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Simple Links and Tangles

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Introduction

In [2] Lickorish produced many prime knots and links in S^3 by using prime tangles. In this paper we give the analogue for simple knots and links.

We work in the piecewise linear category and refer to [2] for the definitions of *tangles*, *prime tangles* and so on. Tangles, in the paper, are always 2-string. We say a tangle (B, t) is *simple* if (B, t) is a prime tangle and $B-t$ contains no incompressible embedded torus. Let F be a submanifold of a manifold M. We denote by $N(F, M)$ a regular neighborhood of F in M. Let L be a link in S^3. Then $E(L)=S^3-\operatorname{int} N(L, S^3)$ is called the *exterior* of L in S^3. We say a link L in S^3 is *simple* if L is non-splittable and every incompressible torus embedded in $E(L)$ is isotopic to a boundary component. A simple link is prime; the converse is not true.

In § 1 we show that a sum of two simple tangles is a simple link (Theorem 1). In § 2 we show that six of seven prime tangles given by Lickorish [2, Figure 2] are simple and a partial sum of two simple tangles is also a simple tangle (Theorem 2). In § 3 we show that a link in S^3 is concordant to a simple link with the same Alexander invariant (Theorem 3). (Compare the last result with those of R. Myers [4] and Y. Nakanishi [5], also S. Bleiler [1].)

I am grateful to Yasutaka Nakanishi for his several improvements of my original proofs.

§ 1. Sums of simple tangles.

First we prove the following two lemmas. Let X be a finite set. We denote by $\#(X)$ the number of elements of X.

LEMMA 1. *Let (A, t) be a prime tangle and T an incompressible*

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torus embedded in $A-t$. *Let* D *be a compressible disk for* T *in* A *such that* D *is transverse to* t. *Then we have* $\#(D\cap t)\geqq 2$.

PROOF. Since T is incompressible in $A-t$, we have $\#(D\cap t)\geqq 1$. We suppose that $\#(D\cap t)=1$. Let S be a 2-sphere in A obtained by doing surgery on T along D and B a 3-ball in A bounded by S. Since (A, t) is prime, $B\cap t$ is an unknotted arc in B. Hence $V=\overline{(B-N(B\cap t, B))}$ is a solid torus in $A-t$. We may assume that $\partial V=T$. Therefore T is compressible in V (hence in $A-t$), a contradiction. This completes the proof. □

Let X be a topological space. We denote by $|X|$ the number of connected components of X. Let F be a properly embedded, compact, 2-sided n-submanifold of compact $(n+1)$-manifold M. Then we denote by M_F the compact $(n+1)$-manifold obtained by splitting M along F.

LEMMA 2. *Let* (B, t) *be a simple tangle,* (C, u) *a prime tangle and* $h: (\partial C, \partial u)\to(\partial B, \partial t)$ *a homeomorphism. We set* $(S^3, L)=(B, t)\cup_h(C, u)$. *Let* T *be an incompressible, non-boundary-parallel torus in* $E(L)$. *Then* T *is isotopic in* S^3-L *to a torus contained in* $C-u$.

PROOF. We set $F=\partial B-\partial t=\partial C-\partial u$. Since F is incompressible in both $B-t$ and $C-u$, so is it in S^3-L. After adjusting by an isotopy, we may assume that T is transverse to F and chosen to minimize $|F\cap T|$. Since S^3-L is irreducible and both F and T are incompressible in S^3-L, each component of $F\cap T$ is essential (i.e., not null-homotopic) in both F and T. We suppose that $|F\cap T|\neq 0$. Then $T_{F\cap T}$ consists of $n(\geqq 2)$ annuli $A_1, \cdots, A_n$. We may assume that $A_1\subset B-t$. Then $\partial B_{\partial A_1}$ consists of two disks D_1, D_2 and an annulus E. Since ∂A_1 is essential in F, we have $D_i\cap t\neq\emptyset$.

