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# Kamke's Theorem and the Dependence of Solutions for Delay Differential Equations

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## Introduction

We consider the delay differential equations including state dependent lags,

$$x_i'(t)=f_i(t, x(t), x(g_2(t, x(t))), \cdots, x(g_m(t, x(t)))) , \tag{E'}$$

$i=1, \cdots, d$, under the assumption that $f_i$ satisfies the Caratheodory condition, where $x(t)$ stands for $(x_1(t), \cdots, x_d(t))$ and prime denotes differentiation with respect to $t$. We assume the existence of a finite number $\alpha$ such that for each $j=2, \cdots, m$, $\alpha\leqq g_j(t, x)\leqq t$, whenever $g_j(t, x)$ is defined; the delays $t-g_j(t, x)$ may be unbounded. This type of system arises in studying a two-body problem of classical electrodynamics [6, 7]. Driver [4] developed the basic theory (existence, uniqueness and dependence of solutions, etc.) for the initial value problem for delay differential equations (E′) with continuous $f_i$ [5]. Since then the theory of delay differential equations (E′) has been studied by many authors. Among them Bullock [1] showed the existence theorem and uniqueness theorem for delay differential equations (E′) of Caratheodory type. On the other hand, Strauss and Yorke [13] constructed a fundamental theory for ordinary differential equations by using the convergence theorem which is a generalization of Kamke's theorem (see [8], Theorem 3.2). Their method proves to be very important in studying the fundamental theory of functional (or delay) differential equations. Costello [3] extended their results to functional differential equations of Caratheodory type with finite delay,

$$x'(t)=F(t, x_t) , \tag{E''}$$

where $x_t(\theta)=x(t+\theta)$, $-r\leqq\theta\leqq0$. Rybakowski [14] also states without

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proof Kamke's theorem for the system (E″) under the very mild condition of Caratheodory type. The object of this paper is to give an extension of these results [13] to delay differential equations (E′) together with the convergence theorem, the existence of noncontinuable solutions and the dependence of solutions on initial conditions and on the functions $f_i$. We emphasize that the equations (E′) cannot be represented by the above form, since the delays $t-g_j(t, x)$ may be unbounded. We introduce a modification of uniform convergence for the sequence of continuous functions $\phi_n: [a_n, b_n]\to R^d$, where $\{a_n\}$ and $\{b_n\}$ converge. This modification plays an essential role in proving the convergence theorem, which is very advantageous when we consider the dependence of solutions on initial conditions and on the right-hand side of (E′).

In section 1, we explain explicit discription of the notations and give the initial value problem for delay differential equations (E′). In section 2, we introduce a modification of uniform convergence mentioned above, and by using its properties, the convergence theorem is established, which is one of our main theorem (Theorem 2.1) in the paper. In section 3, by using a nonstandard argument as pointed out in [13], the existence of noncontinuable solutions and the extension of solutions are given. In section 4, from the convergence theorem, the dependence of solutions is obtained.

## §1. Definition and Notations.

Let $R$ (or $R^1$) denote the set of real numbers and $R^d$ denote the $d$-dimensional real Euclidean space. Let

$$(t, X)=(t, x_{11}, \cdots, x_{1d}, x_{21}, \cdots, x_{2d}, \cdots, x_{md})$$

be a vector in $R^{dm+1}$ and $(t, x)=(t, x_{11}, \cdots, x_{1d})$ be the $d+1$-dimensional vector consisting of the first $d+1$ coordinates of $(t, X)$. Let $D$ be a domain (an open connected set) of $R^{dm+1}$ and $D^*$ the set of vectors $(t, x)$ in $R^{d+1}$ such that $(t, x, y)$ lies in $D$ for some $y$ in $R^{d(m-1)}$. In system (E′), the function $f(t, X)$ is a mapping from $D$ into $R^d$ and the function $g(t, x)=(g_1(t, x), \cdots, g_m(t, x))$ is a mapping from $D^*$ into $R^m$, where $g_1(t, x)\equiv t$ for $(t, x)\in D^*$. For a continuous function $x(t)$ mapping an interval $I$ into $R^d$, we define a function $xg: D^*\to R^{dm}$ by $xg(t, \xi)=(x(g_1(t, \xi)), \cdots, x(g_m(t, \xi)))$ for $(t, \xi)\in D^*$, whenever every composition $x(g_j(t, \xi))$, $j=1, \cdots, m$, has a meaning. Using this notation, we can rewrite the system (E′) as

$$x'(t)=f(t, xg(t, x(t))) . \tag{E}$$

We impose the following conditions on the functions $f$ and $g$;

(F-1) $f(t, X)$ is continuous in $X$ for each fixed $t$.

(F-2) $f(t, X)$ is Lebesgue measurable in $t$ for each fixed $X$.

(F-3) For each compact set $Q$ in $D$, $f(t, X)$ is bounded on $Q$.

(G-1) $g(t, x)$ is continuous in $(t, x) \in D^*$.

(G-2) $g_1(t, x) \equiv t$ for all $(t, x) \in D^*$.

(G-3) there exists an $\alpha \in R$ such that $\alpha \leqq g_j(t, x) \leqq t$, $j=2, \cdots, m$, for all $(t, x) \in D^*$.

Let $M_D$ be the class of functions $f$, defined on $D$ in $R^{dm+1}$, which satisfy the conditions (F-1), (F-2) and (F-3) above.

A real $d$-dimensional vector function $\phi(t)=(\phi_1(t), \cdots, \phi_d(t))$ defined on $[\alpha, \sigma_0]$, $\alpha \leqq \sigma_0$, is said to be an initial function if it is continuous and the point $(\sigma_0, \phi g(\sigma_0, \phi(\sigma_0)))$ lies in $D$. For such a function $\phi$, an initial value problem for the delay differential equations (E) is to find an $R^d$-valued function $x(t)=(x_1(t), \cdots, x_d(t))$ defined for $\alpha \leqq t < \beta$, where $\sigma_0 < \beta \leqq \infty$, which satisfies the following conditions;

(I-1) $x(t)=\phi(t)$ for $\alpha \leqq t \leqq \sigma_0$.

(I-2) $x(t)$ is locally absolutely continuous for $\sigma_0 \leqq t < \beta$.

(I-3) $(t, xg(t, x(t))) \in D$ for $\sigma_0 \leqq t < \beta$.

(I-4) $x'(t)=f(t, xg(t, x(t)))$ almost everywhere on $[\sigma_0, \beta)$.

