TOKYO J. MATH.
VOL. 6, NO. 2, 1983

On Some Groups Which Are Determined by their Subgroup-Lattices

Kazuhiko KOIKE*

Aoyama Gakuin University

(Communicated by Y. Kawada)

Introduction

Sometimes a group G is known to be characterized by its subgroup-lattice $L(G)$. This means that if a group G^* satisfies $L(G)\cong L(G^*)$, then G^* is isomorphic to G. Recently T. Uzawa has proved that a finite Coxeter group of rank$\geqq 3$ has this property. (See [4].)

In this note we prove two propositions. The first is that the alternating group of degree at least 4, $A_n(n\geqq 4)$ is determined by its subgroup-lattice. The second is that the affine Weyl group of rank$\geqq 4$ is determined by its subgroup-lattice.

These two propositions are conjectured by Prof. N. Iwahori. We were noticed recently that the first proposition was already proved by R. Schmidt in more general forms. (See R. Schmidt [3].) But our proof is completely different from that of R. Schmidt. In the second proposition the author owes a great deal to the result of T. Uzawa. We heartily express our gratitude to Prof. N. Iwahori and T. Uzawa.

§1. Characterization of the alternating group $A_n(n\geqq 4)$ by its subgroup-lattice.

First we define a few notations. Let G be a (finite or infinite) group. Then we denote its subgroup-lattice by $L(G)$ and also we denote the symmetric group and the alternating group of degree n by S_n and A_n respectively. Our first proposition is the following.

PROPOSITION 1. *Let G be a group. If the subgroup-lattice $L(G)$ of G is isomorphic to $L(A_n)(n\geqq 4)$ (the subgroup-lattice of A_n), then G is isomorphic to A_n as a group.*

Received October 18, 1982
* Supported by Grant Aid for Scientific Research

REMARK. We can also prove that the symmetric group of degree at least 4, $S_n(n\geqq 4)$ is determined by its subgroup-lattice using the similar argument as in the case of A_n. That is,

PROPOSITION. *Let G be a group. If the subgroup-lattice $L(G)$ of G is isomorphic to $L(S_n)(n\geqq 4)$, then G is isomorphic to S_n as a group.*

The proof of the above proposition is similar to that of Proposition 1, and so we omit it.

We will prove Proposition 1 by induction on n. Before the proof of the proposition, we must prepare a few lemmas.

LEMMA 1.1. *There are exactly n subgroups in $A_n(n\geqq 5, n\neq 6)$ which are isomorphic to A_{n-1}, and each of these subgroups is a maximal subgroup in A_n.*

LEMMA 1.2. *There are twelve subgroups in A_6 which are isomorphic to A_5. And under the inner action of A_6, the number of the conjugate orbits of these subgroups is two, and each orbit consists of 6 subgroups. Moreover each of them is a maximal subgroup in A_6.*

We prove Lemma 1.1 and Lemma 1.2 simultaneously. Let H be a subgroup of $G=A_n(n\geqq 5)$ which is isomorphic to A_{n-1}. Then we have $[G: H]=n$.

PROOF. First we will show that H is a maximal subgroup of G. Suppose H is not a maximal subgroup of G. Then there is a subgroup H^* of G such that $H\subsetneqq H^*\subsetneqq G$.

Making G act on G/H^* from left, we have a permutation representation $f: G\rightarrow S_k$ where k is the index of H^* in G. Since G is a simple group, f is a monomorphism. Therefore the order of G divides the order of S_k, hence we get $k=n$. But because of $H\subsetneqq H^*\subsetneqq G$, k is a non-trivial divisor of n, which is a contradiction.

Now H is a maximal subgroup of G. Since $G=A_n(n\geqq 5)$ is a simple group, we have $N_G(H)=H$. Recalling $[G: H]=n$, we have a permutation representation $f: G\rightarrow S_n$ such that the image of H under f is contained in one of the S_{n-1}'s which are embedded as a stabilizer of a one point in S_n. Since G is a simple group, f is a monomorphism and $[S_n: f(G)]=2$. Therefore $f(G)=A_n$ and f is an automorphism of A_n. Moreover the image of H under f just coincides with one of the A_{n-1}'s which are naturally embedded as a subgroup of the stabilizer group of a one point.