First we show that $E\cap t=\emptyset$. If $E\cap t\neq\emptyset$, we have $\#(E\cap t)\geqq 1$. Since $\#(\partial B\cap t)=4$, either $\#(D_1\cap t)=1$ or $\#(D_2\cap t)=1$. We may assume that $\#(D_1\cap t)=1$. Let l_1 be an inner most loop of $F\cap T\cap D_1$ in D_1. Hence l_1 bounds a disk D_1' in D_1 such that $\#(D_1'\cap t)=1$ and $D_1'\cap T=l_1$. Then it is easy to show that L is a composite link (see [6, § 14, Satz 1] when L is a knot). This contradicts that L is prime (see [2, Theorem 1]). Therefore we have $E\cap t=\emptyset$. Hence $T_1=E\cup A_1$ is a torus embedded in $B-t$. Then T_1 bounds a compact 3-manifold G in $B-t$. Since (B, t) is simple, T_1 is compressible in $B-t$. Let D_0 be a compressing disk for T_1 in $B-t$. Then T_1 bounds a solid torus V in S^3 with a meridian disk D_0.

We suppose that $V=G$. Since ∂D_1 and ∂D_2 are meridians of a solid torus $S^3-\mathrm{int}\, G$, we may assume that $\partial D_0\cap\partial D_i$ (for $i=1, 2$) is a single point, that is, ∂D_1, ∂D_2 are 'longitudes' of V. Hence A_1 is isotopic to E rel ∂A_1 in V. Therefore we can modify T in a small neighborhood of V in S^3-L to contradict our minimality assumption

Hence we have $V=S^3-\text{int}\, G$. Since D_0, D_1 are meridian disks of V, ∂D_1 is homotopic to ∂D_0 in T_1, so is it in S^3-L. Since $D_0\cap L=\emptyset$, ∂D_1 is contractible in S^3-L. Therefore T is compressible in S^3-L, a contradiction. Hence we have $F\cap T=\emptyset$. Since (B, t) is simple, T is contained in $C-u$. This completes the proof. □

The following theorem is straightforward from Lemma 2.

THEOREM 1. *Let (A, t) and (B, u) be simple tangles and $h: (\partial A, \partial t)\rightarrow(\partial B, \partial u)$ a homeomorphism. We set $(S^3, L)=(A, t)\cup_h(B, u)$. Then L is a simple link in S^3.*

§ 2. Examples of simple tangles.

In this section, we give some examples of simple tangles. Let τ be a simplicial 1-subcomplex of a 3-ball B. Then we say $C_\tau=B-\partial B\cup\tau$ is the *open complement* of τ in B.

We prove the following lemma which we use to prove Theorem 3 in § 3.

LEMMA 3. *A tangle $(B, t_1\cup t_2)$ of Figure 1 is simple.*

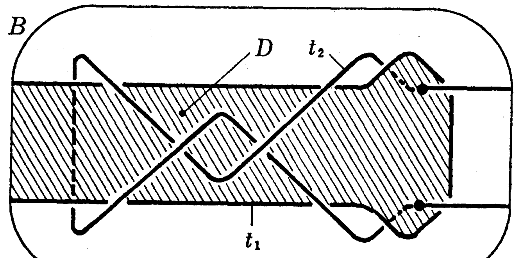

FIGURE 1

PROOF. First we note that $(B, t_1\cup t_2)$ is prime (see [1, Lemma 2.1]). Let C be the open complement of $t_1\cup t_2$ in B. Let D be a 2-disk in B shown in Figure 1 such that $D\cap t_1=t_1$, $D\cap t_2$ consists of two points p_1, p_2 and $\partial D=t_1\cup(D\cap\partial B)$. We set $F=D-\{p_1, p_2\}$.

