/2\sqrt{n_j})e_j-(1/2\sqrt{n_k})e_k)$,

$\alpha(a_{jk}, a_{ik})=0$, $\alpha(a_{ik}, e_i)=-(1/2\sqrt{n_i})a_{ik}$, $\alpha(a_{ik}, e_j)=0$,

$\alpha(a_{ik}, e_k)=(1/2\sqrt{n_k})a_{ik}$, $\alpha(a_{ik}, a_{ij})=0$, $\alpha(a_{ik}, a_{jk})=0$,

$\alpha(a_{ik}, a'_{ik})=\langle a_{ik}, a'_{ik}\rangle((1/2\sqrt{n_i})e_i-(1/2\sqrt{n_k})e_k)$.

2.2. Let R and ∇R be the curvature tensor and its covariant derivative of the canonical metric, respectively. Then R and ∇R are given by the following formulas (cf. [3]):

$$R: \mathfrak{t}\times\mathfrak{t}\times\mathfrak{t}\longrightarrow\mathfrak{t}, \tag{2.2}$$
$$R(a, b, c)=R(a, b)c=\alpha(a, \alpha(b, c))-\alpha(b, \alpha(a, c))-\alpha([a, b], c)$$

and

$$\nabla R: \mathfrak{t}\times\mathfrak{t}\times\mathfrak{t}\times\mathfrak{t}\longrightarrow\mathfrak{t}, \tag{2.3}$$
$$\nabla R(a, b, c, d)=(\nabla_d R)(a, b, c)=\alpha(d, R(a, b, c))-R(\alpha(d, a), b, c)$$
$$-R(a, \alpha(d, b), c)-R(a, b, \alpha(d, c))$$

for every $a, b, c, d\in\mathfrak{t}$.

In the following lemmas, we will obtain the necessary conditions for a homogeneous convex cone to be Riemannian symmetric, in terms of the Lie algebra $\mathfrak{t}$ or of the corresponding T-algebra $\mathfrak{A}$.

LEMMA 2.3. *If there exists a triple* (i, j, k) *with* $1 \leqq i < j < k \leqq r$ *such that* $n_{ij} n_{ik} \neq 0$, $n_{jk} = 0$ *or* $n_{jk} n_{ik} \neq 0$, $n_{ij} = 0$, *then* $\nabla R \neq 0$.

PROOF. First we will show, in the case of $n_{ij} n_{ik} \neq 0$, $n_{jk} = 0$, that $(\nabla_{e_{ij}^\lambda} R) \neq 0$ for $1 \leqq \lambda \leqq n_{ij}$. In fact, by Lemmas 1.1, 2.2 and the formula (2.2) we have $R(e_j, e_{ik}^n, e_{ik}^n) = 0$ for $1 \leqq n \leqq n_{ik}$. Therefore, by the formula (2.3) and Lemma 2.2 we see that

$$\begin{aligned}(\nabla_{e_{ij}^\lambda} R)(e_j, e_{ik}^n, e_{ik}^n) &= -R(\alpha(e_{ij}^\lambda, e_j), e_{ik}^n, e_{ik}^n) = -(1/2\sqrt{n_j}) R(e_{ij}^\lambda, e_{ik}^n, e_{ik}^n) \\ &= -(1/2\sqrt{n_j})(\alpha(e_{ij}^\lambda, \alpha(e_{ik}^n, e_{ik}^n)) - \alpha(e_{ik}^n, \alpha(e_{ij}^\lambda, e_{ik}^n))) \\ &= (1/8 n_i \sqrt{n_j}) e_{ij}^\lambda \neq 0 .\end{aligned}$$

Now we suppose that $n_{jk} n_{ik} \neq 0$, $n_{ij} = 0$. Then we can show analogously as in the above case that $(\nabla_{e_{jk}^m} R)(e_i, e_{ik}^n, e_{ik}^n) = (1/8 n_k \sqrt{n_i}) e_{jk}^m$ for $1 \leqq m \leqq n_{jk}$ and $1 \leqq n \leqq n_{ik}$. q.e.d.

