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Expansion of the Solutions of a Gauss-Manin System at a Point of Infinity

Masatoshi NOUMI

Sophia University

Introduction

Let $f(x)$ be a polynomial, in n complex variables $x=(x_1, \cdots, x_n)$, with an isolated critical point and let $F_0(t, x)$ be a deformation of $f(x)$ with parameters $t=(t_1, \cdots, t_m)$. Setting $F=t_0+F_0$ with a distinguished parameter t_0, we shall investigate the differential system to be satisfied by the integral of type

$$(1) \qquad u=\int \delta^{(\lambda)}(F)dx \quad \text{or} \quad \int F^{-\lambda-1}dx \quad (dx=dx_1\wedge\cdots\wedge dx_n)\ ,$$

where λ is a (generic) complex number. Roughly speaking, such a *Gauss-Manin system* defines a meromorphic connection, on the space S of parameters (t_0, t), at most with poles along its *discriminant variety* D. Thus, our attension will be paid to the many-valued holomorphic solutions on $S\backslash D$ of the Gauss-Manin system. In "simple" examples, one can show that a fundamental system $\Phi(t_0, t)$ of its many-valued holomorphic solutions can be expanded into a power series

$$(2) \qquad \Phi(t_0, t)=\sum_{r=0}^{\infty}\Phi_r(t)t_0^{-\Lambda-(r+1)I}$$

convergent near the point $(t_0, t)=(\infty, 0)$ at infinity, where $-\Lambda$ is the matrix of exponents of f shifted by λ. In the present article, we shall determine such an expansion of Φ in an explicit manner for typical examples of Gauss-Manin systems.

Our computational results will be given in §3. The polynomial $f(x)$ to be deformed is assumed there to belong to either of the types

$$(3) \qquad \begin{array}{ll} (\mathrm{I}) & f(x)=x_1^{p_1}+x_2^{p_2}+\cdots+x_n^{p_n} \quad \text{and} \\ (\mathrm{II}) & f(x)=x_1^{p_1}+x_1x_2^{p_2}+x_3^{p_3}+\cdots+x_n^{p_n}\ . \end{array}$$

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First, we take a deformation F of f defined by

$$(4)\qquad F(t_0, t, x)=t_0+t_1x_1+\cdots+t_nx_n+f(x) \qquad (t=(t_1, \cdots, t_n)).$$

In this case, the expansion of type (2) can be completely determined by solving a system of difference equations (Theorem 3.5). By virtue of such explicit computations, we know that, if f is of type (I), then the many-valued holomorphic solutions of the Gauss-Manin system in question are expressed by certain types of *hypergeometric series* (Theorem 3.7). (The reader will find in 3.2 some examples of such hypergeometric representations of the solutions.) Next we consider the case where F is a *versal* deformation of a canonical form f of *simple sigularity*. In this case, the expansion of Φ can be determined from the result for a deformation of type (4) associated with F by applying an operator of "evolution" (Theorem 3.10). Thanks to this explicit expansion of Φ, we are able to determine the *flat coordinate system*, introduced by K. Saito - T. Yano-J. Sekiguchi [20], for the types A_l, D_l and E_l ($l=6, 7, 8$) from the viewpoint of differential equations (Theorem 3.11).

Our computations will be delivered rather formally. In order to justify such formal computations, we need to formulate Gauss-Manin systems associated with the integral (1) in the *algebraic* (=polynomial) category. Thus §1 will be devoted to the reconstruction, in this category, of certain known results concerning the structure of Gauss-Manin systems. Two types of finite presentation are given there: one as a differential system for a vector of unknown functions, due to K. Saito [17] and F. Pham [14], the other as a differential system for a single unknown function, proposed by S. Ishiura [7]. The former assures the existence of such expansion of Φ as explained above (cf. Theorem 2.2) but does not fit easily for concrete computations, while the latter can be computed immediately from F itself but does not seem to provide so many qualitative informations on its solutions. Thus, the very key to our later arguments lies in making clear the connection between the two presentations, or rather, it might be said that the construction of Φ itself tells us something about it. These circumstances will be explained in §2, as well as how we approach to determining the explicit form of the expansion of Φ. Throughout this article, we will freely use the symbols representing the derivatives of the delta function in place of the complex powers. A superficial reason for this is to avoid the exception special values of λ and to distinguish the constants depending on λ from of the others.

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§1. Presentation of the Gauss-Manin system.

1.0. Notations.

Let $T=C^m$ be the complex affine m-space with canonical coordinates $t=(t_1, \cdots, t_m)$. Then we denote by $\mathscr{R}(T)$ the affine ring $C[t]=C[t_1, \cdots, t_m]$ and by $\mathscr{D}(T)$ the $\mathscr{R}(T)$-algebra $C[t; D_t]=C[t_1, \cdots, t_m; D_{t_1}, \cdots, D_{t_m}]$ of differential operators with polynomial coefficients, where $D_{t_k}=\partial/\partial_{t_k}$ for $1\leqq k\leqq m$. We use the notation

$$(1) \qquad \partial_{t_k}(f):=[D_{t_k}, f]=D_{t_k}f-fD_{t_k} \quad \text{for} \quad f\in\mathscr{R}(T)$$

to distinguish the operation of a vector field on a function from the composition as operators.

