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The Grothendieck Group of a Finite Group Which is a Split Extension by a Nilpotent Group

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Introduction

Let R be a ring. Then the Grothendieck group $G_0(R)$ is the abelian group given by generators $[M]$ where M is a finitely generated R-module, with relations $[M]=[M']+[M'']$ whenever $0\to M'\to M\to M''\to 0$ is an exact sequence of finitely generated R-modules. Let π be a finite group, and $\mathcal{O}$ be a maximal order in $\boldsymbol{Q}\pi$ containing $\boldsymbol{Z}\pi$. Then Swan [4] showed that there is a natural epimorphism from $G_0(\mathcal{O})$ onto $G_0(\boldsymbol{Z}\pi)$. He also gave an example of cyclic group such that $G_0(\boldsymbol{Z}\pi)\not\cong G_0(\mathcal{O})$. In connection with these results of Swan, it is an interesting problem to investigate the relation between $G_0(\boldsymbol{Z}\pi)$ and $G_0(\mathcal{O})$. For an abelian group π, Lenstra [1] gives the description of $G_0(\boldsymbol{Z}\pi)$ which answers the above question. Recently, Miyamoto [2] generalizes Lenstra's result into nilpotent groups.

In this paper, we treat a finite group with a normal nilpotent subgroup which has a complement. For such a group π, we obtain an analogous decomposition of $G_0(\boldsymbol{Z}\pi)$.

THEOREM. *Let π be a finite group with a normal nilpotent subgroup U which has a complement. Then we have*

$$G_0(\boldsymbol{Z}\pi)\cong\bigoplus_{e\in Y} G_0\left(\boldsymbol{Z}\pi e^*\left[\frac{1}{d(e)}\right]\right),$$

where Y is a set of the representatives of the π-conjugacy classes of centrally primitive idempotents of $\boldsymbol{Q}U$, e^* denotes the class sum of the class containing e and $d(e)=|U|/|\mathrm{Ker}\,(U\to\boldsymbol{Q}Ue)|$.

REMARK 1. The idempotent e of the ring R is called centrally primitive, if e is a primitive idempotent of the center of the ring R.

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REMARK 2. $d(e)$ does not depend on the choice of a representative, because $\mathrm{Ker}(U\to \boldsymbol{Q}Ue)$ and $\mathrm{Ker}(U\to \boldsymbol{Q}Uf)$ are conjugate if e and f are conjugate.

REMARK 3. If U is cyclic, each e is also central in $\boldsymbol{Q}\pi$ and $e^*=e$, but not centrally primitive in general.

REMARK 4. If π is nilpotent, applying Theorem with $\pi=U$, we get the same decomposition as in [2].

Applying the above theorem to dihedral groups, we have

COROLLARY 1. *Let* $\pi=\langle\sigma,\tau|\tau^2=\sigma^t=1,\ \tau\sigma\tau^{-1}=\sigma^{-1}\rangle$ *be the dihedral group of order* $2t$ *and* R_d *be the integer ring of the maximal real subfield of* $\boldsymbol{Q}(\zeta_d)$, *where* ζ_d *is a primitive d-th root of unity. Then we have*

$$G_0(\boldsymbol{Z}\pi)\cong\begin{cases}G_0(\boldsymbol{Z})\oplus G_0(\boldsymbol{Z})\oplus\bigoplus_{1\neq d|t}G_0\left(R_d\left[\frac{1}{d}\right]\right) & \text{if } t \text{ is odd}\\ G_0(\boldsymbol{Z})\oplus G_0(\boldsymbol{Z})\oplus G_0(\boldsymbol{Z})\oplus G_0(\boldsymbol{Z})\oplus\bigoplus_{1,2\neq d|t}G_0\left(R_d\left[\frac{1}{d}\right]\right) & \text{if } t \text{ is even}.\end{cases}$$

Another corollary is the following one.

COROLLARY 2. *Let* $\pi=C_m\lhd C_n$ *be a meta-cyclic group such that* $(m,n)=1$ *and* C_n *acts faithfully on each Sylow subgroup of* C_m. *Then we have*

$$G_0(\boldsymbol{Z}\pi)\cong\bigoplus_{k|n}G_0\left(\boldsymbol{Z}\left[\zeta_k,\frac{1}{k}\right]\right)\oplus\bigoplus_{1\neq d|m}G_0\left(R_d\left[\frac{1}{d}\right]\right),$$

where ζ_l *is a primitive l-th root of unity and* $R_d=\boldsymbol{Z}[\zeta_d]^{C_n}$ *is the* C_n*-fixed subring of* $\boldsymbol{Z}[\zeta_d]$ *when we regard* C_n *as an automorphism group of* $\boldsymbol{Q}(\zeta_d)$.