LEMMA 2.4. *If* $\nabla R = 0$ *and* $n_{ij} n_{jk} \neq 0$ *with* $1 \leqq i < j < k \leqq r$, *then the equalities* $n_i = n_j$ *and* $n_{ik} = n_{jk}$ *hold.*

PROOF. From $R(e_j, e_{ik}^n, e_{ik}^{n'}) = R(e_j, e_{ik}^n, \alpha(e_{ij}^\lambda, e_{ik}^{n'})) = 0$ for $1 \leqq n, n' \leqq n_{ik}$ and $1 \leqq \lambda \leqq n_{ij}$, we have

$$(\nabla_{e_{ij}^\lambda} R)(e_j, e_{ik}^n, e_{ik}^{n'}) = -R(\alpha(e_{ij}^\lambda, e_j), e_{ik}^n, e_{ik}^{n'}) - R(e_j, \alpha(e_{ij}^\lambda, e_{ik}^n), e_{ik}^{n'}) .$$

On the other hand,

$$\begin{aligned}R(\alpha(e_{ij}^\lambda, e_j), e_{ik}^n, e_{ik}^{n'}) &= (1/2\sqrt{n_j}) R(e_{ij}^\lambda, e_{ik}^n, e_{ik}^{n'}) \\ &= (1/8\sqrt{n_j}) \sum_{1 \leq \mu \leq n_{ij}} \langle T_\mu {}^t T_\lambda e_{ik}^{n'}, e_{ik}^n \rangle e_{ij}^\mu - (1/8 n_i \sqrt{n_j}) \delta_{nn'} e_{ij}^\lambda\end{aligned}$$

and

$$\begin{aligned}R(e_j, \alpha(e_{ij}^\lambda, e_{ik}^n), e_{ik}^{n'}) &= \frac{1}{2} \alpha([e_j, {}^t T_\lambda e_{ik}^n], e_{ik}^{n'}) \\ &= (1/8\sqrt{n_j}) \sum_{1 \leq \mu \leq n_{ij}} \langle T_\mu {}^t T_\lambda e_{ik}^n, e_{ik}^{n'} \rangle e_{ij}^\mu .\end{aligned}$$

Therefore, we have

$$(\nabla_{e_{ij}^\lambda} R)(e_j, e_{ik}^n, e_{ik}^{n'}) = (1/8\sqrt{n_j})((1/n_i) \delta_{nn'} e_{ij}^\lambda - \sum_{1 \leq \mu \leq n_{ij}} \langle (T_\lambda {}^t T_\mu + T_\mu {}^t T_\lambda) e_{ik}^n, e_{ik}^{n'} \rangle e_{ij}^\mu) .$$

Hence, the system $\{T_\lambda\}_{1 \leq \lambda \leq n_{ij}}$ satisfies the equations:

$$T_\lambda {}^t T_\mu + T_\mu {}^t T_\lambda = (1/n_i) \delta_{\lambda\mu} I_{ik} .$$

Putting $\lambda = \mu$, we have $T_\lambda {}^t T_\lambda = (1/2n_i) I_{ik}$ and by (1.11) ${}^t T_\lambda T_\lambda = (1/2n_j) I_{jk}$. Thus, we have $n_i = n_j$ and $n_{ik} = n_{jk}$. q.e.d.

LEMMA 2.5. *If* $\nabla R=0$ *and* $n_{ij}n_{jk}\neq 0$ *with* $1\leqq i<j<k\leqq r$, *then the equality* $n_{ij}/n_j=n_{ik}/n_k$ *holds.*

PROOF. We want to calculate $(\nabla_{e^m_{jk}}R)(e_i, e^n_{ik}, e^n_{ik})$ for $1\leqq m\leqq n_{jk}$ and $1\leqq n\leqq n_{ik}$. By Lemma 2.2, we have $R(\alpha(e^m_{jk}, e_i), e^n_{ik}, e^n_{ik})=0$. Therefore,

$$(\nabla_{e^m_{jk}}R)(e_i, e^n_{ik}, e^n_{ik})=\alpha(e^m_{jk}, R(e_i, e^n_{ik}, e^n_{ik}))-R(e_i, \alpha(e^m_{jk}, e^n_{ik}), e^n_{ik})$$
$$-R(e_i, e^n_{ik}, \alpha(e^m_{jk}, e^n_{ik}))\ .$$