Let $S=C\times T$ be the affine $(m+1)$-space with coordinates $(t_0, t)=(t_0, t_1, \cdots, t_m)$ and π the canonical projection $S\to T$. Then we define the localization of $\mathscr{D}(S)$ with respect to D_{t_0} by

$$(2) \qquad \mathscr{D}(S)[D_{t_0}^{-1}]=\mathscr{D}(S)\underset{C[D_{t_0}]}{\otimes} C[D_{t_0}, D_{t_0}^{-1}]$$

with the commutation rule

$$(3) \qquad D_{t_0}^{-k}f=\sum_{i=0}^{\infty}\binom{-k}{i}\partial_{t_0}^{i}(f)D_{t_0}^{-k-i}$$

for $k\in Z$ and $f\in\mathscr{R}(S)$. Note that the right side of (3) is a finite sum since f is a polynomial in t_0. We denote by $\mathscr{D}(S)[D_{t_0}^{-1}](0)$ the sub-algebra of $\mathscr{D}(S)[D_{t_0}^{-1}]$ which consists of all operators of order at most zero.

1.1. Gauss-Manin systems.

To formulate Gauss-Manin systems, we follow the framework of K. Saito [17]-[19], while our arguments will be carried within the algebraic category.

Let $X_0=C^n$ be the affine n-space with canonical coordinates $x=(x_1, \cdots, x_n)$. With the notations in 1.0, we define $X=T\times X_0$ and denote by q the canonical projection $X\to T$. Then setting $Z=S\times_T X$, we obtain a cartesian diagram

$$
(1)\qquad
\begin{array}{ccc}
Z & \xrightarrow{\hat{\pi}} & X \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\
S & \xrightarrow{\pi} & T
\end{array}
$$

of affine spaces, where p and $\hat{\pi}$ are canonical projections. Hereafter, the space X_0 is identified with the fibre $q^{-1}(0)$ or $p^{-1}(0)$.

Fix a polynomial $f=f(x)$ in $\mathscr{R}(X_0)$ and its deformation $F_0=F_0(t, x)$ over the parameter space T; $F_0 \in \mathscr{R}(X)$, $F_0|_{t=0}=f$. In view of the diagram (1), we define two morphisms φ: $X \to S$ and ι: $X \to Z$ by $\varphi(t, x)=(-F_0(t, x), t)$ and $\iota(t, x)=(-F_0(t, x), t, x)$ respectively, so that $\hat{\pi}\circ\iota=\mathrm{id}_X$, $p\circ\iota=\varphi$ and $\pi\circ\varphi=q$. The critical variety of φ, denoted by C, has the affine ring $\mathscr{R}(C)=\mathscr{R}(X)/(\partial_x F_0)$, where $(\partial_x F_0)=(\partial_{x_1}F_0, \cdots, \partial_{x_n}F_0)$. The image $\varphi(C)$, denoted by D, is called the *discriminant set* of φ. Note here that the space X is isomorphic to the smooth hypersurface $\iota(X)$ in Z defined by the polynomial $F:=t_0+F_0$ in $\mathscr{R}(Z)$ with the distinguished parameter t_0.

Now, we propose to investigate the differential system of Gauss-Manin associated with the deformation F. For this purpose we use the symbols $\delta_F^{(\lambda)}(\lambda \in C)$ which represent the derivatives $D_{t_0}^{\lambda}\delta_F$ of the "delta function" $\delta_F=\delta_F^{(0)}$. For each complex number λ, we denote by M_F^{λ} the free $\mathscr{R}(X)$-module with basis $(\delta_F^{(\lambda-k)})_{k\in Z}$:

$$
(2)\qquad M_F^{\lambda}=\bigoplus_{k\in Z}\mathscr{R}(X)\delta_F^{(\lambda-k)} .
$$

Here, we define the operation of D_{t_0}, t_0, D_{t_j} $(1\leqq j\leqq m)$ and D_{x_i} $(1\leqq i\leqq n)$ by

$$
(3)\qquad
\begin{cases}
D_{t_0}\delta_F^{(\lambda)}=\delta_F^{(\lambda+1)} & \\
t_0\delta_F^{(\lambda)}=-F_0\delta_F^{(\lambda)}-\lambda\delta_F^{(\lambda-1)} & \\
D_{t_j}\delta_F^{(\lambda)}=\partial_{t_j}(F_0)\delta_F^{(\lambda+1)} & (1\leqq j\leqq m) \\
D_{x_i}\delta_F^{(\lambda)}=\partial_{x_i}(F_0)\delta_F^{(\lambda+1)} & (1\leqq i\leqq n) .
\end{cases}
$$