Such a function $x(t)$ (or $x(t, \sigma_0, \phi)$) is called a solution of the system (E) for the initial function $\phi$ on $[\alpha, \sigma_0]$ or simply a solution of (E) through $(\sigma_0, \phi)$. Clearly, a function $x(t)$ defined on $[\alpha, \beta)$ is a solution of (E) through $(\sigma_0, \phi)$ if and only if it satisfies

$$x(t)=\begin{cases} \phi(t) & \text{for} \quad t \in [\alpha, \sigma_0] \\ \phi(\sigma_0)+\int_{\sigma_0}^{t} f(s, xg(s, x(s)))ds & \text{for} \quad t \in [\sigma_0, \beta) . \end{cases}$$

Finally, for a convenience's sake, we introduce some notations. For a continuous function $\phi$ mapping $[a, b]$ into $R^d$, we set $|\phi|_{[a,b]} = \sup\{|\phi(s)|: s \in [a, b]\}$, where $|\cdot|$ is an arbitrary norm in $R^d$. For a set $Q$ in $R^{k+1}$, $\operatorname{int} Q$ denote the interior of $Q$, $\partial Q$ the boundary of $Q$, $\bar{Q}$ the closure of $Q$ and $\operatorname{pr} Q=\{t: (t, x) \in Q$ for some $x \in R^k\}$.

## §2. Convergence theorem.

The result in this section is the main tool to develop the fundamental theory on the initial value problem of (E) in the following sections.

We say that a sequence of compact intervals $\{[a_n, b_n]\}$ converges to a compact interval $[a, b]$ (or $[a_n, b_n] \to [a, b]$) if $a_n \to a$ and $b_n \to b$ as $n \to \infty$. The function $\tau_n$ mapping $[a, b]$ onto $[a_n, b_n]$ is defined as

$$\tau_n t=\tau_n(b_n, a_n, b, a)t=\frac{b_n-a_n}{b-a}t+\frac{a_nb-ab_n}{b-a} \qquad \text{for all} \quad t\in[a, b] ,$$

if $a<b$. To emphasize the domain of a function, we denote by $\phi|[a, b]$ a function $\phi$ defined on $[a, b]$. Also, this symbol means the restriction of $\phi$ on $[a, b]$, where $\phi$ is defined on a domain containing $[a, b]$. First, we introduce a modification of uniform convergence.

DEFINITION 1. A sequence of (continuous) functions, $\{\phi_n|[a_n, b_n]\}$, is said to converge uniformly to a function $\phi|[a, b]$ if the following conditions are satisfied;

(1) $[a_n, b_n]\to[a, b]$ as $n\to\infty$,

(2) in case $a<b$, $|\tilde{\phi}_n-\phi|_{[a,b]}\to 0$ as $n\to\infty$, where $\tilde{\phi}_n(t)=\phi_n(\tau_n(b_n, a_n, b, a)t)$ for $t\in[a, b]$; in case $a=b$,

$$\sup_{a_n\leq t\leq b_n}|\phi_n(t)-\phi(a)|\longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty .$$

It is obvious from the definition that, if a sequence of continuous functions $\{\phi_n|[a_n, b_n]\}$ converges uniformly to a function $\phi|[a, b]$, then the function $\phi(t)$ is continuous on $[a, b]$. In the case $a_n=a$, $b_n=b$ for all $n$, the uniform convergence of the sequence $\{\phi_n|[a_n, b_n]\}$ coincides with the usual uniform convergence.

EXAMPLE 1. If the function $\phi_n(t)$, $n=1, 2, \cdots$, is given by

$$\phi_n(t)=\begin{cases}-nt & \text{for} \quad t\in[-1/n, 0]\\ 0 & \text{for} \quad t\in[0, 1]\\ nt-n & \text{for} \quad t\in[1, 1+1/n] ,\end{cases}$$

then $\{\phi_n|[0, 1]\}$ converges uniformly to $0|[0, 1]$, but $\{\phi_n|[-1/n, 1+1/n]\}$ does not converge uniformly.

EXAMPLE 2. If the function $\psi_n(t)$, $n=1, 2, \cdots$, is given by

$$\psi_n(t)=\begin{cases}-nt+1 & \text{for} \quad t\in[0, 1/n]\\ 0 & \text{for} \quad t\in[1/n, 1-1/n]\\ nt-n+1 & \text{for} \quad t\in[1-1/n, 1] ,\end{cases}$$

then $\{\psi_n|[1/n, 1-1/n]\}$ converges uniformly to $0|[0, 1]$, but $\{\psi_n|[0, 1]\}$ does not converge uniformly.

DEFINITION 2. A sequence of continuous functions, $\{\phi_n|[a_n, b_n]\}$, $n=1, 2, \cdots$, is said to be uniformly bounded if there is a constant $M\geqq 0$ such that $|\phi_n|_{[a_n,b_n]}\leqq M$ for all $n$. The sequence is said to be equicontinuous if for any $\varepsilon>0$, there exists a $\delta>0$ such that $|t-s|<\delta$, $t, s\in[a_n, b_n]$, implies $|\phi_n(t)-\phi_n(s)|<\varepsilon$ for all $n$.

The following lemma is immediately obtained from Definitions 1, 2.

LEMMA 2.1. *Suppose* $[a_n, b_n]\to[a, b]$ *as* $n\to\infty$, *where* $a_n<b_n$, $a<b$. *Then the sequence* $\{\phi_n|[a_n, b_n]\}$ *is equicontinuous if and only if the sequence* $\{\tilde{\phi}_n\}$ *is equicontinuous on* $[a, b]$, *where* $\tilde{\phi}_n(t)=\phi_n(\tau_n t)$, $\tau_n t=\tau_n(b_n, a_n, b, a)t$.

PROOF. If we set $\sigma_n=\tau_n^{-1}$, then $\phi_n=\tilde{\phi}_n\circ\sigma_n$. Since $[a_n, b_n]\to[a, b]$ as $n\to\infty$, all $\tau_n$ and $\sigma_n$ are Lipschitz continuous with a common Lipschitz constant, that is, $|\tau_n t-\tau_n s|\leqq M|t-s|$ for $t, s\in[a, b]$ and $|\sigma_n t-\sigma_n s|\leqq M|t-s|$ for $t, s\in[a_n, b_n]$, where $M$ is a positive constant. Therefore the relations $\tilde{\phi}_n=\phi_n\circ\tau_n$ and $\phi_n=\tilde{\phi}_n\circ\sigma_n$ imply the equivalence between the equicontinuous of $\{\phi_n|[a_n, b_n]\}$ and $\{\tilde{\phi}_n|[a, b]\}$. This completes the proof.