We show that F is incompressible in C. If not, there is a compressing disk Δ_1 for F in C. Obviously $\partial\Delta_1$ bounds a 2-disk Δ_2 in D such that $\Delta_2\cap t_2=\{p_1, p_2\}$. Then $N=N(D\cup\Delta_1-\text{int}\,\Delta_2, B-t_2)$ is a 3-ball in B such that $t_1\subset N$ and $t_2\cap N=\emptyset$. Hence $\overline{(\partial N-\partial N\cap\partial B)}$ is a 2-disk in B which separates t_1 and t_2. This contradicts that $(B, t_1 \cup t_2)$ is prime.

Next we show that $C-D\cap C$ is homotopy equivalent to $S^1\vee S^1\vee S^1$. It is easily checked that $C-D\cap C$ is homeomorphic to the open complements C_{τ_1}, C_{τ_2} of 1-subcomplexies τ_1, τ_2 in B shown in Figure 2. And obviously C_{τ_2} is homotopy equivalent to $S^1\vee S^1\vee S^1$.

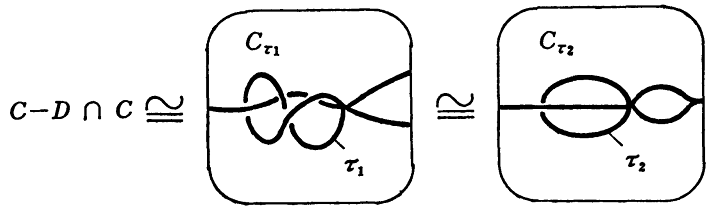

FIGURE 2

Now we suppose that there exists an incompressible torus T in C. After adjusting by an isotopy, we may assume that T is transverse to F and chosen to minimize $|F\cap T|$. Then each component of $F\cap T$ is essential in both F and T.

If $|F\cap T|=0$, then T is contained in $C-D\cap C$. Since T is incompressible in C, so is it in $C-D\cap C$. Hence the homomorphism $\pi_1 T\approx Z\times Z\to \pi_1(C-D\cap C)\approx Z*Z*Z$ induced by the inclusion is injective. This contradicts that any non-trivial subgroup of a free group is also free (see [3, p. 95, Corollary 2.9]). Therefore we have $|F\cap T|\neq 0$.

Let l be an inner most loop of $F\cap T$ in D. By Lemma 1, l bounds a disk D_0 in D such that $D_0\cap t_2=\{p_1, p_2\}$. Hence all components of $F\cap T$ are mutually parallel in F. Note that $T_{F\cap T}$ consists of $n(\geqq 2)$ annuli. Let l_0 be a loop shown in Figure 3. Since $H_1(B; Z)=0$, we have $[l_0]\cdot[T]=$

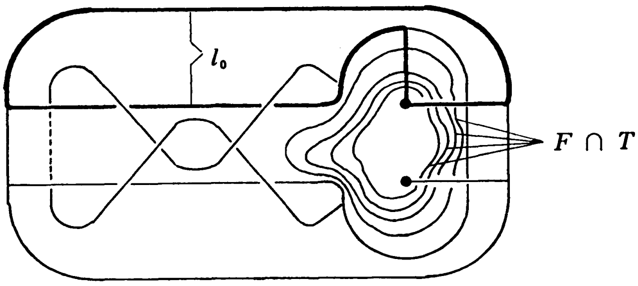

FIGURE 3

0. Therefore there is a component A of $T_{F\cap T}$ such that ∂A bounds an annulus E in F and such that, after adjusting by an isotopy in C, one can assume that torus $T_0=A\cup E$ has no intersection with D and bounds a compact 3-manifold G in $C-D\cap C$ (see Figure 4(a)). By the argument above, T_0 is compressible in $C-D\cap C$. Similarly in Lemma 2, modifying

T by an isotopy in a small neighborhood of G, we can reduce the number $|F \cap T|$ (see Figure 4(b)). This contradicts our minimality assumption. Hence $(B, t_1 \cup t_2)$ must be simple. This completes the proof. □

