TOKYO J. MATH.
VOL. 5, NO. 2, 1982

Some Examples of Analytic Functionals and their Transformations

Kunio YOSHINO

Sophia University

Introduction

Recently the theory of analytic functionals with non-compact carrier has been studied by many mathematicians. For example, see Morimoto [4], [5], Morimoto-Yoshino [6], [7], de Roever [8], Sargos-Morimoto [9], Zharinov [10]. However, it seems only a few examples have been given in the literature. In this paper, we will construct several examples of analytic functionals which are closely related to special functions. Furthermore we calculate the Fourier-Borel transformation and the Avanissian-Gay transformation of the analytic functionals. Some of them were already employed in Morimoto-Yoshino [7]. We will show that, in many cases, the Avanissian-Gay transformation $G_T(w)$ is a generating function of orthogonal polynomials and that the integral representation of the Fourier-Borel transformation by the Avanissian-Gay transformation is a classical formula of special functions. We confine ourselves to one dimensional case.

§1. Fundamental space $Q(L; k')$ and its dual space $Q'(L; k')$.

Let L be a convex compact set or a set of the following form: $L=[a, \infty)+i[-k, k]$, $i=\sqrt{-1}$, $k'\in \boldsymbol{R}$, $a\in \boldsymbol{R}$ and $k\geqq 0$. We denote by $Q_b(L_\varepsilon; k'+\varepsilon')$ the space of all functions $f(\zeta)$ holomorphic in the interior of L_ε which satisfy the following estimate:

$$\sup\{|f(\zeta)|\exp((k'+\varepsilon')\xi);\ \zeta=\xi+i\eta\in L_\varepsilon\}<+\infty\ ,$$

where $L_\varepsilon=[a-\varepsilon, \infty)+i[-k-\varepsilon, k+\varepsilon]$. Taking the inductive limit following the restriction mappings as $\varepsilon>0$ and $\varepsilon'>0$ tending to zero, we define the fundamental space $Q(L; k')$ as follows:

Received March 19, 1981

$$Q(L; k')=\lim_{\varepsilon\downarrow 0, \varepsilon'\downarrow 0} \operatorname{ind} Q_b(L_\varepsilon; k'+\varepsilon') .$$

An analytic functional with carrier in L and of type k' is, by definition, a continuous linear functional on the space $Q(L; k')$. $Q'(L; k')$ will denote the dual space of $Q(L; k')$. An analytic functional T is called to be with carrier at the infinity $\infty+i[-k, k]$ if $T\in Q'(L; k')$, $L=[a, \infty)+i[-k, k]$ for every $a>0$.

§2. Transformations of analytic functionals.

We recall two transformations of analytic functionals:

1) The Fourier-Borel transformation $\tilde{T}$ of $T\in Q'(L; k')$ is defined by

$$\tilde{T}(z)=\langle T_\zeta, \exp(z\zeta)\rangle . \tag{1}$$

Remark that $\tilde{T}(z)$ is defined only for $z\in\{\operatorname{Re} z<-k'\}$ in general. If T is an analytic functional with compact carrier, then the Fourier-Borel transformation $\tilde{T}(z)$ is defined on the whole complex plane $\boldsymbol{C}$. It is known the Fourier-Borel transformation FB: $T\to\tilde{T}$ establishes a linear topological isomorphism of $Q'(L; k')$ onto $\operatorname{Exp}((-\infty, -k')+i\boldsymbol{R}; L)$ the space of all holomorphic functions $F(z)$ on the left half plane $(-\infty, -k')+i\boldsymbol{R}$ which satisfy the following estimate: For every $\varepsilon>0$, $\varepsilon'>0$, there exists a constant $C_{\varepsilon\varepsilon'}\geqq 0$ such that

$$|F(z)|\leqq C_{\varepsilon\varepsilon'} \exp((a-\varepsilon)x+(k+\varepsilon)y)$$

for any $z=x+iy\in(-\infty, -k'-\varepsilon']+i\boldsymbol{R}$ (Theorem 5.1 in Morimoto [5]).