With these operations, each M_F^{λ} becomes a left $\mathscr{D}(Z)[D_{t_0}^{-1}]$-module generated by $\delta_F^{(\lambda)}$ and has a finite presentation over $\mathscr{D}(Z)[D_{t_0}^{-1}]$

$$
(4)\qquad
\begin{array}{ll}
(F+\lambda D_{t_0}^{-1})\delta_F^{(\lambda)}=0 & \\
(D_{t_j}D_{t_0}^{-1}-\partial_{t_j}(F))\delta_F^{(\lambda)}=0 & (1\leqq j\leqq m) \\
(D_{x_i}D_{t_0}^{-1}-\partial_{x_i}(F))\delta_F^{(\lambda)}=0 & (1\leqq i\leqq n) .
\end{array}
$$

Similarly we define

$$
(5)\qquad M_F^{(\lambda)}=\bigoplus_{k\in N}\mathscr{R}(X)\delta_F^{(\lambda-k)}
$$

for $\lambda \in C$. Then each $M_F^{(\lambda)}$ has a natural structure of left $\mathscr{D}(Z)[D_{t_0}^{-1}](0)$-module and has a finite presentation (4) as well. The modules $M_F^{(\lambda+k)}$ $(k \in Z)$ form a filtration of M_F^λ compatible with the order of operators and one has $M_F^\lambda = \bigcup_{k \in Z} M_F^{(\lambda+k)}$.

For each $\lambda \in C$, we define the *Gauss-Manin system* H_F^λ by

$$H_F^\lambda = M_F^\lambda \Big/ \sum_{i=1}^n D_{x_i} M_F^\lambda \tag{6}$$

and denote by $\int dx$ the canonical surjection $M_F^\lambda \to H_F^\lambda$: e.g., $\int \delta_F^{(\lambda)} dx =$ the modulo class of $\delta_F^{(\lambda)}$. H_F^λ has a natural structure of left $\mathscr{D}(S)[D_{t_0}^{-1}]$-module induced by the $\mathscr{D}(Z)[D_{t_0}^{-1}]$-module structure of M_F^λ. Similarly, for each $\lambda \in C$, we define the $\mathscr{D}(S)[D_{t_0}^{-1}](0)$-module $H_F^{(\lambda)}$ by

$$H_F^{(\lambda)} = M_F^{(\lambda)} \Big/ \sum_{i=1}^n D_{x_i} M_F^{(\lambda-1)} . \tag{7}$$

Indeed, the injection $M_F^{(\lambda-1)} \to M_F^{(\lambda)}$ induces a natural homomorphism $H_F^{(\lambda-1)} \to H_F^{(\lambda)}$, but it is no longer injective in general.

A more systematic way to introduce the Gauss-Manin system H_F^λ is to use the the De Rham complex of M_F^λ relative to the projection $Z \to S$. We denote by $\Omega_{Z/S}^p$ the $\mathscr{R}(Z)$-module of differential p-forms relative to $Z \to S$ with coefficients in $\mathscr{R}(Z)$; $\Omega_{Z/S}^1 = \bigoplus_{i=1}^n \mathscr{R}(Z) dx_i$ and $\Omega_{Z/S}^p = \wedge^p \Omega_{Z/S}^1$. The relative De Rham complex of M_F^λ is defined by

$$\begin{aligned} \Omega_{Z/S}(M_F^\lambda)^\cdot : 0 \longrightarrow \Omega_{Z/S}^0 \underset{\mathscr{R}(Z)}{\otimes} M_F^\lambda \xrightarrow{d_{Z/S}} \Omega_{Z/S}^1 \underset{\mathscr{R}(Z)}{\otimes} M_F^\lambda \xrightarrow{d_{Z/S}} \cdots \\ \cdots \xrightarrow{d_{Z/S}} \Omega_{Z/S}^n \underset{\mathscr{R}(Z)}{\otimes} M_F^\lambda \longrightarrow 0 \end{aligned} \tag{8}$$

with the differentiation

$$d_{Z/S}(\omega \otimes u) = d_{Z/S}\omega \otimes u + \sum_{i=1}^n dx_i \wedge \omega \otimes D_{x_i} u , \tag{9}$$

where $\omega \in \Omega_{Z/S}^\cdot$ and $u \in M_F^\lambda$. In terms of this complex, we have an isomorphism

$$H_F^\lambda \xrightarrow{\sim} H^n(\Omega_{Z/S}(M_F^\lambda)^\cdot) \qquad \text{for} \quad \lambda \in C . \tag{10}$$