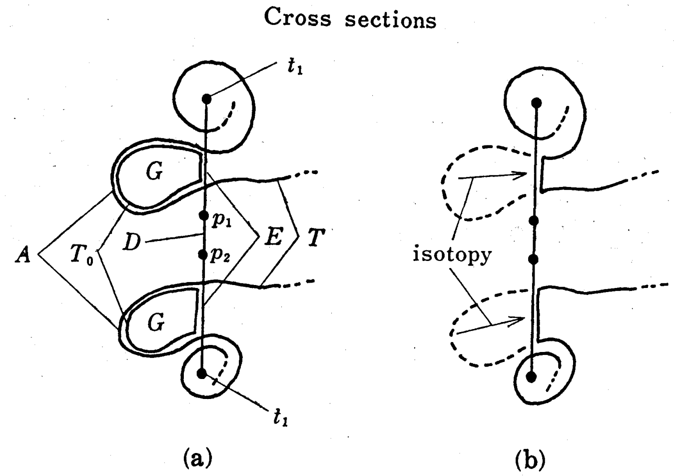

FIGURE 4

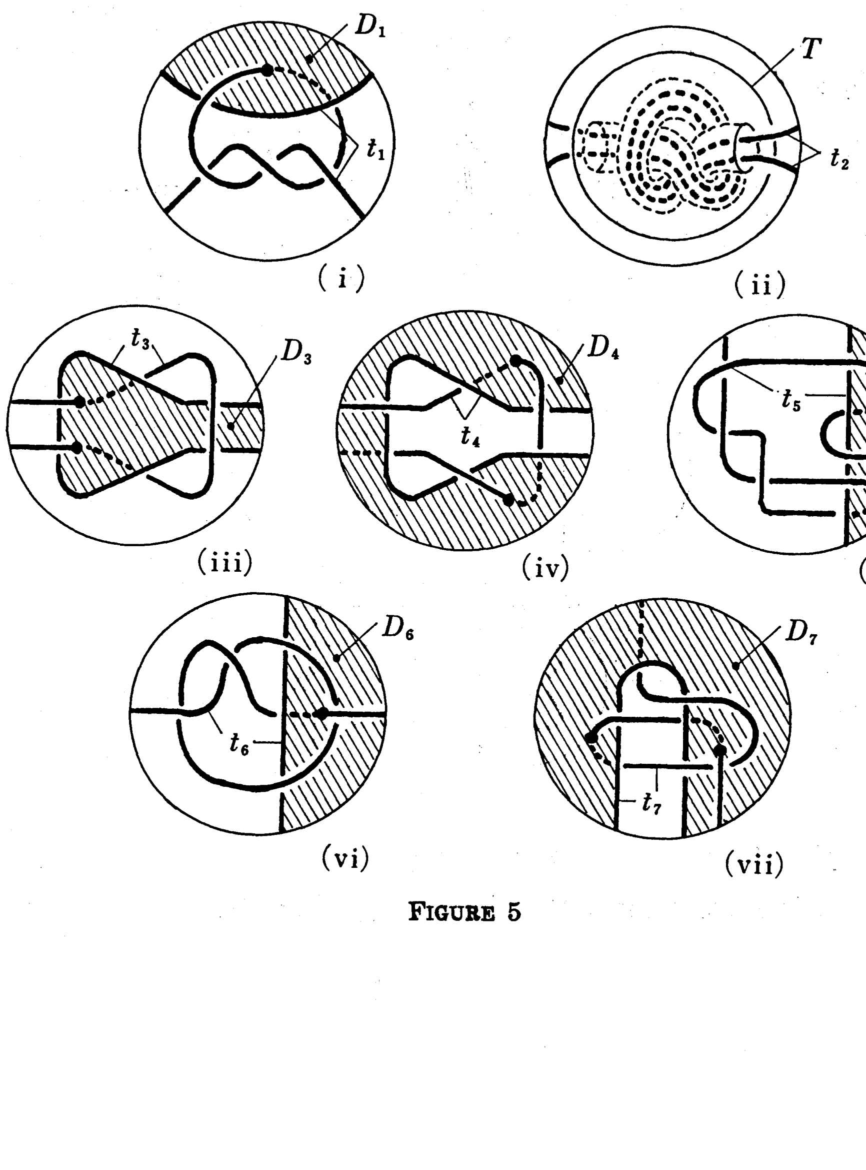

FIGURE 5

Lickorish [2, § 2] showed that seven tangles of Figure 5 are prime.

Let (A, t_1) be a tangle and D_1 a 2-disk shown in (i). Then $A-t_1\cup D_1$ is homotopy equivalent to $S^1\vee S^1$. The argument similar to that in Lemma 3 shows that (A, t_1) is simple.

Let (A, t_2) be a tangle shown in (ii). Since there exists an incompressible torus T in $A-t_2$, tangle (ii) is not simple.

Let (A, t_j) be a tangle and D_j a 2-disk, where $j=3, 4, 5$ or 7. Since $A-t_j\cup D_j$ is homotopy equivalent to $S^1\vee S^1\vee S^1$, (A, t_j) is simple.

Let (A, t_6) be a tangle and D_6 a 2-disk shown in (vi). Then $A-t_6\cup D_6\cup\partial A$ is homeomorphic to the open complement C_1 of tangle (i). Hence tangle (vi) is simple.

Every example (A, t) which we have given has a property that at least one component of t is unknotted in A. The following theorem makes us possible to construct an example of a tangle (A, t) such that each component of t is knotted in A.

THEOREM 2. *Let (C, v) be a tangle and D a 2-disk properly embedded in C that separates (C, v) into two tangles (A, t) and (B, u). If (A, t) is simple or untangle, (B, u) is simple and $D-v\cap D$ is incompressible in $A-t$, then (C, v) is simple.*