2) Suppose in the sequel $0\leqq k<\pi$ and $k'<1$. The Avanissian-Gay transformation of analytic functional $T\in Q'(L; k')$ is defined by

$$G_T(w)=\langle T_\zeta, (1-we^\zeta)^{-1}\rangle . \tag{2}$$

Remark that $G_T(w)$ is a holomorphic function of $w\in\boldsymbol{C}\backslash\exp(-L)$. We have the inversion formula (Theorem 4 in Morimoto-Yoshino [6])

$$\langle T, h\rangle=(2\pi i)^{-1}\int_{\partial L_\varepsilon} G_T(e^{-\zeta})h(\zeta)d\zeta , \tag{3}$$

where $h(\zeta)\in Q(L; k')$ and $\varepsilon>0$ is sufficiently small. Especially $\tilde{T}$ can be expressed by G_T as follows:

$$\begin{aligned}\tilde{T}(z)&=(2\pi i)^{-1}\int_{\partial L_\varepsilon} G_T(e^{-\zeta})\exp(z\zeta)d\zeta \\ &=-(2\pi i)^{-1}\int_{\partial\exp(-L_\varepsilon)} G_T(w)w^{-z-1}dw .\end{aligned} \tag{4}$$

Conversely $G_T(w)$ can be expressed by the Fourier-Borel transformation $\tilde{T}$ of T as follows:

$$(5)\qquad G_T(w)=-\sum_{n=1}^{\infty}\tilde{T}(-n)w^{-n} \qquad \text{for} \quad |w|>e^{-a}\,.$$

If T is an analytic functional with compact carrier, there is another way to represent $G_T(w)$ by $\tilde{T}$:

$$(6)\qquad G_T(w)=\sum_{n=0}^{\infty}\tilde{T}(n)w^{n} \qquad \text{for sufficiently small } |w|\,.$$

Remark that (5) is the Laurent expansion of $G_T(w)$ and that (6) is the Maclaurin expansion of $G_T(w)$. The Avanissian-Gay transformation $G: T\to G_T$ establishes a topological linear isomorphism $Q'(L; k')$ onto $\mathcal{O}_0(\boldsymbol{C}\backslash\exp(-L); k')$ the space of all holomorphic functions on $\boldsymbol{C}\backslash\exp(-L)$, which vanish at $w=\infty$ and satisfy the following condition: For any ε with $0<\varepsilon<\pi-k$, and any ε' with $0<\varepsilon'<1-k'$ there exists a constant $C\geqq 0$ such that

$$|G_T(w)|\leqq C|w|^{-k'-\varepsilon'}$$

for $w\in \boldsymbol{C}\backslash\{0\}$ with $k+\varepsilon\leqq|\arg w|\leqq 2\pi-k-\varepsilon$ (Theorem 6 in Morimoto-Yoshino [6]). Therefore via the space $Q'(L; k')$ of analytic functionals, the spaces $\mathrm{Exp}\,((-\infty, -k')+i\boldsymbol{R}; L)$ and $\mathcal{O}_0(\boldsymbol{C}\backslash\exp(-L); k')$ are linear-topologically isomorphic each other. For the details of the Avanissian-Gay transformation of analytic functionals with compact carriers, we refer the reader to Avanissian-Gay [1]. In many cases the formula (4) turns out to be a classical integral formula, for example, Hankel's integral formula of the Gamma function, Sonine's integral formula of the Bessel function, etc. On the other hand the formulas (5) and (6) are the generating function expansions of orthogonal polynomials, for example, Legendre polynomials, Laguerre polynomials, etc.

§3. Examples of analytic functionals with compact carrier.

In this section we will give several examples of analytic functionals with compact carrier.

EXAMPLE 1 (Dirac's delta function). Let L be the singleton $\{a\}$ and k' be any real number. We put

$$\langle T, h\rangle=h(a)\,,$$

where $h(\zeta)\in\mathcal{O}(\{a\})$. In this case $\tilde{T}(z)=e^{az}$ and $G_T(w)=(1-e^{a}w)^{-1}$.