Combining Definition 1, Lemma 2.1 and Ascoli-Arzela's theorem, we obtain immediately the following lemma.

LEMMA 2.2. *Suppose that the function* $\phi_n|[a_n, b_n]$, $n=1, 2, \cdots$, *is continuous, where* $[a_n, b_n]\to[a, b]$ *as* $n\to\infty$. *Then every subsequence of* $\{\phi_n|[a_n, b_n]\}$ *contains a subsequence which converges uniformly if and only if* $\{\phi_n|[a_n, b_n]\}$ *is uniformly bounded and equicontinuous.*

LEMMA 2.3. *Suppose that the function* $\phi_n(t)$, $n=1, 2, \cdots$, *defined on* $[a_n, b_n]$ *is continuous and* $a_n\to a$, $b_n\to b$ *and* $c_n\to c$ *as* $n\to\infty$, *where* $a_n\leqq c_n\leqq b_n$. *Then* $\{\phi_n|[a_n, b_n]\}$ *converges uniformly to* $\phi|[a, b]$ *if and only if* $\{\phi_n|[a_n, c_n]\}$ *and* $\{\phi_n|[c_n, b_n]\}$ *converge uniformly to* $\phi|[a, c]$ *and* $\phi|[c, b]$, *respectively. In particular, if* $\{\phi_n|[a_n, b_n]\}$ *converges uniformly to* $\phi|[a, b]$, *then* $\phi_n(c_n)$ *converges to* $\phi(c)$.

PROOF. We prove only in the case $a<c<b$. Set $\tau_n t=\tau_n(b_n, a_n, b, a)t$, $\tau_n^1 t=\tau_n(c_n, a_n, c, a)t$ and $\tau_n^2 t=\tau_n(b_n, c_n, b, c)t$. Then we have

$$(2.1)\qquad \tilde{\phi}_n(t)-\phi(t)=\begin{cases}\phi_n(\tau_n t)-\phi_n(\tau_n^1 t)+\phi_n(\tau_n^1 t)-\phi(t) & \text{for} \quad t\in[a, c]\\ \phi_n(\tau_n t)-\phi_n(\tau_n^2 t)+\phi_n(\tau_n^2 t)-\phi(t) & \text{for} \quad t\in[c, b]\,,\end{cases}$$

where $\tilde{\phi}_n(t)=\phi_n(\tau_n t)$. Suppose that $\{\phi_n|[a_n, b_n]\}$ converges uniformly to $\phi|[a, b]$. Then Lemma 2.2 says that the sequence $\{\phi_n|[a_n, b_n]\}$ is equicontinuous. Therefore, since $|\tau_n-\tau_n^1|_{[a,c]}$, $|\tau_n-\tau_n^2|_{[c,b]}\to 0$ as $n\to\infty$, we have $|\phi_n\circ\tau_n-\phi_n\circ\tau_n^1|_{[a,c]}$, $|\phi_n\circ\tau_n-\phi_n\circ\tau_n^2|_{[c,b]}\to 0$ as $n\to\infty$. Relation (2.1) implies $|\phi_n\circ\tau_n^1-\phi|_{[a,c]}$, $|\phi_n\circ\tau_n^2-\phi|_{[c,b]}\to 0$ as $n\to\infty$. This shows the "only if" part of the lemma, and vice versa.

The following lemma is important to prove the main theorem.

LEMMA 2.4. *Let* $f\in M_Q$, *where* $Q$ *is a compact subset of* $D$. *Suppose that a continuous function* $x_n(t)$, $n=1, 2, \cdots$, *defined on* $[\alpha, \beta_n]$ *satisfies*

$$x_n(t)=x_n(\sigma_n)+\int_{\sigma_n}^{t} f(s,\ x_ng(s,\ x_n(s)))ds+G_n(t) \quad for \quad t\in[\sigma_n,\ \beta_n]\ ,$$

*where* $\alpha\leqq\sigma_n<\beta_n$, $\{[\sigma_n, \beta_n]\}$ *converges to* $[\sigma_0, \beta]$, $\sigma_0<\beta$, *and that* $(t, x_ng(t, x_n(t)))\in Q$ *for all* $t\in[\sigma_n, \beta_n]$. *If* $\{x_n|[\alpha, \sigma_n]\}$ *and* $\{|G_n||[\sigma_n, \beta_n]\}$ *converge uniformly to* $\phi|[\alpha, \sigma_0]$ *and* $0|[\sigma_0, \beta]$, *respectively, then there exists a subsequence of* $\{x_n|[\alpha, \beta_n]\}$ *which converges uniformly to a solution* $x|[\alpha, \beta]$ *of* (E) *through* $(\sigma_0, \phi)$.

PROOF. We set

$$\psi_n(t)=\int_{\sigma_n}^{t} f(s,\ x_ng(s,\ x_n(t)))ds \quad \text{for} \quad t\in[\sigma_n,\ \beta_n]\ .$$

Since the function $f$ is bounded on the compact set $Q$, the sequence $\{\psi_n|[\sigma_n, \beta_n]\}$ is uniformly bounded and equicontinuous. Hence, from Lemma 2.2, by taking a subsequence if necessary, we may assume that $\{\psi_n|[\sigma_n, \beta_n]\}$ converges uniformly. Lemma 2.3 implies $x_n(\sigma_n)\to\phi(\sigma_0)$ as $n\to\infty$, while $\{|G_n||[\alpha_n, \beta_n]\}$ converges uniformly to $0|[\sigma_0, \beta]$. Since $x_n(t)=x_n(\sigma_n)+\psi_n(t)+G_n(t)$ for $\sigma_n\leqq t\leqq\beta_n$ the sequence $\{x_n|[\sigma_n, \beta_n]\}$ converges uniformly to a function $x|[\sigma_0, \beta]$. From Lemma 2.3, it follows that $\{x_n|[\alpha, \beta_n]\}$ converges to $x|[\alpha, \beta]$, where $x|[\alpha, \sigma_0]=\phi|[\alpha, \sigma_0]$.