PROOF. The argument similar to that of [2, Theorem 2] implies that (C, v) is prime. Since $D-v\cap D$ is incompressible in $C-v$, one can prove that (C, v) is simple as Lemmas 2 and 3 above. □

By Theorem 2, a tangle (C, v) of Figure 6 is simple and each component of v is knotted in C.

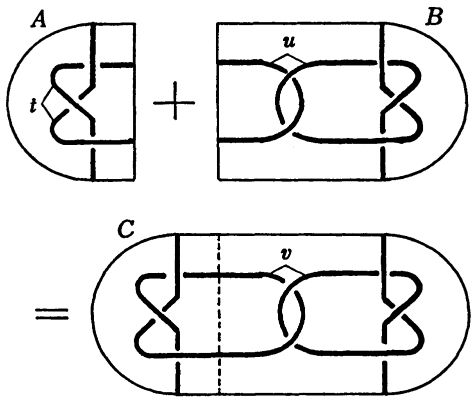

FIGURE 6

§ 3. Simple links and Alexander invariants.

In this section we prove the following theorem.

Theorem 3. *A link in S^3 is concordant to a simple link with the same Alexander invariant.*

Proof. Let L be a link in S^3. By Nakanishi [5, p. 567, Theorem], we may assume that L is prime. Let $\infty \in S^3$ be a point such that $\infty \cap L = \emptyset$, and let $\infty' \in S^3$. Let $\pi\colon S^3 - \infty \to S^2 - \infty'$ be a projection such that π/L is an embedding up to finite double crossing points $p_1, \cdots, p_n$ in $S^2 - \infty'$ (i.e., π is a regular projection). Moreover we suppose that π satisfies the following properties.

(i) For each crossing point p_i, there is a 3-ball B_i in $S^3 - \infty$ such that $(B_i, B_i \cap L)$ is a (2-string) trivial tangle and $B_i \cap B_j = \emptyset$ for i, j, $i \neq j$.