EXAMPLE 2. Let $L=\{0\}$, k' be any real number. We put

$$\langle T, h\rangle=(2\pi i)^{-1}\int_{|\zeta|=\varepsilon}\exp(-te^{-\zeta}/(1-e^{-\zeta}))(1-e^{-\zeta})^{-1}d\zeta$$
$$=-(2\pi i)^{-1}\int_{|w-1|=\varepsilon}\exp(-tw/(1-w))(1-w)^{-1}h(-\log w)dw/w$$

for $h(\zeta)\in\mathcal{O}(\{0\})$. In this case, $\tilde{T}(z)=L_z(t)/\Gamma(z+1)$ and $G_T(w)=\exp(-tw/(1-w))(1-w)^{-1}$, where $L_z(t)$ is the Laguerre function. The formulas (4) and (5) read as follows:

$$(6)\qquad L_z(t)/\Gamma(z+1)=(2\pi i)^{-1}\int_{|w-1|=\varepsilon}\exp(-tw/(1-w))(1-w)^{-1}w^{-z-1}dw\ ,$$

$$(7)\qquad \exp(-tw/(1-w))(1-w)^{-1}=\sum_{n=0}^{\infty}L_n(t)/n!\ w^n\ .$$

EXAMPLE 3. For t $(-1<t<1)$, we define the compact set L as follows: $L=\{-\log(t-i\sqrt{1-t^2}),\ -\log(t+i\sqrt{1-t^2})\}$. For $h(\zeta)\in\mathcal{O}(L)$, we define

$$\langle T, h\rangle=(2\pi i)^{-1}\int_{\Gamma_1}(t-e^{-\zeta})(1-2te^{-\zeta}+e^{-2\zeta})^{-1}h(\zeta)d\zeta$$
$$=-(2\pi i)^{-1}\int_{\Gamma_2}(t-w)(1-2tw+w^2)^{-1}h(-\log w)dw/w\ ,$$

where the integral path Γ_1 (Γ_2) is a sufficiently small closed path enclosing once the points $-\log(t-i\sqrt{1-t^2})$ and $\log(t+i\sqrt{1-t^2})$ $(t-i\sqrt{1-t^2}$ and $t+i\sqrt{1-t^2}$ respectively). See Figure 1.

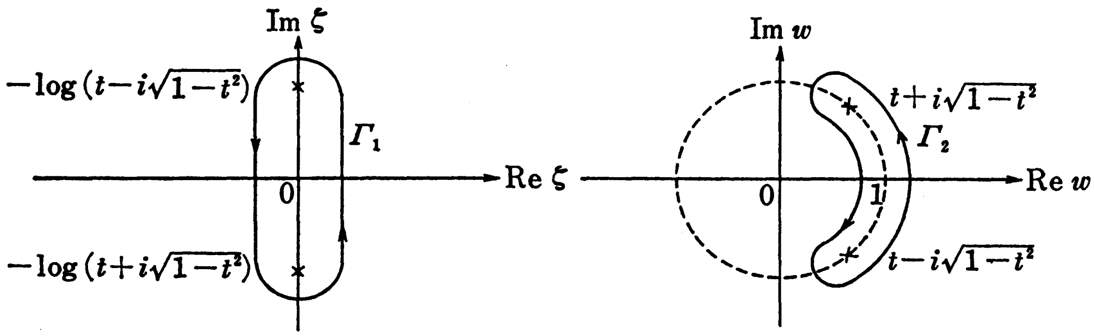

FIGURE 1

In this case $\tilde{T}(z)=T_z(t)$ (the Chebychev function of the first kind) and $G_T(w)=(1-tw)(1-2tw+w^2)^{-1}$. The formulas (4) and (5) read as follows:

$$(8)\qquad T_z(t)=-(2\pi i)^{-1}\int_{\Gamma_2}(1-w^2)(1-2tw+w^2)^{-1}w^{-z-1}dw\ ,$$

$$(1-tw)(1-2tw+w^2)^{-1}=\sum_{n=0}^{\infty} T_n(t)w^n\ . \tag{9}$$

EXAMPLE 4. Let L, Γ_1 and Γ_2 be as in Example 3. For $h(\zeta)\in\mathcal{O}(L)$, we define

$$\begin{aligned}\langle T, h\rangle &= -(2\pi i)^{-1}\int_{\Gamma_2}(1-2tw+w^2)^{-1}\ h(-\log w)dw/w\\ &=(2\pi i)^{-1}\int_{\Gamma_1}(1-2te^{-\zeta}+e^{-2\zeta})^{-1}\ h(\zeta)d\zeta\ .\end{aligned}$$