Let $t$ be a fixed number in $(\sigma_0, \beta]$. Then for any $\varepsilon$, $(1/3)(t-\sigma_0)>\varepsilon>0$, there exists an $N_1$ such that $n>N_1$ implies $|\sigma_n-\sigma_0|<\varepsilon$ and $|\tau_nt-t|<\varepsilon$, where $\tau_nt=\tau_n(\beta_n, \sigma_n, \beta, \sigma_0)t$. Thus for all $n>N_1$, the function $x_n(t)$ is defined on $[\alpha, t-\varepsilon]$. For a fixed $s$, $\sigma_0+\varepsilon\leqq s\leqq t-\varepsilon$, it follows from the hypotheses (G-1) and (G-3) that $\alpha\leqq g_j(s, x_n(s))\leqq s$ and $g_j(s, x_n(s))\to g_j(s, x(s))$ as $n\to\infty$. Hence from Lemma 2.3 we have $x_ng_j(s, x_n(s))\to xg_j(s, x(s))$ as $n\to\infty$, that is, $x_ng(s, x_n(s))\to xg(s, x(s))$ as $n\to\infty$. Using the hypothesis (F-1), we obtain $f(s, x_ng(s, x_n(s)))\to f(s, xg(s, x(s)))$ as $n\to\infty$. Moreover, by using the Lebesgue dominated convergence theorem, for any $\varepsilon_1>0$, we can find $N_2$ such that $n>N_2$ implies

$$\left|\int_{\sigma_0+\varepsilon}^{t-\varepsilon} f(s,\ x_ng(s,\ x_n(s)))ds-\int_{\sigma_0+\varepsilon}^{t-\varepsilon} f(s,\ xg(s,\ x(s)))ds\right|<\varepsilon_1\ .$$

If we set

$$\psi(t)=\int_{\sigma_0}^{t} f(s,\ xg(s,\ x(s)))ds \quad \text{for} \quad t\in[\sigma_0,\ \beta]\ ,$$

then, by dividing the intervals of integrals, for all $n>N$, $N=\max\{N_1, N_2\}$, we have

$$\begin{aligned}&|\psi_n(\tau_nt)-\psi(t)|\\&\leqq\left|\int_{\sigma_n}^{\tau_nt} f(s,\ x_ng(s,\ x_n(s)))ds-\int_{\sigma_0}^{t} f(s,\ xg(s,\ x(s)))ds\right|\end{aligned}$$

$$<6M\varepsilon+\varepsilon_1\ ,$$

where $M=\sup_Q|f|$. Since $t\in[\sigma_0, \beta]$ is arbitrary, we have $x(t)=\phi(\sigma_0)+\psi(t)$ for $t\in[\sigma_0, \beta]$. Namely $x(t)$ is a solution of (E) through $(\sigma_0, \phi)$. This proves the lemma.

PROPOSITION. *Let $f\in M_Q$, where $Q$ is a compact subset of domain $D\subset R^{d+1}$ and $(\sigma_0, x_0)\in \operatorname{int} Q$. Let $x_n(t)$ be defined on $[a_n, b_n]$ and let $(\sigma_n, x_n)\to(\sigma_0, x_0)$, $[a_n, b_n]\to[a, b]$, $a<b$, as $n\to\infty$, where $\sigma_n\in[a_n, b_n]$. Suppose a function $G_n(t)$ is defined by*

$$G_n(t)=x_n(t)-x_n-\int_{\sigma_n}^{t} f(s, x_n(s))ds$$

*for all $t\in[a_n, b_n]$ and that $(t, x_n(t))\in Q$ for $t\in[a_n, b_n]$. If $x_n(t)$ is measurable on $[a_n, b_n]$ and if $|G_n|_{[a_n,b_n]}\to 0$ as $n\to\infty$, then $\{x_n|[a_n, b_n]\}$ contains a subsequence which converges uniformly on $[a, b]$ to a solution $x_0(t)$ of*

$$(\mathrm{E}_0)\qquad x'(t)=f(t, x(t))\ ,\qquad x(\sigma_0)=x_0\ .$$

PROOF. Since $x_n(t)$ is measurable on $[a_n, b_n]$, $x_n-G_n|[a_n, b_n]$ is continuous. Therefore $\{x_n-G_n|[a_n, b_n]\}$ is uniformly bounded and equicontinuous. From Lemma 2.2, by taking a subsequence if necessary, we may assume that $\{x_n-G_n|[a_n, b_n]\}$ converges uniformly to a function $x_0(t)$ defined on $[a, b]$. From the same argument used in the proof of Lemma 2.4, it follows that $x_0(t)$ is a solution of $(\mathrm{E}_0)$.

This proposition generalizes Lemma 1 in [13]. In fact, if $[a_n, b_n]=[a, b]$ for all $n$, proposition coincides with Lemma 1 in [13].

DEFINITION 3. Let $D$ be a domain of $R^{dm+1}$ and let $g$ be an $R^m$-valued function defined on $D^*$, which satisfies (G-1, 2, 3). Let $x(t)$ be a continuous function mapping some interval $I$ into $R^d$. Then $x(t)$ is said to be a noncontinuable function (with respect to $D$ and $g$) if $(t, xg(t, x(t)))\in D$ for all $t\in I$ and if there exists a sequence $\{t_n\}$ in $I$ such that $t_n\to\sup I$ as $n\to\infty$ and that for each compact subset $Q$ of $D$, there exists a number $N=N(Q)$ such that

$$(t_n, xg(t_n, x(t_n)))\in D-Q\qquad \text{for all}\quad n\geqq N\ .$$

Furthermore, if $x(t)$ is a solution of (E) and $D$ is the domain of $f$, then it is called a noncontinuable solution of (E). If a noncontinuable solution $x(t)$ of (E) through $(\sigma_0, \phi)$ is defined on $[\sigma_0, \beta)$, $\sigma_0<\beta$, then we sometimes write $D_x(\sigma_0)=[\sigma_0, \beta)$ or $D_x=[\sigma_0, \beta)$.

Although Definition 3 above may look slightly different from the usual definition for noncontinuable solution (cf. [7, 9]), but as we shall see later they are equivalent to each other (Theorem 3.2).

DEFINITION 4 [13]. Let $x(t)$ be a continuous function mapping some (possibly unbounded) interval $I$ into $R^d$. A sequence of continuous functions $\{x_n(t)\}$ is said to converge compactly to $x(t)$ if for every compact subset $J$ of $I$, $x_n(t)$ is defined on $J$ for sufficiently large $n$ and $\{x_n|J\}$ converges uniformly to $x|J$.

The following is the main result of this section.