As to $M_F^{(\lambda)}$, we define the relative De Rham complex by

$$\begin{aligned} \Omega_{Z/S}(M_F^{(\lambda)})^\cdot : 0 \longrightarrow \Omega_{Z/S}^0 \underset{\mathscr{R}(Z)}{\otimes} M_F^{(\lambda-n)} \xrightarrow{d_{Z/S}} \Omega_{Z/S}^1 \underset{\mathscr{R}(Z)}{\otimes} M_F^{(\lambda-n+1)} \xrightarrow{d_{Z/S}} \cdots \\ \cdots \xrightarrow{d_{Z/S}} \Omega_{Z/S}^n \underset{\mathscr{R}(Z)}{\otimes} M_F^{(\lambda)} \longrightarrow 0 . \end{aligned} \tag{11}$$

Then we have

$$H_F^{(\lambda)} \xrightarrow{\sim} H^n(\Omega_{Z/S}(M_F^{(\lambda)})^\cdot) \qquad \text{for } \lambda \in \boldsymbol{C} . \tag{12}$$

Note that the complexes $\Omega_{Z/S}(M_F^{(\lambda+k)})^\cdot$ ($k \in \boldsymbol{Z}$) define an exhaustive filtration of $\Omega_{Z/S}(M_F^\lambda)^\cdot$.

REMARK. Assume that λ is not an integer and consider the many-valued holomorphic function $F^{-\lambda-1}$ on $Z \backslash \iota(X)$. Then the $\mathscr{D}(Z)$-module $\mathscr{D}(Z)F^{-\lambda-1}$ has an expression

$$\mathscr{D}(Z)F^{-\lambda-1} = \mathscr{R}(Z)[F^{-1}]F^{-\lambda-1} = \bigoplus_{k \in \boldsymbol{Z}} \mathscr{R}(X)F^{-\lambda-1+k} . \tag{13}$$

In this case, as the correspondence

$$\delta_F^{(\lambda)} \longmapsto \frac{\Gamma(\lambda+1)}{2\pi\sqrt{-1}} \cdot (-F)^{-\lambda-1} \tag{14}$$

defines a $\mathscr{D}(Z)$-isomorphism $M_F^\lambda \xrightarrow{\sim} \mathscr{D}(Z)F^{-\lambda-1}$, we have an $\mathscr{D}(S)$-isomorphism

$$H_F^\lambda \xrightarrow{\sim} \mathscr{D}(Z)F^{-\lambda-1} \Big/ \sum_{i=1}^n D_{x_i} \mathscr{D}(Z)F^{-\lambda-1} . \tag{15}$$

Thus the Gauss-Manin system H_F^λ provides the differential system that should be verified by the integral of the complex power $F^{-\lambda-1}$

$$u(t_0, t) = \int_{\Gamma(t_0,t)} F(t_0, t, x)^{-\lambda-1} dx \qquad (dx = dx_1 \wedge \cdots \wedge dx_n) , \tag{16}$$

where the integral is presupposed to allow the integration by parts and the differentiation under the sign of integral with respect to the parameters (t_0, t). In this sense, our formulation is the same as that of K. Aomoto [1] and M. Kita-M. Noumi [11], where a larger class of complexes than ours are studied under the name of "twisted rational De Rham complexes".

1.2. Structure of the Gauss-Manin system H_F^λ.

This paragraph contains an algebraic analogue to a result of K. Saito [17] or F. Pham [14]. Ideas of proof are borrowed from M. Kita-M. Noumi [11]. Here we impose two types of condition on f and F.

First, we assume

(A.1) *$f(0)=0$ and $f: X_0 = \boldsymbol{C}^n \longrightarrow \boldsymbol{C}$ has an only isolated critical point at the origin .*

On this assumption, the quotient ring $C[x]/(\partial_x f)$, where $(\partial_x f)=(\partial_{x_1}f, \cdots, \partial_{x_n}f)$, is a finite dimensional vector space over C, whose dimension will be called the *Milnor number* of f and denoted by $\mu=\mu(f)$. Let

$$(1) \qquad (\Omega^{\cdot}_{X_0}; df): 0 \longrightarrow \Omega^0_{X_0} \xrightarrow{df\wedge} \Omega^1_{X_0} \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \Omega^n_{X_0} \longrightarrow 0$$

be the Koszul complex of df, where $\Omega^1_{X_0}=\bigoplus_{i=1}^n C[x]dx_i$ and $\Omega^p_{X_0}=\wedge^p \Omega^1_{X_0}$. Recall that the partial derivatives $\partial_{x_1}f, \cdots, \partial_{x_n}f$ form a regular sequence in the regular local ring $C[x]_{(x)}$. Hence, one obtains

$$(2) \qquad \begin{cases} H^p(\Omega^{\cdot}_{X_0}; df)=0 & \text{for } p\neq n \\ H^n(\Omega^{\cdot}_{X_0}; df) \xleftarrow{\sim} C[x]/(\partial_x f) . \end{cases}$$

The n-th cohomology $H^n(\Omega^{\cdot}_{X_0}; df)$ will also be denoted by Ω_f, so that Ω_f is a $\mu(f)$-dimensional vector space over C.