In this case $\tilde{T}(z)=U_z(t)$ and $G_T(w)=(1-2tw+w^2)^{-1}$, where $U_z(t)$ is the Chebychev function of the second kind. The formulas (4) and (5) read as follows:

$$U_z(t)=(2\pi i)^{-1}\int_{\Gamma_2}(1-2tw+w^2)^{-1}w^{-z-1}dw\ , \tag{10}$$

$$(1-2tw+w^2)^{-1}=\sum_{n=0}^{\infty} U_n(t)w^n\ . \tag{11}$$

EXAMPLE 5 (Characteristic function of a closed interval $[a, b]$). Let $L=[a, b]$. For $h(\zeta)\in\mathcal{O}(L)$, we define

$$\langle T, h\rangle=\int_a^b h(\zeta)d\zeta\ .$$

We have $\tilde{T}(z)=(e^{bz}-e^{az})z^{-1}$ and $G_T(w)=\log(e^{-b}-w)-\log(e^{-a}-w)$. In this example, we omit the formulas (4) and (5).

EXAMPLE 6. For $t\ (-1<t<1)$, we define the compact interval L as follows: $L=[-\log(t-i\sqrt{1-t^2}),\ -\log(t+i\sqrt{1-t^2})]$. For $h(\zeta)\in\mathcal{O}(L)$, we define

$$\begin{aligned}\langle T, h\rangle &=(2\pi i)^{-1}\int_{\Gamma_1}(1-2te^{-\zeta}+e^{-2\zeta})^{-1/2}h(\zeta)d\zeta\\ &=-(2\pi i)^{-1}\int_{\Gamma_2}(1-2tw+w^2)^{-1/2}h(-\log w)dw/w\ ,\end{aligned}$$

where Γ_1 and Γ_2 are the same integral paths as in Example 3. We have $\tilde{T}(z)=P_z(T)$ and $G_T(w)=(1-2tw+w^2)^{-1/2}$, where $P_z(t)$ is the Legendre function. In this case the formulas (4) and (5) read as follows:

$$P_z(t)=-(2\pi i)^{-1}\int_{\Gamma_2}(1-2tw+w^2)^{-1/2}w^{-z-1}dw\ , \tag{12}$$

$$(1-2tw+w^2)^{-1/2}=\sum_{n=0}^{\infty} P_n(t)w^n . \tag{13}$$

In fact, by the definition of this functional and the formula (4), we have

$$\tilde{T}(z)=-(2\pi i)^{-1}\int_{\Gamma_2}(1-2tw+w^2)^{-1/2}w^{-z-1}dw . \tag{14}$$

The right hand side of (14) equals to the following integral:

$$(2\pi i)^{-1}\int^{(1+,t+)}(w^2-1)^z 2^{-z}(w-t)^{-z-1}dw ,$$

which, in turn, coincides with the Legendre function $P_z(t)$ by the Schläfli's integral formula. Therefore, we obtain (12). It is clear that the Avanissian-Gay transformation $G_T(w)$ of the functional T is $(1-2tw+w^2)^{-1/2}$. If we apply the formula (6), then we have the generating function expansion (13).

EXAMPLE 7 (Generalization of Example 6). Let L, Γ_1 and Γ_2 be as in Example 6. For $h\in\mathcal{O}(L)$, we define

$$\langle T, h\rangle=-(2\pi i)^{-1}\int_{\Gamma_2}(1-2tw+w^2)^{-\nu}h(-\log w)dw/w ,$$

where 2ν is an integer. We have $\tilde{T}(z)=C_z^\nu(t)$ and $G_T(w)=(1-2tw+w^2)^{-\nu}$, where $C_z^\nu(t)$ is the Gegenbauer function. The formulas (4) and (5) read as follows:

$$C_z^\nu(t)=-(2\pi i)^{-1}\int_{\Gamma_2}(1-2tw+w^2)^{-\nu}w^{-z-1}dw , \tag{15}$$

$$(1-2tw+w^2)^{-\nu}=\sum_{n=0}^{\infty} C_n^\nu(t)w^n . \tag{16}$$

(16) is a generating function expansion of the Gegenbauer polynomials.

§4. Examples of analytic functional with real carrier.

In this section we will show four examples.