Let us recall the following well-known result.

THEOREM. *If* $n \neq 6$, $\mathrm{Aut}(A_n)$ *is isomorphic to* S_n *and the action of* S_n *on* A_n *is given by the conjugation. If* $n=6$, $\mathrm{Aut}(A_6)$ *is isomorphic to* $\mathrm{Aut}(S_6)$ *and* $[\mathrm{Aut}(S_6)\colon \mathrm{Int}(S_6)]=2$.

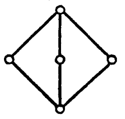
Using this theorem, if $n \neq 6$, f belongs to $\mathrm{Aut}(A_n)=S_n$. Hence H coincides with one of the A_{n-1}'s embedded as a subgroup of the stabilizer of a one point, which complete the proof of Lemma 1.1. If $n=6$, f belongs to $\mathrm{Aut}(A_6)=\mathrm{Aut}(S_6)$ and since $[\mathrm{Aut}(S_6)\colon \mathrm{Int}(S_6)]=2$, there are two possibilities. The one is that H coincides with one of the A_5's naturally embedded in S_6, and the another is that H coincides with one of the images of the naturally embedded A_5's under the outer automorphism of S_6. It can be easily verified that the two possibilities really occur and the proof of Lemma 1.2 is completed.

The next lemma is due to T. Uzawa.

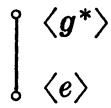
LEMMA 1.3. *Let* G *be a non-cyclic 2-group (of order* 2^n*), and let* G^* *be any group. If the subgroup-lattice* $L(G^*)$ *of* G^* *is isomorphic to the subgroup-lattice* $L(G)$*, then* G^* *is also a 2-group, and the order of* G^* *is equal to that of* G.

PROOF. In general let G_0 be a group whose order is 2^n. Then G_0 has a subgroup-sequence

$$G_0=H_0 \supsetneqq H_1 \supsetneqq H_2 \cdots \supsetneqq H_n=\{e\}\ , \quad [H_i\colon H_{i+1}]=2$$

and n is characterized by the maximal length of such subgroup-sequences. Therefore once we can show that G^* is also a 2-group, the order of G^* must be equal to that of G. We shall prove Lemma 1.3 by induction on n, (where 2^n is the order of G). If the order of G is four, G is isomorphic to $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ and the Hasse diagram of the subgroup-lattice of $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ is . In this case it can be easily proved that G^* (whose subgroup-lattice is isomorphic to that of $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$) is isomorphic to $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ as a group.

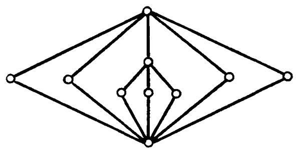
Now assume that Lemma 1.3 holds for all G which satisfies the condition of Lemma 1.3 and whose order is smaller than 2^n. We shall prove Lemma 1.3 for G of order 2^n. Let g^* be any element of G^* whose order is a prime number. It is sufficient to show that the order of g^* must be two. Let H be the subgroup of G which corresponds to the subgroup $\langle g^* \rangle$ (the group which is generated by g^*) of G^* under the isomorphism of the subgroup-lattice $L(G) \cong L(G^*)$. Since the order of g^*

is a prime number, the subgroup-lattice of $\langle g^* \rangle$ is 

, therefore H is a cyclic group of order p where p is a prime number. We denote a generator of H by g. Recalling that G is a 2-group, p must be two, and moreover the centralizer of g in G is non-trivial, i.e., $Z_G(g) \neq \{e\}$. If an involution h different from g exists in $Z_G(g)$, then $\langle g, h \rangle$ (the group generated by g and h) is isomorphic to $Z_2 \times Z_2$, hence the subgroup H_1^* of G^* which corresponds to $\langle g, h \rangle$ under the isomorphism $L(G) \cong L(G^*)$ is isomorphic to $Z_2 \times Z_2$. Since $H_1^* \supsetneqq \langle g^* \rangle$, the order of g^* must be two, and the proof is completed. If an involution different from g does not exist in $Z_G(g)$, g belongs to the center of G, and g is a unique involution in G. Therefore H is characterized as the non-trivial smallest subgroup of G. It follows that $\langle g \rangle^*$ is also a normal subgroup of G^*. From the assumptions on G and g, $G/\langle g \rangle$ cannot be a cyclic group. Since the subgroup-lattice of $G/\langle g \rangle$ is isomorphic to that of $G^*/\langle g^* \rangle$, $G^*/\langle g^* \rangle$ is also a 2-group by the induction hypothesis. According to the Sylow's theorem, there exists an involution h^* in G^*. But recalling that $\langle g^* \rangle$ is the non-trivial smallest subgroup of G^*, h^* is equal to g^*, and the order of g^* must be two. The proof of Lemma 1.3 is completed.