§1. Proof of Theorem.

In this section, we prove the theorem. Let π be a finite group with a normal nilpotent subgroup U which has a complement H. For a $\boldsymbol{Z}\pi$-module M and a set S of prime divisors of $|U|$, we define N_SM to be a $\boldsymbol{Z}\pi$-module which is equal to M as a $\boldsymbol{Z}$-module, and the actions of U_SH on N_SM and M coincide, but $U_{\pi(U)-S}$ acts trivially, where U_S is the S-part of U and $\pi(U)$ is the set of all prime divisors of $|U|$. Since U_T is normal in π and has a complement for any $T\subseteq\pi(U)$, this is well-defined. In other words, N_S is the exact functor from the category of $\boldsymbol{Z}\pi$-modules to itself induced from composite of the canonical group homomorphisms

$\pi \to \pi/U_{\pi(U)-S} \xrightarrow{\sim} U_S H \hookrightarrow \pi$. For a centrally primitive idempotent e of $\boldsymbol{Q}U$ and $S \subseteq \pi(U)$, e_S denotes the S-part of e, so e_S is a centrally primitive idempotent of $\boldsymbol{Q}U_S$. On the other hand, e^S denotes a centrally primitive idempotent of $\boldsymbol{Q}U$ such that the S-part of e^S is e_S and the $\pi(U)-S$ part of e^S corresponds to the trivial representation. Then it is easily seen that $N_S M$ is a $\boldsymbol{Z}\pi(e^S)^*$-module if M is a $\boldsymbol{Z}\pi e^*$-module. To prove the theorem, we construct the group homomorphisms which are inverse to each other as given in [1], [2]. So we need some lemmas analogous to those given in [1], [2].

LEMMA 1. *Let M be a $\boldsymbol{Z}\pi e^*$-module with $d(e)M=0$. Then there exists a filtration $0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_t = M$ such that each M_j/M_{j-1} is annihilated by a prime number q_j dividing $d(e)$ and $U_{\{q_j\}}$ acts trivially on M_j/M_{j-1}.*

PROOF. We can assume that $qM=0$ for some prime number q dividing $d(e)$. Then M is an $\boldsymbol{F}_q\pi$-module annihilated by Ker $(\boldsymbol{Z}\pi \to \boldsymbol{Z}\pi e^*)$. Since $U_{\{q\}}$ is a q-group, $M^{U\{q\}} \neq 0$. So we define M_j $(1 \leqq j \leqq t)$ inductively by $M_j/M_{j-1} = (M/M_{j-1})^{U\{q\}}$ where $M_1 = M^{U\{q\}}$ and $M_t = M$ if $(M/M_{t-1})^{U\{q\}} = M/M_{t-1}$. Since $U_{\{q\}}$ is normal in π, $0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_t = M$ is a filtration of $\boldsymbol{Z}\pi$-modules. And $U_{\{q\}}$ acts trivially on each M_j/M_{j-1}, so this is the desired filtration.

LEMMA 2. *Let M be a $\boldsymbol{Z}\pi$-module. Suppose that M is both a $\boldsymbol{Z}\pi e^*$-module and a $\boldsymbol{Z}\pi e'^*$-module with $e^* \neq e'^*$. Then there exists a natural number t such that $(d(e)d(e'))^t M = 0$.*

PROOF. Put $\mathscr{S} = \{\emptyset \neq S \subseteq \pi(U) \mid e_S \underset{\pi}{\not\sim} e'_S\}$, where $e_1 \underset{\pi}{\sim} e_2$ means that e_1 and e_2 are π-conjugate. Since $e \underset{\pi}{\not\sim} e'$, $\mathscr{S} \neq \emptyset$. Let S be a minimal element of $\mathscr{S}$ with respect to the inclusion. Then it is easily seen that any p in S divides $d(e)d(e')$. On the other hand, M is both a $\boldsymbol{Z}U_S e_S^*$-module and a $\boldsymbol{Z}U_S e_S'^*$-module. Since e_S^* and $e_S'^*$ are central idempotents of $\boldsymbol{Q}U_S$ such that $e_S^* e_S'^* = 0$, $M[1/p_1 p_2 \cdots p_r] = 0$ where $\{p_1, p_2, \cdots, p_r\} = S$. Thus we are done.