EXAMPLE 8 (Characteristic function of an infinite interval $[a, \infty)$). Let $k'\geqq 0$ and $L=[a, \infty)$. We define $T\in Q'(L; k')$ as follows:

$$\langle T, h\rangle=\int_a^\infty h(\zeta)d\zeta$$

for $h(\zeta)\in Q(L;k')$. We have $\tilde{T}(z)=-\exp(az)z^{-1}$ and $G_T(w)=\log(1-e^{-a}w^{-1})$. The formulas (4) and (5) read as follows:

$$-\exp(az)z^{-1}=(2\pi i)^{-1}\int_{\partial L_\varepsilon}\log(1-e^{\zeta-a})\exp(z\zeta)d\zeta\ , \tag{17}$$

$$\log(1-e^{-a}w^{-1})=-\sum_{n=1}^{\infty}(e^{-a}w^{-1})^n n^{-1}\ . \tag{18}$$

(18) is the ordinary Maclaurin expansion of $\log(1-w)$.

EXAMPLE 9 (Mellin transformation of test functions). Let $s\in C$, $k'\geqq 0$ and $L=[0,\infty)$. For $h(\zeta)\in Q(L;k')$, we put

$$\langle T,h\rangle=(2i\sin(s-1)\pi)^{-1}\int_{\partial L_\varepsilon}(-\zeta)^{s-1}h(\zeta)d\zeta\ .$$

We have $\tilde{T}(z)=\Gamma(s)(-z)^{-s}$ and $G_T(w)=-\sum_{n=1}^{\infty}\Gamma(s)n^{-s}w^{-n}=-\Gamma(s)F(w^{-1},s)$. $F(w,s)$ is the Jonquière function (Appell's function). For the Jonquière function, see Magnus [3] p. 33. Especially $G_T(1)=\Gamma(s)\zeta(s)$.

EXAMPLE 10. Let $L=[0,\infty)$ and $1/2\leqq k'<1$. We put

$$\begin{aligned}\langle T,h\rangle&=(2\pi i)^{-1}\int_{\partial L_\varepsilon}(e^{-\zeta}(e^{-\zeta}-1))^{-1/2}h(\zeta)d\zeta\\&=-(2\pi i)^{-1}\int_{\partial\exp(-L_\varepsilon)}(w(w-1))^{-1/2}h(-\log w)dw/w\ .\end{aligned}$$

Remark that this functional is not a Fourier-hyperfunction, while those functionals in Examples 8, 9 are Fourier-hyperfunctions. We have $\tilde{T}(z)=2(2z+1)^{-1}B(-z,1/2)^{-1}$ and $G_T(w)=(w(w-1))^{-1/2}$, where $B(x,y)$ is the Beta function. In fact, by the definition of this functional and the formula (4), we have

$$\tilde{T}(z)=(2\pi i)^{-1}\int_{\partial\Gamma_\varepsilon}(e^{-\zeta}(e^{-\zeta}-1))^{-1/2}\exp(z\zeta)d\zeta\ .$$

By Cauchy's integral theorem and elementary estimates, we can change the integral contour ∂L_ε to the new contour shown in Figure 2.

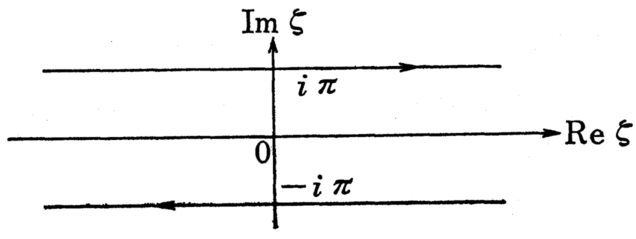

FIGURE 2

Therefore we have

$$\begin{aligned}\tilde{T}(z) &= (2\pi i)^{-1}\int_{-\infty}^{+\infty}(e^{-\xi}(e^{-\xi}-1))^{-1/2}\exp(z\xi)(e^{\pi iz}-e^{-\pi iz})d\xi \\ &= \pi^{-1}\sin\pi z\int_{0}^{+\infty}t^{-z-3/2}(1+t)^{-1/2}dt\,.\end{aligned}$$