The second assumption is concerned with the homogeneity of f. Fix an n-vector $\rho=(\rho_1, \cdots, \rho_n)$ of positive rational numbers, called the *weight* of $x=(x_1, \cdots, x_n)$. For a multi-index $\nu=(\nu_1, \cdots, \nu_n)\in N^n$, define $\langle\rho, \nu\rangle=\sum_{i=1}^n \rho_i\nu_i$. Then, for a rational number ε, a polynomial g in $C[x]$ is called *weighted homogeneous of ρ-degree* ε if it has an expression

$$(3) \qquad g(x)=\sum_{\langle\rho,\nu\rangle=\varepsilon} a_\nu x^\nu \qquad (a_\nu \in C) ,$$

where $x=x_1^{\nu_1}\cdots x_n^{\nu_n}$. In terms of the Euler vector field $\theta_x:=\sum_{i=1}^n \rho_i x_i D_{x_i}$, it is equivalent to the condition $\theta_x g=\varepsilon g$. A polynomial g in x is called *of ρ-degree*$\leqq\varepsilon$, if it can be written as a sum of weighted homogeneous polynomials in x of ρ-degree$\leqq\varepsilon$. With this terminology, the second assumption is stated as

(A.2) (i) *$f(x)$ is a weighted homogeneous polynomial of ρ-degree* 1.
(ii) *The ρ-degree of $F(t_0, t, x)-f(x)$ is strictly less than* 1.

By an effect of the assumption (A.2.i), the Koszul complex $(\Omega^{\cdot}_{X_0}; df)$ has a direct decomposition into the eigenspaces of the Lie derivative L_{θ_x}. For $\varepsilon\in Q$, we define the space of *weighted homogeneous p-forms of ρ-degree* ε by

$$(4) \qquad \Omega^p_{X_0,\varepsilon}=\{\omega\in\Omega^p_{X_0}; L_{\theta_x}\omega=\varepsilon\omega\} \qquad (0\leqq p\leqq n) .$$

Then we have $\Omega^p_{X_0}=\bigoplus_{\varepsilon\in Q}\Omega^p_{X_0,\varepsilon}$. (Note that $\Omega^p_{X_0,\varepsilon}=0$ if $\varepsilon<0$.) Moreover, for each $\varepsilon\in Q$, we define a complex

$$(5)\qquad (\Omega^{\cdot}_{X_0}(\varepsilon); df)\colon 0 \longrightarrow \Omega^0_{X_0,\varepsilon-n} \xrightarrow{df\wedge} \Omega^1_{X_0,\varepsilon-n+1} \xrightarrow{df\wedge} \cdots$$
$$\cdots \xrightarrow{df\wedge} \Omega^n_{X_0,\varepsilon} \longrightarrow 0\,,$$

so that $(\Omega^{\cdot}_{X_0}; df) = \bigoplus_{\varepsilon\in Q} (\Omega^{\cdot}_{X_0}(\varepsilon); df)$. Thus we obtain a direct decomposition of cohomology groups

$$(6)\qquad H^p(\Omega^{\cdot}_{X_0}; df) = \bigoplus_{\varepsilon\in Q} H^p(\Omega^{\cdot}_{X_0}(\varepsilon); df) \qquad (0\leqq p\leqq n)\,.$$

As to the n-th cohomology group, set $\Omega_f(\varepsilon) = H^n(\Omega^{\cdot}_{X_0}(\varepsilon); df)$. Then we have a direct decomposition $\Omega_f = \bigoplus_{\varepsilon\in Q} \Omega_f(\varepsilon)$ by the ρ-degree. Noting that $\Omega_f \xleftarrow{\sim} C[x]/(\partial_x f)$, one sees that the quotient ring $C[x]/(\partial_x f)$ has a C-basis consisting of the residue classes of polynomials all weighted homogeneous with respect to the weight ρ. A rational number ε is called an *exponent* of f if $\Omega_f(\varepsilon)\neq 0$. Since $1\notin(\partial_x f)$, the rational number $\varepsilon_* := \sum_{i=1}^n \rho_i$ is necessarily an exponent of f and that minimal. In fact, we have $L_{\theta_x} dx = \varepsilon_* dx$ $(dx = dx_1\wedge\cdots\wedge dx_n)$.

The assumption (A.2.ii) requires that F should have the form

$$(7)\qquad F(t_0, t, x) = t_0 + \sum_{\langle\rho,\nu\rangle<1} a_\nu(t)x^\nu + f(x) \qquad (a_\nu(t)\in C[t])\,.$$