(ii) $D_i = \pi(B_i)$ is a regular neighborhood of p_i in $S^2 - \infty'$ such that $D_i \cap D_j = \emptyset$ for $i, j, i \neq j$, and $\pi(L) \cap D_i$ consists of two proper arcs in D_i which intersect each other in one point p_i.

(iii) Of all projections which have the properties (i), (ii), π has the minimal number of the crossing point of π/L.

We set $M = S^3 - \operatorname{int}(B_1 \cup \cdots \cup B_n)$ and $F = S^2 - \operatorname{int}(D_1 \cup \cdots \cup D_n)$. Let $i\colon F \to M$ be the natural inclusion such that $i(\infty') = \infty$ and $i(\pi(L) \cap F) = L \cap M$. We identify F and $i(F)$. We set $E_i = B_i - L \cap B_i$ and $G_i = D_i - \pi(L) \cap D_i$. Hence $\partial E_i = \partial B_i - \{\text{four points}\}$ and $\partial G_i = \partial D_i - \{\text{four points}\}$.

(3.1) Let l be a simple loop in F such that l meets $\pi(L)$ transversely in two points. Since L is prime, the property (iii) of π implies that l bounds a 2-disk D in S^2 such that $D \cap (D_1 \cup \cdots \cup D_n) = \emptyset$, i.e., $D \subset F$. Then, obviously, $D \cap \pi(L)$ is an arc.

(3.2) Similarly, for any proper arc α in $F - \pi(L) \cap F$ such that $\partial\alpha \subset \partial G_i$, there is an arc $\alpha_0 \subset \partial G_i$ such that $\partial\alpha_0 = \partial\alpha$ and a loop $\alpha_0 \cup \alpha$ bounds a 2-disk in $F - \pi(L) \cap F$.

First we show that ∂E_i is incompressible in $M - L \cap M$. If not, there is a compressible disk D for ∂E_i in $M - L \cap M$. Then ∂D bounds two 2-disks Δ_1, Δ_2 in ∂B_i such that $\Delta_1 \cap \Delta_2 = \partial D$ and $\#(\Delta_1 \cap L) = \#(\Delta_2 \cap L) = 2$. We may assume that D is transverse to F and has the minimal $|D \cap F|$ of all 2-disks which are properly isotopic to D in $M - L \cap M$. Now we suppose that $|D \cap F| \neq 0$. Then $D \cap F$ consists of proper arcs in $F - \pi(L) \cap F$ whose boundaries are contained in ∂G_i. By (3.2), for each component of $D \cap F$, there is an arc in ∂G_i such that the union of these two arcs is a loop which bounds a 2-disk in $F - \pi(L) \cap F$. Let D_0 be inner most one of such 2-disks. Then $D_0 \cap D$ is a proper arc in D. By doing surgery on D along D_0, we obtain two 2-disks D_1, D_2 such that $D_j \subset M - L \cap M$ and

$\partial D_j \subset \partial E_i$ for $j=1, 2$. We may assume that $\Delta_1 \supset \partial D_1 \cup \partial D_2$, that is, an arc $D_0 \cap \partial B_i$ separates Δ_1 into two 2-disks Δ_{11}, Δ_{12} such that $\partial\Delta_{11}=\partial D_1$, $\partial\Delta_{12}=\partial D_2$. Since $\#(\Delta_{1j} \cap L) \neq 1$, we may assume that $\#(\Delta_{11} \cap L)=0$ and $\#(\Delta_{12} \cap L)=2$. Since $M-L\cap M$ is irreducible, 2-sphere $D_1 \cup \Delta_{11}$ bounds a 3-ball B in $M-L\cap M$. Then one can modify D by a proper isotopy in a small neighborhood of B to contradict our minimality assumption. Therefore we have $|D \cap F|=0$. Then ∂D is contained in $\partial E_i - \partial E_i \cap F$ ($\cong$ two open hemispheres). Hence ∂D bounds a 2-disk in $\partial E_i - \partial E_i \cap F \subset \partial E_i$. This contradicts that D is a compressing disk for ∂E_i. Therefore ∂E_i must be incompressible in $M-L\cap M$.