THEOREM 2.1. *Let $f \in M_D$ and $x_n(t)$ be a noncontinuable function, defined on $[\alpha, \beta_n)$, which satisfies*

$$(\mathrm{E}_1) \qquad x_n(t) = x_n(\sigma_n) + \int_{\sigma_n}^{t} f(s, x_n g(s, x_n(s)))ds + G_n(t) \qquad \text{for} \quad t \in [\sigma_n, \beta_n)\,,$$

*where $\alpha \leqq \sigma_n < \beta_n$, $\alpha \leqq \sigma_0$, and $\sigma_n \to \sigma_0$ as $n \to \infty$. Suppose that the sequence $\{x_n|[\alpha, \sigma_n]\}$ converges uniformly to $\phi|[\alpha, \sigma_0]$, $(\sigma_0, \phi g(\sigma_0, \phi(\sigma_0))) \in D$, and that for every compact subset $Q$ of $D$ with $(\sigma_0, \phi g(\sigma_0, \phi(\sigma_0))) \in \operatorname{int} Q$, there exists a sequence $\{\omega_n(Q)\}$ such that $|G_n(t)| \leqq \omega_n(Q)$ on $[\sigma_n, t_n^*]$ and $\omega_n(Q) \to 0$ as $n \to \infty$, where $t_n^* = t_n^*(Q) = \inf\{t > \sigma_n;\ (t, x_n g(t, x_n(t))) \in \partial Q\}$. Then there exists a noncontinuable solution $x(t)$ of* (E) *through $(\sigma_0, \phi)$ and a subsequence of $\{x_n(t)\}$ which converges compactly to $x(t)$.*

PROOF. Since $(\sigma_n, x_n g(\sigma_n, x_n(\sigma_n))) \to (\sigma_0, \phi g(\sigma_0, \phi(\sigma_0))) \in D$ *as* $n \to \infty$, we can choose a sequence of compact sets $\{Q_k\}$ such that $Q_k \in \operatorname{int} Q_{k+1}$, $D = \cup_{k=0}^{\infty} Q_k$ and $(\sigma_n, x_n g(\sigma_n, x_n(\sigma_n))) \in \operatorname{int} Q_0$ for all $n$. Put $t_n(k) = t_n^*(Q_k)$. Then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{t_{n_i}(k)\}$ converges as $i \to \infty$ for all $k$. In fact, since

$$(2.2) \qquad (t_n(k), x_n g(t_n(k), x_n(t_n(k)))) \in \partial Q_k \qquad \text{for all } n\,,$$

$\{t_n(k)\}_{n=1}^{\infty}$ is contained in the compact set $\operatorname{pr} Q_k$. Thus first we can choose a subsequence $\{t_{n(i,0)}(0)\}_{i=1}^{\infty}$ of $\{t_n(0)\}$ which converges. Next we can also choose a subsequence $\{n(i, 1)\}_{i=1}^{\infty}$ of $\{n(i, 0)\}_{i=1}^{\infty}$ such that $\{t_{n(i,1)}(1)\}_{i=1}^{\infty}$ converges. Continuing in this fashion, one obtains the required subsequence by the well known diagonal method. We denote such a subsequence by $\{x_n\}$ again. Furthermore, since $Q_{k-1} \subset \operatorname{int} Q_k$, we have $t_n(k-1) < t_n(k)$ for all $n$. If we set $W_k = \lim_{n\to\infty} t_n(k)$, then we obtain

$$(2.3) \qquad \sigma_0 \leqq W_0 \leqq W_1 \leqq \cdots \leqq W_k \leqq \cdots\,.$$

Put $W = \lim_{k\to\infty} W_k$.

Next, by taking a subsequence if necessary, we shall show that for all $k$, $\{x_n|[\alpha, t_n(k)]\}$ converges uniformly. Since $t_n(k) \to W_k$ as $n \to \infty$, Lemma 2.4 assures that there exists a subsequence of $\{x_n|[\alpha, t_n(k)]\}$ which converges uniformly. Thus, by the familiar diagonal method, we can extract a convergent sequence. Now, let the limit of the sequence $\{x_n|[\alpha, t_n(k)]\}$ be denoted by $x^k|[\alpha, W_k]$. Since $\sigma_0 \leqq t_n(k-1) < t_n(k)$ for all $n$, we have $x^{k-1}|[\alpha, W_{k-1}] = x^k|[\alpha, W_{k-1}]$ by Lemma 2.3. From this fact, there exists a unique function $x|[\alpha, W)$ such that $x|[\alpha, W_k] = x^k|[\alpha, W_k]$ for all $k$. Consequently, for all $k$ the sequence $\{x_n|[\alpha, t_n(k)]\}$ converges uniformly to $x|[\alpha, W_k]$.

By using this fact and Lemma 2.3, we have $(t, xg(t, x(t))) \in Q_k$ for all $t \in [\sigma_0, W_k]$, and hence $(t, xg(t, x(t))) \in D$ for all $t \in [\sigma_0, W)$. Moreover, since $x_n(t_n(k)) \to x(W_k)$ as $n \to \infty$, it follows from the continuity of $g$ that $g(t_n(k), x_n(t_n(k))) \to g(W_k, x(W_k))$ as $n \to \infty$. However, from the hypothesis (G-3), we have $\alpha \leqq g(t_n(k), x_n(t_n(k))) \leqq t_n(k)$. Thus by Lemma 2.3 again, we have $x_n g(t_n(k), x_n(t_n(k))) \to xg(W_k, x(W_k))$ as $n \to \infty$. Therefore it follows from (2.2) that

$$(W_k, xg(W_k, x(W_k))) \in \partial Q_k . \tag{2.4}$$

From (2.3) and $Q_{k-1} \subset \operatorname{int} Q_k$, we have

$$\sigma_0 \leqq W_0 < W_1 < \cdots < W_k < W_{k+1} < \cdots \to W . \tag{2.5}$$

Thus from (2.5) and $t_n(k+1) \to W_{k+1}$ as $n \to \infty$, it follows that $[\alpha, W_k] \subset [\alpha, t_n(k+1)]$ for safficiently large $n$. By Lemma 2.3 again, $\{x_n|[\alpha, W_k]\}$ converges uniformly to $x|[\alpha, W_k]$, and hence $\{x_n|[\alpha, \beta_n)\}$ converges compactly to $x|[\alpha, W)$.

Finally, we show that $x(t)$ is a noncontinuable solution of (E). For any compact set $Q$ in $D$ there exists an $N = N(Q)$ such that $Q$ is contained in $\operatorname{int} Q_k$ for all $k \geqq N$. Then the relation (2.4) means that $(W_k, xg(W_k, x(W_k))) \in D - Q$ for all $k \geqq N$. In view of Lemma 2.4 the function $x(t)$ is a noncontinuable solution of (E). This completes the proof.