PROOF OF PROPOSITION 1. First we will show that the proposition holds for $n=4, 5, 6$ and later for general n.

In case of $n=4$, the subgroup-lattice of A_4 is just the following.



And it can be easily proved that if $L(G)$ is isomorphic to $L(A_4)$, then G is isomorphic to A_4 as a group.

In the case of $n=5$, let G be a group whose subgroup-lattice $L(G)$ is isomorphic to $L(A_5)$. Then according to the case of $n=4$ and Lemma 1.1, there are exactly 5 subgroups which are isomorphic to A_4. We denote these subgroups by H_1, H_2, H_3, H_4 and H_5. We shall show that H_i is not a normal subgroup of G for any i. Suppose there is some i such that H_i is a normal subgroup of G. Then for any j $H_i \cap H_j$ is a normal subgroup of H_j. But according to the subgroup-lattice $L(A_5)$, there is some j such that $H_i \cap H_j$ is isomorphic to Z_3, which is a contradiction.

Since H_i is a maximal subgroup of G, we can get $N_G(H_i)=H_i$ for any i and each conjugate orbit of H_1, H_2, H_3, H_4, H_5 under the action of G consists of N elements, where $N=|G/H_i|=|G|/|H_i|=|G|/(4!/2)$. Now considering that 5 is a prime number, we can get a homomorphism $f: G\rightarrow S_5$ by making G act on H_1, H_2, H_3, H_4, H_5 by conjugations. According to the subgroup-lattice $L(G)$, f is a monomorphism, and the order of G is $5\times|H_i|=5!/2$. Consequently the image of G under f is equal to A_5. The case of $n=5$ is proved.

In case of $n=6$, let G be a group whose subgroup-lattice is isomorphic to that of A_6. Then according to the case $n=5$ and Lemma 1.2, there are exactly 12 subgroups which are isomorphic to A_5. We denote these subgroups by $H_1, H_2, \cdots, H_{12}$. As before we shall show that H_i is not a normal subgroup of G for any i. Suppose there is some i such that H_i is a normal subgroup of G. For any j $H_i\cap H_j$ is a normal subgroup of H_j. Since H_j is isomorphic to the simple group A_5, $H_i\cap H_j=\{e\}$ for any $j\neq i$. According to the structure of the subgroup-lattice $L(G)$, this is a contradiction. Since H_i is a maximal subgroup of G, we have $N_G(H_i)=H_i$ and each conjugate orbit of $H_1, H_2, \cdots, H_{12}$ under the action of G consists of N elements, where $N=|G/H_i|=|G|/|H_i|=|G|/(5!/2)$ for any i. Therefore N is a divisor of 12. We shall show that $N=6$. Suppose $N=2$. We may assume H_1 and H_2 are conjugate. Then we have a homomorphism $f: G\rightarrow S_2$. Since the kernel of f is equal to $H_1\cap H_2$, $H_1\cap H_2$ is anormal subgroup of G, and consequently $H_1\cap H_2$ is also a normal subgroup of H_1. Since H_1 is a simple group, we have $H_1\cap H_2=\{e\}$ and f is a monomorphism. But comparing the order of G with that of S_2, this is a contradiction. In case of $N=3$ or $N=4$, the proof is similar. Suppose $N=12$. Then the order of G is equal to $12\times|H_1|=12\times|A_5|=720$. Hence the order of 2-sylow subgroup of G must be equal to 2^4. On the other hand, the order of 2-sylow subgroup of A_6 is equal to 2^3. According to Lemma 1.3, this is a contradiction. The only remaining possibility is 6. Therefore $N=6$, and we have a homomorphism $f: G\rightarrow S_6$. It can be easily proved as before that f is a monomorphism, hence considering the order of G, we have $f(G)=A_6$. The case $n=6$ is completed.