For a $\boldsymbol{Z}\pi e^*$-module M, $[M, \langle e^* \rangle]$ means that $[M]$ is considered as an element in $G_0(\boldsymbol{Z}\pi e^*[1/d(e)])$.

LEMMA 3. *For a $\boldsymbol{Z}\pi$-module M which is both a $\boldsymbol{Z}\pi e^*$-module and a $\boldsymbol{Z}\pi e'^*$-module, we have*

$$\sum_{S \subseteq \pi(e)} [N_S M, \langle (e^S)^* \rangle] = \sum_{S' \subseteq \pi(e')} [N_{S'} M, \langle (e'^{S'})^* \rangle]$$

in $\bigoplus_e G_0(\boldsymbol{Z}\pi e^*[1/d(e)])$, where $\pi(e)$ is the set of all prime divisors of $d(e)$.

PROOF. Suppose that $[N_S M, \langle (e^S)^* \rangle] \neq 0$. If $S \not\subseteq \pi(e')$, we can find a prime number p in S which is not contained in $\pi(e')$. Then by the definition of $d(e')$, $e'_{\{p\}}$ corresponds to the trivial representation of $QU_{\{p\}}$. On the other hand, $e_{\{p\}}$ does not correspond to the trivial representation since $p \in S$. Thus M is both a $ZU_{\{p\}}e^*_{\{p\}}$-module and a $ZU_{\{p\}}e'^*_{\{p\}}$-module with $e^*_{\{p\}}e'^*_{\{p\}} = 0$, and we have $p^t M = 0$ for some natural number t. But since p divides $d(e^S)$, this contradicts the hypothesis. Hence $S \subseteq \pi(e')$, and S appears in the right hand side. Assume that $(e^S)^* \neq (e'^S)^*$. Then $N_S M$ is both a $Z\pi(e^S)^*$-module and a $Z\pi(e'^S)^*$-module with $(e^S)^* \neq (e'^S)^*$. So by Lemma 2, $(d(e^S)d(e'^S))^t N_S M = 0$ for some natural number t. Noting that $\pi(e^S) = \pi(e'^S) = S$, this implies that $(d(e^S))^{t'} N_S M = 0$ with some natural number t'. But this contradicts the assumption. Hence we have $(e^S)^* = (e'^S)^*$. By the symmetric argument, the lemma is proved.

Now, we are ready to prove the theorem.

Define $\Phi(e)\colon G_0(Z\pi e^*[1/d(e)]) \to G_0(Z\pi)$ by $\Phi(e)([M]) = \sum_{S \subseteq \pi(e)} (-1)^{\#\{\pi(e) - S\}} [N_S M]$, where M is $Z\pi e^*$-module. Applying Lemma 1, in the same way as Lenstra's proof, we find that $\Phi(e)$ is compatible with the defining relation of $G_0(Z\pi e^*[1/d(e)])$ and is a well-defined group homomorphism. Put $\Phi = \sum_e \Phi(e)$. Then Φ is the desired homomorphism.

Next, we define a map in the other direction. For a $Z\pi$-module M which is also a $Z\pi e^*$-module, we put $\Psi([M]) = \sum_{S \subseteq \pi(e)} [N_S M, \langle (e^S)^* \rangle]$. Then by Lemma 3, Ψ is a well-defined additive map. Since any $Z\pi$-module has a filtration such that each factor module is a $Z\pi e^*$-module for some e^*, by the same argument as in [1], Ψ is extended to a group homomorphism $\Psi\colon G_0(Z\pi) \to \bigoplus_e G_0(Z\pi e^*[1/d(e)])$.

Finally, by the same calculation as in [1], it is checked that Φ and Ψ are inverse to each other. This completes the proof of the theorem.

§2. Proofs of corollaries.