Consider the Koszul complex of dF_0

$$(8)\qquad (\Omega^{\cdot}_{X/T}; dF_0)\colon 0 \longrightarrow \Omega^0_{X/T} \xrightarrow{dF_0\wedge} \Omega^1_{X/T} \xrightarrow{dF_0\wedge} \cdots \xrightarrow{dF_0\wedge} \Omega^n_{X/T} \longrightarrow 0$$

and define $\Omega_F := H^n(\Omega^{\cdot}_{X/T}; dF_0)$. Then we have

PROPOSITION 1.1. *Assume that* (A.1) *and* (A.2) *hold. Then*

a) $$H^p(\Omega^{\cdot}_{X/T}; dF_0) = 0 \qquad \text{for} \quad p\neq n\,,$$

b) *The $C[t]$-module $\Omega_F = H^n(\Omega^{\cdot}_{X/T}; dF_0)$ is free of rank $\mu = \mu(f)$. Moreover,let $\omega_1, \cdots, \omega_\mu$ be weigted homogeneous n-forms in $\Omega^n_{X_0}$ whose residue calsses form a C-basis of Ω_f. If we consider $\omega_1, \cdots, \omega_\mu$ as n-forms in $\Omega^n_{X/T}$, then their residue classes form a free $C[t]$-basis of Ω_F.*

PROOF. Choose a positive integer h such that $\rho_i h\in Z$ for $1\leqq i\leqq n$. For each integer r, set

$$F_r\Omega^{\cdot}_{X/T} := \bigoplus_{i\leqq r} C[t] \otimes_C \Omega^{\cdot}_{X_0}\Big(\frac{i}{h}\Big) \subset \Omega^{\cdot}_{X/T}\,.$$

Then it is easy to check that $(F_r\Omega^{\cdot}_{X/T})_{r\in Z}$ define an increasing filtration of $\Omega^{\cdot}_{X/T}$ such that

$$\Omega^{\cdot}_{X/T} = \bigcup_{r \in Z} F_r \Omega^{\cdot}_{X/T} \quad \text{and} \quad F_r \Omega^{\cdot}_{X/T} = 0 \quad \text{if} \quad r < 0 \ .$$

Moreover, this filtration is compatible with the exterior product $dF_0 \wedge$, so we have a convergent spectral sequence

$$E_1^{-r, r+p} = H^p(gr_r^F(\Omega^{\cdot}_{X/T}; dF_0)) \Longrightarrow H^p(\Omega^{\cdot}_{X/T}; dF_0) \ .$$

(See Cartan-Eilenberg [5], XV, 4.) The assumption (A.2.ii) then implies an isomorphism of complexes

$$gr_r^F(\Omega^{\cdot}_{X/T}; dF_0) \xrightarrow{\sim} \left(C[t] \underset{C}{\otimes} \Omega^{\cdot}_{X_0}\left(\frac{r}{h}\right); df \right) ,$$

for $r \in Z$. Since $H^p(C[t] \otimes_C \Omega^{\cdot}_{X_0}(r/h); df) = C[t] \otimes_C H^p(\Omega^{\cdot}_{X_0}(r/h); df)$, by (A.1) and (A.2.i) we have

$$H^p(gr_r^F(\Omega^{\cdot}_{X/T}; dF_0)) = \begin{cases} 0 & \text{if} \quad p \neq n \\ C[t] \underset{C}{\otimes} \Omega_f\left(\frac{r}{h}\right) & \text{if} \quad p = n \ . \end{cases}$$

(See (2) and (6) above.) This means $E_1^{-r, r+p} = 0$ for $p \neq n$, hence, by the machinery of spectral sequences, we obtain

$$H^p(\Omega^{\cdot}_{X/T}; dF_0) = 0 \qquad \text{for} \quad p \neq n \ ,$$

which proves a), and

$$H^n(gr^F(\Omega^{\cdot}_{X/T}; dF_0)) \xrightarrow{\sim} gr^F(H^n(\Omega^{\cdot}_{X/T}; dF_0)) \ .$$

The last isomorphism says that, with respect to the natural filtration of Ω_F, we have an isomorphism

$$C[t] \underset{C}{\otimes} \Omega_f\left(\frac{r}{h}\right) \xrightarrow{\sim} gr_r^F(\Omega_F) \qquad \text{for} \quad r \in Z \ .$$

Since $\Omega_f(r/h) = 0$ except for a finite number of r, we may conclude that Ω_F is a free $C[t]$-module of rank μ and that a weighted homogeneous C-basis of Ω_f induces a free $C[t]$-basis of Ω_F, as desired. Q.E.D.

Now we turn to the Gauss-Manin system H_F^{λ}. Recall that

(9) $$H_F^{\lambda} \xrightarrow{\sim} H^n(\Omega_{Z/S}(M_F^{\lambda})^{\cdot}) \quad \text{and} \quad H_F^{(\lambda)} \xrightarrow{\sim} H^n(\Omega_{Z/S}(M_F^{(\lambda)})^{\cdot})$$

for each $\lambda \in C$. In view of 1.1.(5), we obtain an exact sequence

(10) $$0 \longrightarrow M_F^{(\lambda-1)} \longrightarrow M_F^{(\lambda)} \xrightarrow{\sigma_\lambda} \mathscr{R}(X) \longrightarrow 0 \ ,$$

where σ_λ is defined by

$$\sigma_\lambda\Big(\sum_{k=0}^{\infty} a_k(t,x)\delta_F^{(\lambda-k)}\Big)=a_0(t,x)\,. \tag{11}$$