REMARK. Under the same assumptions as in Theorem 2.1, if a subsequence $\{x_{n_i}(t)\}$ of $\{x_n(t)\}$ converges compactly to noncontinuable solution $x(t)$ of (E) through $(\sigma_0, \phi)$, then we have $\sup D_x \leqq \liminf_{i\to\infty} \sup D_{x_{n_i}}$.

COROLLARY 2.1. *Let $f^n, f \in M_D$, $n = 1, 2, \cdots$. Let a noncontinuable function $x_n(t)$ defined on $[\alpha, \beta_n)$ satisfies*

$$x_n(t) = x_n(\sigma_n) + \int_{\sigma_n}^{t} f^n(s, x_n g(s, x_n(s)))ds \quad \text{for} \quad t \in [\sigma_n, \beta_n) \tag{$E_2$}$$

*where, $\alpha \leqq \sigma_n < \beta_n$, $\alpha \leqq \sigma_0$ and $\sigma_n \to \sigma_0$ as $n \to \infty$. If the sequence $\{x_n|[\alpha, \sigma_n]\}$*

*converges uniformly to* $\phi|[\alpha, \sigma_0]$, $(\sigma_0, \phi g(\sigma_0, \phi(\sigma_0))) \in D$, *and if*

$$\text{(2.6)} \qquad \lim_{n\to\infty} \int_{\mathrm{pr}\, Q} \sup_{X\in Q_t} |f^n(t, X)-f(t, X)|\, dt=0,\ Q_t=\{X: (t, X)\in Q\}\ ,$$

*for every compact set* $Q\in D$ *such that* $(\sigma_0, \phi g(\sigma_0, \phi(\sigma_0)))$ *is in* $\mathrm{int}\, Q$, *then there exists a subsequence of* $\{x_n(t)\}$ *which converges compactly to a noncontinuable solution of* (E) *through* $(\sigma_0, \phi)$.

PROOF. If we set

$$G_n(t)=\int_{\sigma_n}^{t} |f^n(s, x_n g(s, x_n(s)))-f(s, x_n g(s, x_n(s)))|\, ds\ ,$$

then we can rewrite $(E_2)$ as $(E_1)$. Given any such compact set $Q\subset D$, put

$$\omega_n(Q)=\int_{\mathrm{pr}\, Q} \sup_{X\in Q_t} |f^n(t, X)-f(t, X)|\, dt\ .$$

Since $\omega_n(Q)\to 0$ as $n\to\infty$, from (2.6) and $G_n(t)\leqq\omega_n(Q)$, we have the conclusion.

COROLLARY 2.2. *Let* $f^n$, $f$ *be continuous functions on a domain* $D$ *in* $R^{dm+1}$. *If we replace condition* (2.6) *in Corollary* 2.1 *by the condition that* $f^n$ *converges compactly to* $f$ *on* $D$, *then the conclusion of Corollary* 2.1 *is true.*

We note that recently, Kato [12] shows Kamke's theorem in functional differential equations with infinite delay on an abstract phase space (cf. [10]).

## §3. Existence of noncontinuable solutions.

In this section, by applying Theorem 2.1 we show that a noncontinuable solution of (E) through $(\sigma_0, \phi)$ always exists.

THEOREM 3.1. *Suppose* $f\in M_D$. *Then for every initial function* $\phi|[\alpha, \sigma_0]$, *there exists a noncontinuable solution of* (E) *through* $(\sigma_0, \phi)$.

PROOF. For each $n$, define $x_n(t)$ as follows;

$$x_n(t)=\begin{cases} \phi(t) & \text{for } \alpha\leqq t\leqq\sigma_0 \\ \phi(\sigma_0) & \text{for } \sigma_0\leqq t\leqq\sigma_0+\dfrac{1}{n} \\ \phi(\sigma_0)+\displaystyle\int_{\sigma_0}^{t-1/n} f(s, x_n g(s, x_n(s)))\, ds & \text{for } \sigma_0+\dfrac{1}{n}\leqq t\ . \end{cases}$$

Note first that this formula is meaningful for $n$ large enough since functions $\phi$, $g$ are continuous and $(\sigma_0, \phi g(\sigma_0, \phi(\sigma_0))) \in D$. In this recursive way $x_n$ is defined on $[\sigma_0+k/n, \sigma_0+(k+1)/n]$ for $k=0, 1, \cdots,$ provided $(s, x_n g(s, x_n(s)))$ belongs to $D$ on the previous interval. Thus this process can be continued either for all $t$ or until the first point $\beta_n$ at which $(\beta_n, x_n g(\beta_n, x_n(\beta_n))) \in \partial D$. Hence each $x_n(t)$ is continuous on $[\alpha, \beta_n)$ and noncontinuable with respect to $D$ and $g$. Furthermore, $x_n(t)$ satisfies

$$x_n(t)=\begin{cases}\phi(t) & \text{for } \alpha\leqq t\leqq \sigma_0\\ \phi(\sigma_0)+\int_{\sigma_0}^t f(s, x_n g(s, x_n(s)))ds+G_n(t) & \text{for } \sigma_0\leqq t<\beta_n\,,\end{cases}$$

where

$$G_n(t)=\begin{cases}-\int_{\sigma_0}^t f(s, x_n g(s, x_n(s)))ds & \text{for } \sigma_0\leqq t\leqq\sigma_0+\dfrac{1}{n}\\ -\int_{t-1/n}^t f(s, x_n g(s, x_n(s)))ds & \text{for } \sigma_0+\dfrac{1}{n}\leqq t<\beta_n\,.\end{cases}$$

For every compact set $Q$ in $D$ with $(\sigma_0, \phi g(\sigma_0, \phi(\sigma_0))) \in \operatorname{int} Q$, we have $|G_n(t)|\leqq(1/n)M_Q$ for $t\in[\sigma_0, t_n^*(Q)]$, where $M_Q=\sup_Q|f|$. Therefore we obtain the conclusion from Theorem 2.1.

Bullock [1] stated without proof the local existence of solution for (E) of Caratheodory type, but we obtain immediately the existence of noncontinuable solutions of (E) without appealing to the local existence.

REMARK. Let $f\in M_D$, $D=(a, b)\times R^{dm}$, and let $x(t)$ is a noncontinuable solution of (E). We note from Definition 3 that if $c=\sup D_x<b$, then $\limsup_{t\to c-0}|x(t)|=+\infty$. This conclusion does not necessarily imply $\lim_{t\to c-0}|x(t)|=+\infty$. Papers [2, 11] pointed out that, in general, for delay differential equations a noncontinuable solution $x(t)$ may not have the property

$$\lim_{t\to c-0} x(t)=\limsup_{t\to c-0} x(t)=\infty\ .$$

In particular, Herdman [11] constructed a counter example.