PROOF OF COROLLARY 1. Let $\pi = \langle \sigma, \tau \mid \tau^2 = \sigma^t = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the dihedral group of order $2t$ and e_d be a centrally primitive idempotent of $Q\langle\sigma\rangle$ corresponding to the irreducible representation given by $\sigma \mapsto \zeta_d$ $(d|t)$. Then $|\langle\sigma\rangle|/|\mathrm{Ker}\,(\langle\sigma\rangle \to Q\langle\sigma\rangle e_d)| = d$. Applying Theorem with $U = \langle\sigma\rangle$, we get

$$G_0(Z\pi) \cong \bigoplus_{d|t} G_0\left(Z\pi e_d\left[\frac{1}{d}\right]\right) \cong G_0(Z) \oplus G_0(Z) \oplus \bigoplus_{1 \neq d|t} G_0\left(Z\pi e_d\left[\frac{1}{d}\right]\right).$$

Assume that t is odd. Then each e_d $(d \neq 1)$ is also a centrally primitive idempotent of $Q\pi$ and $Z\pi e_d$ is a twisted group ring over $Z[\zeta_d]$

with the center R_d. Since $\boldsymbol{Z}[\zeta_d, 1/d]$ is unramified over $R_d[1/d]$, $\boldsymbol{Z}\pi e_d[1/d]$ is a maximal order (cf. [3], Theorem (40.14)), and $\boldsymbol{Z}\pi e_d[1/d]\cong M_2(R_d[1/d])$.

Next, suppose that t is even. Then $e_2=1/t(1-\sigma+\sigma^2-\cdots-\sigma^{t-1})$ and $e_2=e_2(1+\tau)/2+e_2(1-\tau)/2$ is a decomposition of e_2 into centrally primitive idempotents of $\boldsymbol{Q}\pi$. But since $d_2=2$, $\boldsymbol{Z}\pi e_2[1/d_2]=\boldsymbol{Z}\pi e_2(1+\tau)/2[1/2]\oplus \boldsymbol{Z}\pi e_2(1-\tau)/2[1/2]$ as rings. So, noting that $\boldsymbol{Z}\pi e_2(1+\tau)/2\cong \boldsymbol{Z}\pi e_2(1-\tau)/2\cong \boldsymbol{Z}$, we have

$$G_0\left(\boldsymbol{Z}\pi e_2\left[\frac{1}{d_2}\right]\right)=G_0\left(\boldsymbol{Z}\left[\frac{1}{2}\right]\right)\oplus G_0\left(\boldsymbol{Z}\left[\frac{1}{2}\right]\right)\cong G_0(\boldsymbol{Z})\oplus G_0(\boldsymbol{Z}) .$$

Because e_d $(d\neq 1, 2)$ is a centrally primitive idempotent of $\boldsymbol{Q}\pi$, by the same argument as in the odd case, we complete the proof of Corollary 1.

PROOF OF COROLLARY 2. For any $d|m$, let e_d be a centrally primitive idempotent of $\boldsymbol{Q}C_m$ which corresponds to the irreducible representation given by $\sigma\mapsto\zeta_d$, where $\langle\sigma\rangle=C_m$. Then we have $|C_m|/|\mathrm{Ker}\,(C_m\to \boldsymbol{Q}C_m e_d)|=d$. Applying Theorem with $U=C_m$, we have

$$\begin{aligned} G_0(\boldsymbol{Z}\pi)&\cong G_0(\boldsymbol{Z}\pi e_1)\bigoplus_{1\neq d|m} G_0\left(\boldsymbol{Z}\pi e_d\left[\frac{1}{d}\right]\right) \\ &\cong G_0(\boldsymbol{Z}C_n)\bigoplus_{1\neq d|m} G_0\left(\boldsymbol{Z}\pi e_d\left[\frac{1}{d}\right]\right) . \end{aligned}$$

By the assumption, each e_d $(d\neq 1)$ is also a centrally primitive idempotent of $\boldsymbol{Q}\pi$, and $\boldsymbol{Z}\pi e_d$ is a twisted group ring over $\boldsymbol{Z}[\zeta_d]$ with the center R_d. Since $\boldsymbol{Z}[\zeta_d, 1/d]$ is unramified over $R_d[1/d]$, in the same way as in the proof of Corollary 1, we have $G_0(\boldsymbol{Z}\pi e_d[1/d])\cong G_0(R_d[1/d])$ if $d\neq 1$.

On the other hand, $G_0(\boldsymbol{Z}C_n)$ is calculated in [1]. This completes the proof of Corollary 2.

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