Now using the induction on n, we will prove the proposition for general n. If $n=6$, the proposition is valid from the above argument. Assume that the proposition is valid for $n-1$ $(n\geqq 7)$. Let G be a group whose subgroup-lattice is isomorphic to that of A_n. According to Lemma 1.1 and the induction hypothesis, there exist n subgroups which are isomorphic to A_{n-1}. We denote these subgroups by $H_1, H_2, \cdots, H_n$. We can prove similarly as in the case $n=6$ that H_i is not a normal subgroup of G for any i. Therefore we get $N_G(H_i)=H_i$ and each conjugate orbit of $H_1, H_2, \cdots, H_n$ under the action of G consists of N elements, where $N=|G/H_i|=|G|/|H_i|=|G|/((n-1)!/2)$. Hence N divides n. Exchanging the suffix if necessary, we may assume that the G-conjugate orbit of H_1 is $\{H_1, H_2, \cdots, H_N\}$. Then we have a permutation representation $f: G \rightarrow S_N$ by making G act on $H_1, H_2, \cdots, H_N$ by conjugations. Since the kernel of f is contained in $N_G(H_1)=H_1$ and H_1 is a simple group, f is a monomorphism and comparing the order of G with that of S_N, $|G|=(n-1)! \times N/2$ divides $|S_N|=N!$. Now recalling that N divides n, we have $N=n$ and $|G|=n!/2$. Therefore $f(G)=A_n$, which complete the proof of Proposition 1.

§ 2. Characterization of affine Weyl groups by their subgroup-lattices.

First let us recall the definition and the structure of affine Weyl groups. (See Iwahori-Matsumoto [2], N. Bourbaki [1].) Let V be a finite dimensional Euclidean space over $\boldsymbol{R}$, and we denote the inner product on V by $(\ ,\ )$. Now let us recall the definition of the reduced root system in V.

DEFINITION. A subset Δ of V is called the reduced root system in V if Δ satisfies the following conditions.

(R1) Δ is a finite set in V and the linear span of Δ is the total space V. Also Δ does not contain $0 \in V$.

(R2) For $\alpha \in \Delta$, we denote the orthogonal reflection relative to α by s_α, i.e.,

$$s_\alpha(x)=x-\frac{2\cdot(\alpha, x)}{(\alpha, \alpha)}\alpha .$$

Then s_α leaves Δ stable i.e., $s_\alpha(\Delta)=\Delta$.

(R3) If $\alpha \in \Delta$, the only multiple of α in Δ are $\pm\alpha$.

(R4) For any $\alpha, \beta \in \Delta$, $2\cdot(\alpha, \beta)/(\alpha, \alpha)$ is integer.

In the above definition, the group generated by the s_α's is called the Weyl group of Δ and is denoted by $\boldsymbol{W}$. $\boldsymbol{W}$ is known to be a finite group. For $\alpha \in \Delta$ and $k \in \boldsymbol{Z}$, we define the hyperplane $L_{\alpha,k}$ in V by