By the formula of the Beta function

$$B(x, y)=\int_{0}^{+\infty}t^{x-1}(1+t)^{-x-y}dt \qquad \text{(Inui [2] p. 13 or Magnus [3] p. 7)}\,,$$

we have

$$\begin{aligned}\tilde{T}(z) &= \pi^{-1}\sin\pi z \quad B(-z-1/2, z+1) \\ &= -\Gamma(-z-1/2)\Gamma(-z)^{-1}\Gamma(1/2)^{-1} \\ &= 2(2z+1)^{-1}B(-z, 1/2)^{-1}\,,\end{aligned}$$

where we used the formula $B(x, y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ and $\Gamma(x)\Gamma(1-x)=\pi/\sin\pi x$. It is clear that the Avanissian-Gay transformation $G_T(w)$ of T is equal to $(w(w-1))^{-1/2}$.

EXAMPLE 11 (Generalization of Example 10). Suppose the same assumption as in Example 10. Let $k'=m-\lambda<1$ $(m=1, 2, 3, \cdots)$ and $L=[0, \infty)$. We define $T\in Q'(L; k')$ as follows:

$$\langle T, h\rangle=-(2\pi i)^{-1}\int_{\partial\exp(-L_\varepsilon)}w^{-m}(1-w^{-1})^{-\lambda}h(-\log w)dw/w$$

for $h(\zeta)\in Q(L; k')$. We have $\tilde{T}(z)=-\Gamma(\lambda)^{-1}\Gamma(-z-m+1)^{-1}\Gamma(-z-m+\lambda)$ and $G_T(w)=w^{-m}(1-w^{-1})^{-\lambda}$. The formulas (4) and (5) read as follows:

$$\begin{aligned}&\Gamma(\lambda)^{-1}\Gamma(-z-m+1)^{-1}\Gamma(-z-m+1) \\ &\qquad =(2\pi i)^{-1}\int_{\partial\exp(-L_\varepsilon)}w^{-m-z-1}(1-w^{-1})^{-\lambda}dw\,,\end{aligned} \tag{19}$$

$$w^{-m}(1-w^{-1})^{-\lambda}=\sum_{n=1}^{\infty}\Gamma(\lambda)^{-1}\Gamma(n-m+1)^{-1}\Gamma(n-m+\lambda)w^{-n}\,. \tag{20}$$

§5. Examples of analytic functionals with non-compact carrier.

In this section we calculate the transformations of analytic functionals with non-compact carrier.

EXAMPLE 12. Let $L=[0, \infty)+i[-\pi/2, \pi/2]$ and $0\leqq k'<1$. We define $T\in Q'(L; k')$ as follows:

$$\langle T, h\rangle = -(2\pi i)^{-1}\int_{\partial L_\varepsilon} 2^{-1}\log(1-2te^{\zeta}+e^{2\zeta})h(\zeta)d\zeta$$
$$=(2\pi i)^{-1}\int_{\partial \exp(-L_\varepsilon)} 2^{-1}\log(1-2tw^{-1}+w^{-2})h(-\log w)dw/w\ .$$

We will show following four equalities:

$$\tilde{T}(z) = -T_z(t)z^{-1}\ , \tag{21}$$

$$G_T(w) = -\log(1-2tw^{-1}+w^{-2})\ , \tag{22}$$

$$T_{-z}(t)z^{-1} = -(2\pi i)^{-1}\int_{\partial\exp(-L_\varepsilon)} 2^{-1}\log(1-2tw^{-1}+w^{-2})w^{-z-1}dw\ , \tag{23}$$

$$-2^{-1}\log(1-2tw+w^2) = \sum_{n=1}^{\infty} T_n(t)n^{-1}w^n\ , \tag{24}$$

where $T_z(t)$ is the Chebychev function of the first kind.

First we will show equality (21). In the definition of this functional, we take the principal branch of the logarithmic function. Therefore the function $\log(1-2tw^{-1}+w^{-2})$ is holomorphic in the complement of the closed set $\{w=u+iv;\ (u-1/2t)^2+v^2=1/4t^2,\ u^2+v^2\leqq 1\}$. See Figure 3.