Noting that $\Omega_{Z/S}(M_F^{(\lambda)})^p=\Omega_{X/T}^p\otimes_{\mathscr{R}(X)} M_F^{(\lambda-n+p)}$, we define the "symbol" homomorphism

$$\sigma_\lambda^p\colon \Omega_{Z/S}(M_F^{(\lambda)})^p \longrightarrow \Omega_{X/T}^p \tag{12}$$

by $\sigma_\lambda^p(\omega\otimes u)=\omega\sigma_{\lambda-n+p}(u)$ for $\omega\in\Omega_{X/T}^p$ and $u\in M_F^{(\lambda-n+p)}$. Then we obtain an exact sequence of complexes

$$0\longrightarrow \Omega_{Z/S}(M_F^{(\lambda-1)})^\cdot \longrightarrow \Omega_{Z/S}(M_F^{(\lambda)})^\cdot \xrightarrow{\sigma_\lambda^\cdot} \Omega_{X/T}^\cdot \longrightarrow 0\,, \tag{13}$$

where $\Omega_{X/T}^\cdot$ stands for the Koszul complex $(\Omega_{X/T}^\cdot; dF_0)$. So far as (A.1) and (A.2) hold, the cohomology groups of this Koszul complex are known by Proposition 1.1. Hence, passing to the long exact sequence of cohomology groups, (13) induces natural isomorphisms

$$H^p(\Omega_{Z/S}(M_F^{(\lambda-1)})^\cdot) \xrightarrow{\sim} H^p(\Omega_{Z/S}(M_F^{(\lambda)})^\cdot) \qquad \text{for} \quad p\neq n \tag{14}$$

and an exact sequence

$$0\longrightarrow H^n(\Omega_{Z/S}(M_F^{(\lambda-1)})^\cdot)\longrightarrow H^n(\Omega_{Z/S}(M_F^{(\lambda)})^\cdot)\longrightarrow H^n(\Omega_{X/T}^\cdot; dF_0)\longrightarrow 0\,. \tag{15}$$

Thus we have proved

Proposition 1.2. *On the assumptions* (A.1) *and* (A.2),

a) *There is an exact sequence*

$$0\longrightarrow H_F^{(\lambda-1)}\longrightarrow H_F^{(\lambda)}\xrightarrow{r^{(\lambda)}}\Omega_F\longrightarrow 0$$

for each $\lambda\in \boldsymbol{C}$, *where* $r^{(\lambda)}$ *is an* $\mathscr{R}(T)$*-homomorphism characterized by*

$$r^{(\lambda)}(\omega\delta_F^{(\lambda)})=[\omega] \qquad for \quad \omega\in\Omega_{X/T}^n\,.$$

b) *For each* $\lambda\in\boldsymbol{C}$, $H_F^{(\lambda+k)}$ $(k\in\boldsymbol{Z})$ *define an exhaustive increasing filtration of* H_F^λ *such that* $D_{t_0}\colon H_F^{(\lambda+k)}\to H_F^{(\lambda+k+1)}$ *is an isomorphism for each* $k\in\boldsymbol{Z}$.

The assertion b) is clear.

The next theorem is compared to Proposition 6.2.2 of F. Pham [14], 2ème partie.

Theorem 1.3. *On the assumptions* (A.1) *and* (A.2), *let* $\omega_1,\cdots,\omega_\mu$ *be* *weighted homogeneous n-forms in* $\Omega^n_{X_0}$ *whose residue classes form a* C*-basis of* Ω_f. *For* $1 \leqq i \leqq \mu$, *we define*

$$u_i := \int \delta_F^{(\lambda)} \omega_i \in H_F^{(\lambda)} ,$$

considering $\omega_1, \cdots, \omega_\mu$ *as n-forms in* $\Omega^n_{X/T}$. *Then* $u_1, \cdots, u_\mu$ *form a free basis of* $H_F^{(\lambda)}$ *over the ring* $C[t][D_{t_0}^{-1}]$, *so that the Gauss-Manin system* $H_F^{(\lambda)}$ *(resp.* H_F^λ*) is a free* $C[t][D_{t_0}^{-1}]$*-module (resp.* $C[t][D_{t_0}, D_{t_0}^{-1}]$*-module) of rank* $\mu = \mu(f)$.