From Lemma 2.2 it follows that

$$\alpha(e^m_{jk}, R(e_i, e^n_{ik}, e^n_{ik}))=(1/8n_k\sqrt{n_i})e^m_{jk}$$

and

$$R(e_i, \alpha(e^m_{jk}, e^n_{ik}), e^n_{ik})=R(e_i, e^n_{ik}, \alpha(e^m_{jk}, e^n_{ik}))$$
$$=(1/8\sqrt{n_i})\sum_{1\leqq\lambda\leqq n_{ij}}\langle T_\lambda e^m_{jk}, e^n_{ik}\rangle{}^tT_\lambda e^n_{ik}\ .$$

Therefore,

$$(\nabla_{e^m_{jk}}R)(e_i, e^n_{ik}, e^n_{ik})=(1/8\sqrt{n_i})((1/n_k)e^m_{jk}-2\sum_{1\leqq\lambda\leqq n_{ij}}\langle T_\lambda e^m_{jk}, e^n_{ik}\rangle{}^tT_\lambda e^n_{ik})=0\ .$$

Summing these qualities with respect to n, we have

$$(n_{ik}/n_k)e^m_{jk}=2\sum_{1\leqq\lambda\leqq n_{ij}}{}^tT_\lambda T_\lambda e^m_{jk}=(n_{ij}/n_j)e^m_{jk}\ .$$

From this, it follows that $n_{ij}/n_j=n_{ik}/n_k$. q.e.d.

From Lemmas 2.4 and 2.5, we get the following

LEMMA 2.6. *If* $\nabla R=0$ *and* $n_{ij}n_{jk}\neq 0$ *with* $1\leqq i<j<k\leqq r$, *then the equalities* $n_{ij}=n_{jk}=n_{ik}$ *hold.*

PROOF. First we want to show that the equality

$$2(1/n_k-1/n_j)=(n_{jk}/n_j-n_{ij}/n_j)$$

holds. We note that from the assertion $n_{ik}=n_{jk}$ in Lemma 2.4, the condition (1.11) is equivalent to the following condition:

$$T_\lambda{}^tT_\mu+T_\mu{}^tT_\lambda=(1/n_j)\delta_{\lambda\mu}I_{jk}\quad(1\leqq\lambda,\ \mu\leqq n_{ij})\ . \tag{2.4}$$

Now, we calculate for $1\leqq\lambda\leqq n_{ij}$, $1\leqq m\leqq n_{jk}$ and $1\leqq n\leqq n_{ik}$,

$$(\nabla_{e^m_{jk}}R)(e^\lambda_{ij}, e^m_{jk}, e^n_{ik})=\alpha(e^m_{jk}, R(e^\lambda_{ij}, e^m_{jk}, e^n_{ik}))-R(\alpha(e^m_{jk}, e^\lambda_{ij}), e^m_{jk}, e^n_{ik})$$
$$-R(e^\lambda_{ij}, \alpha(e^m_{jk}, e^m_{jk}), e^n_{ik})-R(e^\lambda_{ij}, e^m_{jk}, \alpha(e^m_{jk}, e^n_{ik}))\ .$$

By direct calculations using Lemmas 1.1 and 2.2, we can see that

$$R(e_{ij}^{\lambda}, e_{jk}^{m}, e_{ik}^{n}) = (1/4\sqrt{n_k})\langle T_\lambda e_{jk}^{m}, e_{ik}^{n}\rangle e_k - (1/4\sqrt{n_i})\langle T_\lambda e_{jk}^{m}, e_{ik}^{n}\rangle e_i .$$