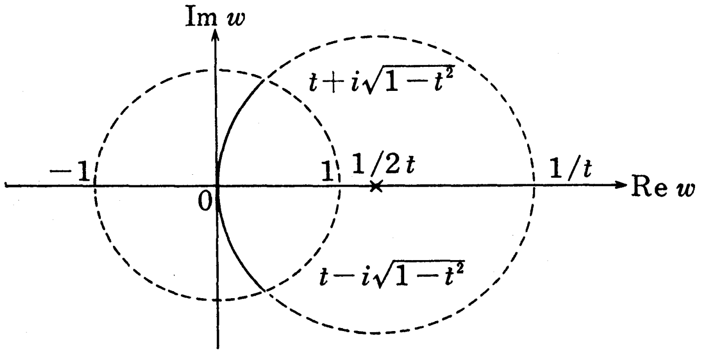

FIGURE 3

From the definition of this functional and the formula (4), we have

$$\tilde{T}(z) = -(2\pi i)^{-1}\int_{\partial\exp(-L_\varepsilon)} 2^{-1}\log(1-2tw^{-1}+w^{-2})w^{-z-1}dw\ .$$

By the equality

$$w^{-z-1}\log(1-2tw^{-1}+w^{-2}) = d/dw\{\log(1-2tw^{-1}+w^{-2})w^{-z}(-z)^{-1}\}$$
$$-2(1-2tw+w^2)^{-1}(tw-1)w^{-z-1}(-z)^{-1}\ ,$$

we obtain

$$\tilde{T}(z)=(2\pi i)^{-1}\int_{\partial\exp(-L_\varepsilon)}(1-2tw+w^2)^{-1}(tw-1)w^{-z-1}(-z)^{-1}dw\ .$$

By the residue theorem, we have

$$\begin{aligned}\tilde{T}(z)&=-z^{-1}\{(t+i\sqrt{1-t^2}\,)^z+(t-i\sqrt{1-t^2}\,)^z\}/2\\&=-z^{-1}T_z(t)\ .\end{aligned}$$

For the last equality, see Inui [2] p. 204. It is easily seen that the Avanissian-Gay transformation $G_T(w)$ is equal to $-1/2\log(1-2tw^{-1}+w^{-2})$. By the formulas (4) and (5), we can obtain equalities (23) and (24). Remark that (24) is a generating function expansion of Chebychev function of the first kind (Magnus [3] p. 259). From (23), we can deduce the following integral representation of the Chebychev polynomial of the first kind (Magnus [3] p. 260):

$$T_n(t)=(4\pi i)^{-1}\int^{(+0)}(1-w^2)(1-2tw+w^2)^{-1}w^{-n-1}dw\ .$$

This integral representation of $T_z(t)$ is given in the literature only if z is a positive integer, as far as the author knows.

EXAMPLE 13. Let $L=[0,\infty)+i[-\pi/2,\pi/2]$. $0\leqq k'<1$. We define an analytic functional as follows:

$$\langle T, h\rangle=(2\pi i)^{-1}\int_{\partial L_\varepsilon}\log(1+e^{2\zeta})h(\zeta)d\zeta\ ,$$

where $h(\zeta)\in Q(L;k')$. We have $\tilde{T}(z)=-2z^{-1}\cos(\pi z/2)$ and $G_T(w)=\log(1+w^{-2})$.

§6. Examples of analytic functionals with carrier at the infinity.

In this section we will show three examples of analytic functionals with carrier at the infinity. For the details of these examples, see Morimoto-Yoshino [7].

EXAMPLE 14. Let $a\in\boldsymbol{R}$, $L=[a,\infty)+i[-\pi/2,\pi/2]$ and k' be an arbitrary real number. We put

$$\langle T, h\rangle=(2\pi i)^{-1}\int_{\partial L_\varepsilon}\exp(e^\zeta)h(\zeta)d\zeta\ .$$

We have $\tilde{T}(z)=-\Gamma(1-z)^{-1}$ and $G_T(w)=\exp(w^{-1})-1$. In this example, the formulas (4) and (5) read as follows:

$$\Gamma(1-z)^{-1} = -(2\pi i)^{-1} \int_{\partial L_\varepsilon} \exp(e^\zeta) \exp(z\zeta) d\zeta\,, \tag{25}$$

$$\exp(w^{-1}) - 1 = \sum_{n=1}^{\infty} (n!)^{-1} w^{-n}\,. \tag{26}$$

(25) is equivalent to Hankel's integral formula of the Gamma function. (26) is the Laurent expansion of $\exp(w^{-1})-1$.