THEOREM 3.2. *Let $D$ be a domain of $R^{dm+1}$ and let $x(t)$ be a continuous function mapping some interval $I$ into $R^d$. Then the following statements are equivalent:*

(1) *$x(t)$ is not a noncontinuable function with respect to $D$ and $g$.*

(2) *Put $S=\{(t, xg(t, x(t))): t\in I\}\subset D$. Then the closure $\bar{S}$ of $S$ is compact and $\bar{S}\subset D$.*

*Moreover, if* $f \in M_D$ *and* $x(t)$ *is a solution of* (E), *then they are equivalent to the following statement*:

(3) $\lim_{t\to b-0} xg(t, x(t)) = X_0$ *and* $(b, X_0) \in D$, *where* $b = \sup I$.

PROOF. If (1) is satisfied, then $S$ is bounded, and hence $\bar{S}$ is compact. Let now $(t_n, Y_n) \in S \to (b, Y_0) \in \partial\bar{S}$ as $n \to \infty$, where $Y_n = xg(t_n, x(t_n))$. Then it follows from (1) that the point $(b, Y_0)$ belongs to $D$, that is, $\bar{S} \subset D$. One easily sees that (2) implies (1) and that (3) implies (2). Thus we show that (2) implies (3). Since $f$ is bounded on the compact set $\bar{S}$ from (F-3), we have $|x(t)-x(s)| \leqq M(\bar{S})|t-s|$ for all $t, s \in I$, where $M(\bar{S}) = \sup_{\bar{S}} |f|$. Hence, by the Cauchy convergence criterion $\lim_{t\to b-0} x(t)$ exists, so that $X_0 = \lim_{t\to b-0} xg(t, x(t))$ exists and $(b, X_0)$ is in $D$. This completes the proof.

REMARK. It follows from Theorem 3.2 that Definition 3 is equivalent to the standard definition for noncontinuable solutions (cf. [7, 9]).

THEOREM 3.3. (Extension). *Let* $f \in M_D$. *Then every solution of* (E) *can be extended to a noncontinuable solution.*

PROOF. Let $x(t)$, defined on $[\sigma_0, b)$, be not a noncontinuable solution of (E) through $(\sigma_0, \phi)$. Then it follows from Theorem 3.2 that $\lim_{t\to b-0} xg(t, x(t)) = X_0$ and $(b, X_0) \in D$. Thus by Theorem 3.1 we obtain that there is a noncontinuable solution of (E) through $(b, \tilde{x})$, where $\tilde{x}(t) = x(t)$ for $t \in [\alpha, b)$ and $\tilde{x}(b) = x_0$.

THEOREM 3.4. *Suppose* $f \in M_D$. *Then of all intervals on which some noncontinuable solution* (E) *through* $(\sigma_0, \phi)$ *is defined, there is a smallest interval on which such a noncontinuable solution is defined.*

PROOF. Let $NS$ denote the family of all the noncontinuable solutions of (E) through $(\sigma_0, \phi)$ and set $J = \bigcap_{x \in NS} D_x$. If we set $\beta = \sup J$, then there exists a sequence $\{\beta_n\}$ such that $\beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_n \geqq \cdots \to \beta$ as $n \to \infty$, where $\beta_n$ is the right end point of a noncontinuable solution $x_n(t)$, that is, $D_{x_n} = [\sigma_0, \beta_n)$. By applying Theorem 2.1, we find a subsequence of $\{x_n\}$ which converges compactly to a noncontinuable solution $x(t)$ of (E) through $(\sigma_0, \phi)$. Then in view of Remark of Theorem 2.1, we have $D_x \subseteqq J$. From the definition of $J$ we have $J \subseteqq D_x$, and hence $D_x = J$. This implies that a noncontinuable solution of (E) through $(\sigma_0, \phi)$ defined on $J$ exists. This completes the proof.

## §4. Dependence of solutions.

In this section, we shall discuss the dependence of solutions on the initial condition and on the right hand sides of the delay differential equations (E). In particular, the result on the dependence of solutions due to Yoshizawa [15. Theorem 5.1] for ordinary differential equations shall be extended to our system (E).

THEOREM 4.1. *Let* $f \in M_D$ *and let* $I=[\sigma_0, b]$, $\sigma_0<b$, *be the subinterval of* $J$, *where* $J$ *is the smallest interval given in Theorem* 3.4. *Then for any* $\varepsilon>0$, *there exists a* $\delta>0$ *for which the following condition holds: if* $|\sigma-\sigma_0|<\delta$, $|\tilde{\phi}-\phi_0|_{[\alpha,\sigma_0]}<\delta$ *in case* $\alpha<\sigma_0(|\phi(\alpha)-\phi_0(\alpha)|<\delta$ *in case* $\alpha=\sigma_0)$, *where* $\tilde{\phi}=\phi(\tau(\sigma, \alpha, \sigma_0, \alpha)t)$, *and if* $q(t)$, $t \in J_1=[\alpha, \sigma_0] \cup J$, *is a bounded and integrable function with* $\int_{J_1}|q(t)|dt<\delta$, *then every noncontinuable solution* $y(t, \sigma, \phi)$ *of*

$$(\mathrm{E}_3) \qquad y'(t)=f(t, yg(t, y(t)))+q(t) ,$$

*is always continued beyond* $t=b$ *and satisfies*

$$(4.1) \qquad |y(t, \sigma, \phi)-x(t, \sigma_0, \phi_0)|<\varepsilon \quad \text{for} \quad t \in [\max\{\sigma, \sigma_0\}, b]$$

*for some noncontinuable solution* $x(t, \sigma_0, \phi_0)$ *of* (E) *which may depend on* $y(t, \sigma, \phi)$.