Therefore, we have

$$\alpha(e_{jk}^{m}, R(e_{ij}^{\lambda}, e_{jk}^{m}, e_{ik}^{n})) = (1/8n_k)\langle T_\lambda e_{jk}^{m}, e_{ik}^{n}\rangle e_{jk}^{m} .$$

The second term is given by

$$R(\alpha(e_{jk}^{m}, e_{ij}^{\lambda}), e_{jk}^{m}, e_{ik}^{n}) = (1/8)\sum_{1\leq\mu\leq n_{ij}} \langle T_\mu e_{jk}^{m}, e_{ik}^{n}\rangle {}^tT_\mu T_\lambda e_{jk}^{m} - (1/8n_k)\langle T_\lambda e_{jk}^{m}, e_{ik}^{n}\rangle e_{jk}^{m} .$$

Similarly we have

$$R(e_{ij}^{\lambda}, \alpha(e_{jk}^{m}, e_{jk}^{m}), e_{ik}^{n}) = (1/8n_j){}^tT_\lambda e_{ik}^{n}$$

and

$$R(e_{ij}^{\lambda}, e_{jk}^{m}, \alpha(e_{jk}^{m}, e_{ik}^{n})) = (1/8)\sum_{1\leq\mu\leq n_{ij}} \langle {}^tT_\mu e_{ik}^{n}, e_{jk}^{m}\rangle {}^tT_\lambda T_\mu e_{jk}^{m}$$
$$+ (1/4)\sum_{1\leq\mu\leq n_{ij}} \langle {}^tT_\mu e_{ik}^{n}, e_{jk}^{m}\rangle {}^tT_\mu T_\lambda e_{jk}^{m} - (1/8n_j)\langle {}^tT_\lambda e_{ik}^{n}, e_{jk}^{m}\rangle e_{jk}^{m} .$$

Summing the equalities $(\nabla_{e_{jk}^{m}}R)(e_{ij}^{\lambda}, e_{jk}^{m}, e_{ik}^{n})=0$ with respect to m, we get the following:

$$((1/4n_k)+(1/8n_j)-(n_{jk}/8n_j)){}^tT_\lambda - (1/8)\sum_{1\leq\mu\leq n_{ij}} (3\,{}^tT_\mu T_\lambda {}^tT_\mu + {}^tT_\lambda T_\mu {}^tT_\mu) = 0 .$$

By the equalities (1.11) and (2.4) it follows that

$$(2(1/n_k - 1/n_j) - (n_{jk}/n_j - n_{ij}/n_j)){}^tT_\lambda = 0$$

for $1\leqq\lambda\leqq n_{ij}$. From $2(1/n_k - 1/n_j) = (n_{jk}/n_j - n_{ij}/n_j)$ and the equalities obtained in Lemmas 2.4 and 2.5, we conclude that $n_{ij}=n_{jk}=n_{ik}$. q.e.d.

§ 3. Main result.

In this section we will prove the theorem stated in Introduction and give an application.

LEMMA 3.1. *Let $\mathfrak{A}=\sum_{1\leq i,j\leq r}\mathfrak{A}_{ij}$ be a T-algebra of rank r with $r\geqq 3$ satisfying the following two conditions:*

(1) *For each pair (i, j) with $1\leqq i<j\leqq r$, there exists a series $i_0, i_1, \cdots, i_m$ such that $i_0=i$, $i_m=j$ and $n_{i_{\lambda-1}i_\lambda}\neq 0$ for $1\leqq\lambda\leqq m$.*

(2) *For each triple (i, j, k) with $1\leqq i<j<k\leqq r$ satisfying the conditions $n_{ij}n_{ik}\neq 0$, or $n_{jk}n_{ik}\neq 0$, the equalities $n_{ij}=n_{jk}=n_{ik}$ hold.*

Then n_{ij} is constant for $1\leqq i<j\leqq r$.