EXAMPLE 15. Let L and k' be as in Example 14. Suppose $\lambda > 0$. We put

$$\langle T, h \rangle = (2\pi i)^{-1} \int_{\partial L_\varepsilon} \exp(\lambda \sinh \zeta) h(\zeta) d\zeta\,.$$

We have $\tilde{T}(z) = J_{-z}(\lambda)$ and $G_T(w) = \exp(\lambda(w - w^{-1})/2) - \sum_{n=1}^{\infty} J_n(\lambda) w^n$, where $J_z(\lambda)$ is the Bessel function. The formulas (4) and (5) read as follows:

$$J_{-z}(\lambda) = (2\pi i)^{-1} \int_{\partial L_\varepsilon} \exp(\lambda \sinh \zeta) \exp(z\zeta) d\zeta\,, \tag{27}$$

$$\exp(\lambda(w - w^{-1})/2) = \sum_{n=-\infty}^{\infty} J_n(\lambda) w^n\,. \tag{28}$$

(27) is equivalent to Sonine's integral representation of the Bessel function. And (28) is the generating function expansion of the Bessel function

EXAMPLE 16. Suppose $L = [a, \infty) + i[-3/4\pi, 3/4\pi]$ and k' be an arbitrary real number. We put

$$\langle T, h \rangle = (2\pi i)^{-1} \int_{\partial L_\varepsilon} \exp(2te^\zeta - e^{2\zeta}) h(\zeta) d\zeta\,.$$

Then we have $\tilde{T}(z) = -H_{-z}(t)/\Gamma(1-z)$ and $G_T(w) = \exp(2tw^{-1} - w^{-2})$. In this case the formulas (4) and (5) read as follows:

$$-H_{-z}(t)/\Gamma(1-z) = (2\pi i)^{-1} \int_{\partial L_\varepsilon} \exp(2te^\zeta - e^{2\zeta}) \exp(z\zeta) d\zeta\,, \tag{29}$$

$$\exp(2tw^{-1} - w^{-2}) = \sum_{n=1}^{\infty} H_n(t)(n!)^{-1} w^{-n}\,. \tag{30}$$

(30) is the generating function expansion of the Hermite polynomial $H_n(t)$.

ACKNOWLEDGEMENT. The author wishes to express his hearty gratitude to Prof. M. Morimoto for the valuable suggestions. Especially Example 7 and Example 11 are due to him.

References

[1] V. AVANISSIAN and R. GAY, Sur une transformation des fonctionnelles analytiques et applications aux fonctions entières de plusieurs variables, Bull. Soc. Math. France, **103** (1975), 341–384.

[2] T. INUI, Special Functions, Iwanami Press, Tokyo, 1962 (in Japanese).

[3] W. MAGNUS, F. OBERHETTINGER and R. P. SONI, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer Verlag, New York, 1966.

[4] M. MORIMOTO, On the Fourier-Ultrahyperfunctions 1, Sûrikaiseki Kenkyûjo Kôkyûroku, **192** (1973), 10–34.

[5] M. MORIMOTO, Analytic functionals with non-compact carrier, Tokyo J. Math., **1** (1978), 72–103.

[6] M. MORIMOTO and K. YOSHINO, A uniqueness theorem for the holomorphic functions of exponential type, Hokkaido Math. J., **7** (1978), 259–270.

[7] M. MORIMOTO and K. YOSHINO, Some examples of analytic functionals with carrier at the infinity, Proc. Japan Acad, **56** (1980), 357–361.

[8] J. W. DE ROEVER, Complex Fourier Transformation and Analytic Functionals with Unbounded Carriers, Mathematisch Centrum, Amsterdam 1977.

[9] P. SARGOS and M. MORIMOTO, Transformations des fonctionnelles analytiques à porteurs non-compacts, Tokyo J. Math., **4** (1981), 457–492.

[10] V. V. ZHARINOV, Laplace transformation of Fourier hyperfunctions and related classes of analytic functional, Teoret. Mat. Fiz., **33** (1977), 291–309, (in Russian); English translation, Theoret. and Math. Phys., **33** (1978), 1027–1039.

Present Address:
DEPARTMENT OF MATHEMATICS
SOPHIA UNIVERSITY
KIOI-CHO, CHIYODA-KU, TOKYO 102