$$L_{\alpha,k}=\{x \in V \mid (\alpha, x)=k\} .$$

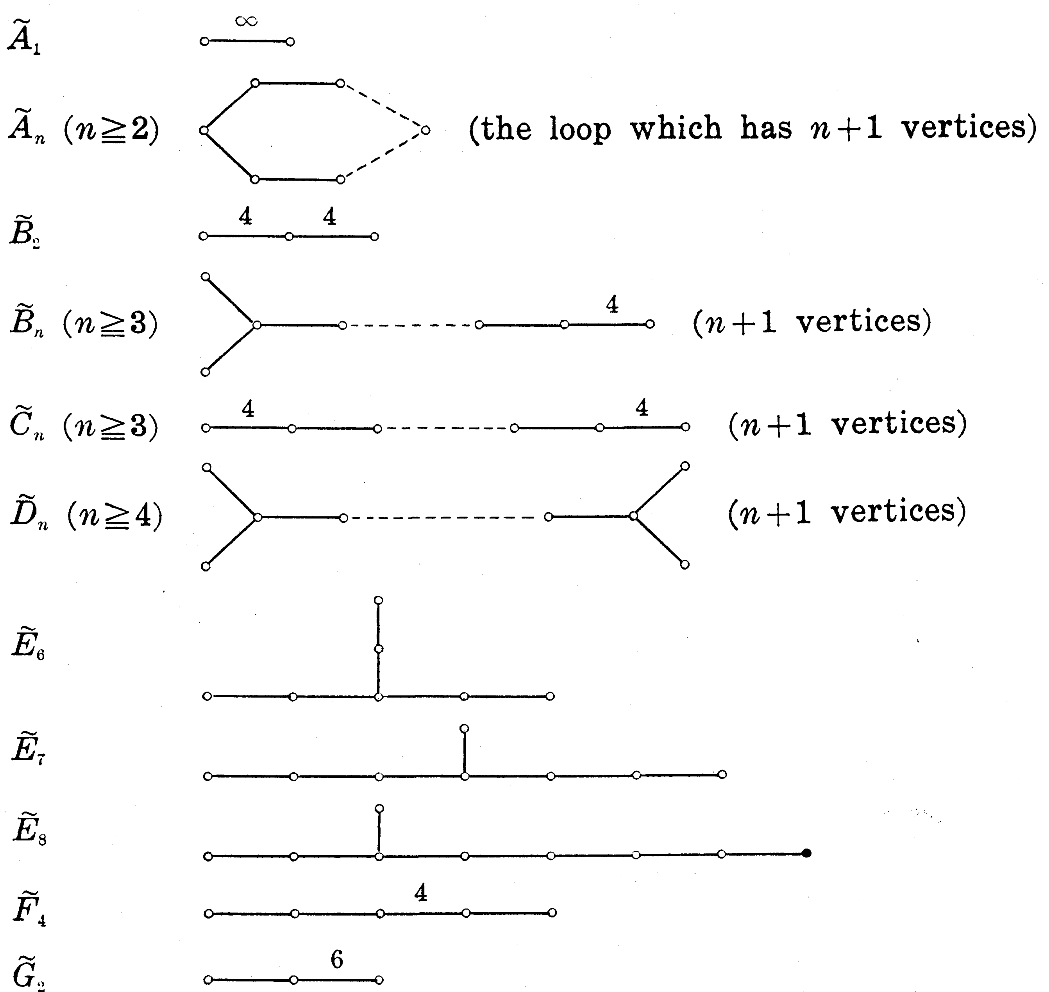
We denote the orthogonal reflection relative to $L_{\alpha,k}$ by $s_{\alpha,k}$. Affine Weyl group $\boldsymbol{W}_a$ (with respect to the root system Δ) is defined to be the affine transformation group, which is generated by the $s_{\alpha,k}$'s for all $\alpha \in \Delta$ and all $k \in \boldsymbol{Z}$. We denote the translation on V given by a vector $v \in V$ by

$T(v)$, and also we denote by D^* the group generated by the $T(\alpha^*)$'s for all $\alpha \in \Delta$, where $\alpha^* = 2\cdot\alpha/(\alpha, \alpha)$. D^* becomes a free $\boldsymbol{Z}$-module and the free generator of D^* are given by $T(\alpha_1^*)$, $T(\alpha_2^*)$, $\cdots$, $T(\alpha_n^*)$, where $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a simple root system in Δ with respect to some lexicographic order of V. As for the following proposition, see N. Bourbaki [1].

Proposition. *$W_a = W\cdot D^*$ and $W \cap D^* = \{e\}$, and D^* is a normal subgroup of W_a.*

An affine Weyl group is called irreducible if the underlying root system is irreducible. In other words, if the root system does not admit an orthogonal decomposition $\Delta = \Delta_1 \cup \Delta_2$, where $\Delta_1(\neq\emptyset)$ and $\Delta_2(\neq\emptyset)$ are subroot systems and mutually orthogonal, W_a is called irreducible.