PROOF. First we show that $n_{ij}\neq 0$ for every pair (i, j) with $1\leqq i<$

$j \leqq r$. Suppose that there exists a pair (i, j) $(i<j)$ with $n_{ij}=0$. Then we can take a series $i_0, i_1, \cdots, i_m$ $(m \geqq 2)$ of different indices satisfying the condition (1). We consider the three indices i_0, i_1, i_2. From the conditions (2) and (1.8) it follows that $n_{i_0i_1}n_{i_1i_2} \neq 0$ implies the equalities $n_{i_0i_1}=n_{i_1i_2}=n_{i_0i_2}$. Therefore, the series $i_0, i_2, \cdots, i_m$ satisfies the condition (1). Again, starting from the series $i_0, i_2, \cdots, i_m$, we get the condition $n_{i_0i_3} \neq 0$ in the same way. This procedure finally leads to the condition $n_{i_0i_m} \neq 0$, which contradicts the assumption.

Next we show that $n_{ij}=n_{12}$ for every (i, j) with $1 \leqq i<j \leqq r$. By the condition (2), we can assume that $i \geqq 3$. Let us consider the triple (n_{12}, n_{2i}, n_{1i}). Then by the conditions (2) and (1.8) we have $n_{12}=n_{1i}$. On the other hand, for the triple (n_{1i}, n_{ij}, n_{1j}) we have $n_{1i}=n_{ij}$. Therefore $n_{ij}=n_{12}$. q.e.d.

Now we are in a position to prove the main theorem which is an affirmative answer to the Matsushima's problem stated in Introduction.

THEOREM 3.2. *Let V be a homogeneous convex cone. If V is Riemannian symmetric with respect to the canonical metric, then V is self-dual.*

PROOF. A reducible homogeneous convex cone V is linearly isomorphic to a direct product $V_1 \times V_2 \times \cdots \times V_s$ of irreducible homogeneous convex cones V_i; the decomposition is unique up to an order. In this case, from the definition of the characteristic function (cf. Introduction), it follows that $\varphi_V=\varphi_{V_1} \cdot \varphi_{V_2} \cdots \varphi_{V_s}$ and the Riemannian manifold (V, g_V) is isomorphic to the direct product of homogeneous Riemannian manifolds (V_i, g_{V_i}). Therefore V is Riemannian symmetric if and only if each V_i is Riemannian symmetric. It is known that V is self-dual if and only if each irreducible component V_i is self-dual (cf. [7]). A homogeneous convex cone of rank one is the cone of positive real numbers and an irreducible homogeneous convex cone of rank two is the circular cone. These two cones are self-dual. Therefore, in order to prove the theorem we can assume that V is irreducible and the rank r of the T-algebra $\mathfrak{A}$ satisfying $V=V(\mathfrak{A})$ is greater than or equal to 3. Then it is known in Asano [1] that for an irreducible cone $V(\mathfrak{A})$, the condition (1) in Lemma 3.1 is satisfied. On the other hand, from Lemmas 2.3 and 2.6 it follows that the condition (2) in Lemma 3.1 is satisfied. We obtain, by Lemma 3.1, that n_{ij} is constant for every i, j with $1 \leqq i<j \leqq r$. It was proved in Vinberg [7] that an irreducible homogeneous convex cone $V(\mathfrak{A})$ is self-dual if and only if n_{ij} is constant for every i, j with $1 \leqq i<j \leqq r$.

Therefore, V is self-dual. q.e.d.

Finally we give an application of the above theorem. It is known that every homogeneous self-dual cone is Riemannian symmetric with respect to the canonical metric (cf. [2], [4]). It was proved by Rothaus [5] that a homogeneous convex cone V in $\boldsymbol{R}^n$ is self-dual if and only if the tube domain $D(V)=\{z\in\boldsymbol{C}^n; \operatorname{Im} z\in V\}$ over V is Hermitian symmetric with respect to the Bergman metric of $D(V)$. Therefore, as an application of the above theorem, we have the following

COROLLARY 3.3. *For a homogeneous convex cone V, the following three conditions are equivalent:*

(1) *V is Riemannian symmetric with respect to the canonical metric.*

(2) *V is self-dual.*

(3) *The tube domain $D(V)$ over V is Hermitian symmetric with respect to the Bergman metric.*

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