PROOF. It is sufficient to prove only in the case $\alpha<\sigma_0$. First, we show that there exists a $\delta_0>0$ such that, if $|\sigma-\sigma_0|<\delta_0$, $|\tilde{\phi}-\phi_0|_{[\alpha,\sigma_0]}<\delta_0$ and $\int_{J_1}|q(t)|dt<\delta_0$, then $\sup D_y>b$ for every noncontinuable solution $y(t, \sigma, \phi)$ of $(\mathrm{E}_3)$. Suppose not. Then there exist sequences $\{\sigma_k\}$, $\{\phi_k|[\alpha, \sigma_k]\}$ and $\{q_k(t)\}$ such that

$$(4.2) \qquad \sigma_k \longrightarrow \sigma_0,\ \phi_k|[\alpha, \sigma_k] \longrightarrow \phi|[\alpha, \sigma_0] \quad \text{and} \quad \int_{J_1}|q_k(t)|dt \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty$$

and moreover $\sup D_{y_k} \leqq b$, where $y_k(t)$ is a noncontinuable solution of

$$(4.3) \qquad y(t)=\begin{cases} \phi_k(t) & \text{for} \quad \alpha \leqq t \leqq \sigma_k \\ \phi_k(\sigma_k)+\int_{\sigma_k}^t \hat{f}(s, yg(s, y(s)))ds+\int_{\sigma_k}^t q_k(s)ds & \text{for} \quad \sigma_k<t , \end{cases}$$

where $\hat{f}$ denotes the restriction of $f$ on $D \cap (J_1 \times R^{dm})$. Since

$$\left|\int_{\sigma_k}^t q_k(s)ds\right| \leqq \int_{J_1} |q_k(s)|\, ds \equiv \omega_k \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty ,$$

it follows from Theorem 2.1 that there exists a subsequence of $\{y_k(t)\}$, denoted by $\{y_k(t)\}$ again, which converges compactly to a noncontinuable solution $y_0(t, \sigma_0, \phi_0)$ of (E). By Remark of Theorem 2.1, we have

$$\sup D_{y_0} \leqq \liminf_{k\to\infty} [\sup D_{y_k}] \leqq b ,$$

a contradiction.

Next we show the inequality (4.1) holds. Now suppose the conclusion is false. Then there exist $\{\sigma_k\}$ $\{\phi_k\}$ and $\{q_k\}$ such that the condition (4.2) holds and that the equation (4.3) has a noncontinuable solution $y_k(t)$ having the property that for any noncontinuable solution $x(t, \sigma_0, \phi_0)$ of (E),

$$|y_k(t_k) - x(t_k, \sigma_0, \phi_0)| \geqq \varepsilon \qquad \text{for some} \quad t_k \in [\max\{\sigma_k, \sigma_0\}, b] ,$$

where $t_k$ may depend on $x(t, \sigma_0, \phi_0)$. By repeating the same argument, it follows that for a sufficiently large $k$,

$$|y_k(t) - y_0(t, \sigma_0, \phi_0)| < \varepsilon \qquad \text{for all} \quad t \in [\alpha, b] ,$$

a contradiction. This completes the proof.

THEOREM 4.2. *Let $F$, $f \in M_D$ and let $I = [\sigma_0, b]$, $\sigma_0 < b$, be the subinterval of $J$, where $J$ is the smallest interval in Theorem* 4.1. *Then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $|\sigma - \sigma_0| < \delta$, $|\tilde{\phi} - \phi_0|_{[\alpha, \sigma_0]} < \delta$ in case $\alpha < \sigma_0$ ($|\phi(\alpha) - \phi_0(\alpha)| < \delta$ in case $\alpha = \sigma_0$), where $\tilde{\phi}(t) = \phi(\tau(\sigma, \alpha, \sigma_0, \alpha)t)$, and if $F$ satisfies*

$$\int_{\mathrm{pr}\, Q} \sup_{X \in Q_t} |f(t, X) - F(t, X)| \, dt < \delta$$

*for every compact set $Q \subset D$, then every noncontinuable solution $y(t, \sigma, \phi)$ of*

$$y'(t) = F(t, yg(t, y(t)))$$

*satisfies*

$$|y(t, \sigma, \phi) - x(t, \sigma_0, \phi_0)| < \varepsilon \qquad \textit{for all} \quad t \in [\max\{\sigma, \sigma_0\}, b] ,$$

*where $x(t, \sigma_0, \phi_0)$ is a noncontinuable solution of* (E) *which may depend on $y(t, \sigma, \phi)$.*

We can easily prove this theorem by Corollary 2.1 and by the argument used in the proof of Theorem 4.1.

COROLLARY 4.1. *Suppose that $f$ is in the class $M_D$ and that a noncontinuable solution $x(t, \sigma_0, \phi_0)$ of* (E) *through $(\sigma_0, \phi_0)$ is defined on $D_x$ and is unique. Let $I = [\sigma_0, b]$, $\sigma_0 < b$, be the subinterval of $D_x$. Then for $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $|\sigma - \sigma_0| < \delta$, $|\tilde{\phi} - \phi_0|_{[\alpha, \sigma_0]} < \delta$ in case $\alpha < \sigma_0$*

($|\phi(\alpha)-\phi_0(\alpha)|<\delta$ *in case* $\alpha=\sigma_0$), *where* $\tilde{\phi}(t)=\phi(\tau(\sigma, \alpha, \sigma_0, \alpha)t)$, *and if* $q(t)$, $t\in J_1=[\alpha, \sigma_0]\cup D_x$, *is a bounded and integrable function with* $\int_{J_1}|q(t)|dt<\delta$, *then every noncontinuable solution* $y(t, \sigma, \phi)$ *of* ($E_3$) *satisfies* (4.1).

Note that the result corresponding to Corollary 4.1 for Theorem 4.2 can be obtained similary.

The following is the familiar theorem on the continuous dependence of solutions on initial conditions.

COROLLARY 4.2. *Let* $f\in M_D$. *Suppose that a solution* $x(t, \sigma_0, \phi_0)$ *of* (E) *through* $(\sigma_0, \phi_0)$ *is defined on* $[\sigma_0, \sigma_0+A]$, $A>0$, *and is unique. Then for any* $\varepsilon>0$, *there exists a* $\delta>0$ *such that, if* $|\sigma-\sigma_0|<\delta$ *and* $|\tilde{\phi}-\phi_0|_{[\alpha,\sigma_0]}<\delta$ *in case* $\alpha<\sigma_0$ ($|\phi(\alpha)-\phi_0(\alpha)|<\delta$ *in case* $\alpha=\sigma_0$), *where* $\tilde{\phi}(t)=\phi(\tau(\sigma, \alpha, \sigma_0, \alpha)t)$, *then every solution* $x(t, \sigma, \phi)$ *of* (E) *satisfies* $|x(t, \sigma, \phi)-x(t, \sigma_0, \phi_0)|<\varepsilon$ *for all* $t\in[\max\{\sigma, \sigma_0\}, \sigma_0+A]$.

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