Table I



The list of Coxeter diagrams of irreducible affine Weyl groups is given in Table I. (See Bourbaki [1].)

Now we will state the main result of this section.

PROPOSITION 2. *Let W_a be an irreducible affine Weyl group whose rank as a Coxeter group is at least 4. Let G be any (finite or infinite) group. If the subgroup-lattice $L(G)$ is isomorphic to that of W_a, then G is isomorphic to W_a as a group.*

In the above proposition, the condition that W_a is irreducible is not essential, and we can easily deduce the reducible cases from the irreducible ones.

Before the proof of Proposition 2, we must prepare a few lemmas. Following the idea of T. Uzawa in [4] and using the results in the case of finite dihedral groups, we obtain the next lemma, inspecting the list of Coxeter diagrams of irreducible affine Weyl groups. So we will omit the proof.

LEMMA 2.1. *Let W_a be an affine Weyl group whose rank as a Coxeter group is at least 4. Let G be any (finite or infinite) group. If the subgroup-lattice $L(G)$ of G is isomorphic to that of W_a, then there exists a surjective group homomorphism $f: W_a \to G$.*

The next lemma essentially asserts that the homomorphism f given by Lemma 2.1 is a monomorphism.

LEMMA 2.2. *Any non-trivial normal subgroup H of an irreducible affine Weyl group has a finite index in G.*

Using Lemma 2.1 and Lemma 2.2, we shall prove the proposition. From Lemma 2.1 there is a surjective homomorphism $f: W_a \to G$. If f is not a monomorphism, then the kernel of f will be a non-trivial normal subgroup of W_a. In view of Lemma 2.2, G becomes a finite group, and therefore $L(G)$ must be a finite lattice. But this is a contradiction, since $L(W_a)$ is an infinite lattice and $L(G) \cong L(W_a)$.

Hence, for the proof of the proposition we have enough to prove Lemma 2.2.

PROOF OF LEMMA 2.2. Let H be a non-trivial normal subgroup of W_a. It is sufficient to show that $H \cap D^* \neq \{0\}$ and $H \cap D^*$ is a free Z-module of rank n (where n is the dimension of V). First we shall show that $H \cap D^* \neq \{0\}$. Suppose $H \cap D^* = \{0\}$. Let $w \cdot T(d)$ be an element of H, where $w \in W$, $w \neq e$ and $d \in Z\alpha_1^* + Z\alpha_2^* + \cdots + Z\alpha_n^*$. Since H is a normal subgroup of W_a, for any $v \in Z\alpha_1^* + Z\alpha_2^* + \cdots + Z\alpha_n^*$. We have

$$(T(-v) \cdot w \cdot T(d) \cdot T(v)) \cdot (w \cdot T(d))^{-1} = T(w \cdot v - v) \in H .$$

Therefore $w \cdot v = v$ for any $v \in Z\alpha_1^* + Z\alpha_2^* + \cdots + Z\alpha_n^*$. But this is a contradiction since W acts on V faithfully. $H \cap D^*$ is also a free $\boldsymbol{Z}$-module as the subgroup of the free $\boldsymbol{Z}$-module D^*. Let $T(d)$ be an element of $H \cap D^*$. For $w \in W$, we have $w \cdot T(d) \cdot w^{-1} = T(w \cdot d)$, and so W leaves $H \cap D^*$ stable. Since W acts on V irreducibly, the rank of $H \cap D^*$ must be equal to the dimension of V. The proof of Lemma 2.2 is completed.

References

[1] N. Bourbaki, Groupes et Algèbres de Lie, Chapitres 4, 5, 6, Hermann, Paris, 1968.

[2] N. Iwahori and H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups, Publ. Math. I.H.E.S., **25** (1965), 237–280.

[3] R. Schmidt, Untergruppenverbände zweifach transitiver Permutationsgruppen, Math. Z., vol. **144**, no. 2 (1975), 161–168.

[4] T. Uzawa, Finite Coxeter groups and their subgroup-lattices, to appear.

Present Address:
Department of Mathematics
Aoyama Gakuin University
Chitosedai, Setagaya-ku, Tokyo