Let L_0 be a link obtained from (S^3, L) by removing trivial tangles $(B_i, B_i \cap L)$ and sewing back new tangles (B_i, t_i) which are n copies of the tangle of Figure 1. By [5, Lemma 3], L is concordant to L_0 and they have the same Alexander invariant. Since, for each i, ∂E_i is incompressible in both $B_i - t_i$ and $M-L\cap M$, so is it in $S^3 - L_0$.

ASSERTION 1. *L_0 is prime.*

PROOF. Let S be a 2-sphere in S^3 such that S meets L_0 transversely in two points. We may assume that S is transverse to $\bigcup_{i=1}^n \partial E_i$ and has the minimal $|S \cap (\bigcup_{i=1}^n \partial E_i)|$ of all 2-spheres S' in S^3 such that $(S', S' \cap L_0)$ is isotopic to $(S, S \cap L_0)$ in (S^3, L_0). Since both ∂E_i and $S - L_0 \cap S$ are incompressible in $S^3 - L_0$, each component of $S \cap (\bigcup_{i=1}^n \partial E_i)$ is essential in both ∂E_i and $S - L_0 \cap S$.

If $|S \cap (\bigcup_{i=1}^n E_i)| \neq 0$, then there is an inner most loop l of $S \cap (\bigcup_{i=1}^n \partial E_i)$ in S. Then l bounds a 2-disk D in S such that $D \cap (\bigcup_{i=1}^n \partial E_i)=l$ and $D \cap L_0$ is a single point. We may assume that $l \subset \partial E_1$. By the primeness of (B_1, t_1) and the minimality of $|S \cap (\bigcup_{i=1}^n \partial E_i)|$, we have $D \subset M$. If l bounds a 2-disk D_1 in ∂B_1 such that $\#(D_1 \cap L_0)=2$, 2-sphere $S_1 = D \cup D_1$ meets L_0 transversely in three points. This contradicts that S_1 separates S^3 into two 3-balls. Hence l bounds a 2-disk D_2 in ∂B_1 such that $\#(D_2 \cap L_0)=1$. We set $S_2 = D \cup D_2$. Then we may assume that $S_2 \cap F$ is a simple loop l_0 such that $\#(l_0 \cap L_0)=2$. By (3.1), l_0 bounds a 2-disk D_3 in F such that $D_3 \cap L_0$ is an arc. Then S_2 bounds a 3-ball B in M such that $(D_3, \partial D_3) \subset (B, \partial B)$. Then one can modify $(S, S \cap L_0)$ by an isotopy in a small neighborhood of $(B, B \cap L_0)$ to contradict our minimality assumption. Hence we have $|S \cap (\bigcup_{i=1}^n \partial E_i)|=0$, i.e., either $S \subset M$ or $S \subset B_i$ for some i. By (3.1) and the primeness of (B_i, t_i), S bounds a 3-ball B in S^3 such that $B \cap L_0$ is an unknotted arc in B. Therefore L_0 is prime. □

ASSERTION 2. *L_0 is simple.*

Proof. We suppose that there exists an incompressible torus T in $\mathrm{int}\, E(L_0)$ which is not boundary-parallel in $E(L_0)$. Since L_0 is prime and (B_i, t_i) is simple (Lemma 3), after modifying T by an isotopy in $S^3 - L_0$, we may assume $T \subset M - L_0 \cap M$ by the argument similar to that in Lemma 2. Since $\pi_1(M - L_0 \cap M)$ is a free group, T is compressible in $M - L_0 \cap M$, so is in $S^3 - L_0$, a contradiction. Hence L_0 is simple. This completes the proof of Theorem 3. □

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