First, let us show that $u_1, \cdots, u_\mu$ are independent over the ring $C[t][D_{t_0}^{-1}]$. Let $P_1, \cdots, P_\mu$ be operators in $C[t][D_{t_0}^{-1}]$, not all zero, such that $P_1 u_1 + \cdots + P_\mu u_\mu = 0$. Set $k = -\max\{\mathrm{ord}(P_i); 1 \leqq i \leqq \mu\}$ and denote by $a_i = a_i(t)$ the coefficient of $D_{t_0}^{-k}$ of P_i for $1 \leqq i \leqq \mu$. Then through the "symbol" homomorphism $r^{(\lambda-k)}: H_F^{(\lambda-k)} \to \Omega_F$, one sees $a_1\omega_1 + \cdots + a_\mu \omega_\mu = 0$, which contradicts to the fact that $\omega_1, \cdots, \omega_\mu$ form a free $C[t]$-basis of Ω_F (Proposition 1.1.b)). Next we will prove that each element $u \in H_F^{(\lambda)}$ has an expression

$$u = P_1 u_1 + \cdots + P_\mu u_\mu \quad \text{for some} \quad P_i \in C[t][D_{t_0}^{-1}] . \tag{16}$$

To do so, it is enough to consider the case where $u = \int \delta_F^{(\lambda)} \omega$ with $\omega \in \Omega^n_{X/T}$, since $\int \delta_F^{(\lambda-k)} \omega = D_{t_0}^{-k} \int \delta_F^{(\lambda)} \omega$.

LEMMA 1.4. *Let* ω *be an n-form in* $\Omega^n_{X/T}$ *of* ρ*-degree* ε. *Then there exist polynomials* $a_1, \cdots, a_\mu$ *in* $C[t]$ *and an n-form* ζ *in* $\Omega^n_{X/T}$ *of* ρ*-degree* $\leqq \varepsilon - 1$ *such that*

$$\int \delta_F^{(\lambda)} \omega = \sum_{i=1}^{\mu} a_i \int \delta_F^{(\lambda)} \omega_i + D_{t_0}^{-1} \int \delta_F^{(\lambda)} \zeta .$$

PROOF. We use the notation in the proof of Proposition 1.1. Set $r := \varepsilon h$ so that $\omega \in F_r \Omega^n_{X/T}$. Recall that

$$H^n(gr_r^F(\Omega^{\bullet}_{X/T}; dF_0)) = C[t] \underset{C}{\otimes} \Omega_f\left(\frac{r}{h}\right) .$$

This implies that there exist $a_1, \cdots, a_\mu \in C[t]$ and $\eta \in F_r \Omega^{n-1}_{X/T}$ such that

$$\omega = \sum_{i=1}^{\mu} a_i \omega_i + dF_0 \wedge \eta .$$

Here we have

$$\int \delta_F^{(\lambda)} dF_0 \wedge \eta = \int d\delta_F^{(\lambda-1)} \wedge \eta = -\int \delta_F^{(\lambda-1)} d\eta$$

by the integration by parts. Hence

$$\int \delta_F^{(\lambda)} \omega = \sum_{i=1}^{\mu} a_i \int \delta_F^{(\lambda)} \omega_i - D_{t_0}^{-1} \int \delta_F^{(\lambda)} d\eta \ .$$

By the definition of $F_r \Omega_{X/T}^{n-1}$, η is an $(n-1)$-form of ρ-degree$\leqq (r/h)-1=\varepsilon-1$. Since $[L_{\theta_x}, d]=0$, one sees that $d\eta$ is an n-form of ρ-degree$\leqq\varepsilon-1$. This proves Lemma. Q.E.D.

By applying Lemma 1.4 repeatedly, we see that, for any integer $N\geqq 0$, there exist $a_i^k \in C[t]$ $(1\leqq i\leqq\mu,\ 0\leqq k\leqq N)$ and a n-form $\zeta\in\Omega_{X/T}^n$ of ρ-degree$\leqq\varepsilon-(N+1)$ such that

$$\int \delta_F^{(\lambda)} \omega = \sum_{i=1}^{\mu} \sum_{k=0}^{N} a_i^k D_{t_0}^{-k} u_i + N_{t_0}^{-N-1} \int \delta_F^{(\lambda)} \zeta \ . \tag{17}$$

Let $\varepsilon_* = \sum_{i=1}^n \rho_i$ be the minimal exponent of f and set $N=[\varepsilon-\varepsilon_*]$ so that $\varepsilon-(N+1)<\varepsilon_*$. Then we have $\zeta=0$. Thus we have proved that $u_1, \cdots, u_\mu$ form a free $C[t][D_{t_0}^{-1}]$-basis of $H_F^{(\lambda)}$. The "resp." part follows easily from this by Proposition 1.2.b), which completes the proof of Theorem 1.3.

COROLLARY. *Let ω be an n-form in $\Omega_{X/T}^n$ of ρ-degree$\leqq\varepsilon$. Then there exist operators $P_1, \cdots, P_\mu$ in $C[t][D_{t_0}^{-1}]$ of the form*

$$P_i = \sum_{k=0}^{N} a_i^k D_{t_0}^{-k} \qquad (a_i^k \in C[t]) \quad \textit{with} \quad N=[\varepsilon-\varepsilon_*]\ ,$$

where ε_* stands for the minimal exponent, such that

$$\int \delta_F^{(\lambda)} \omega = \sum_{i=1}^{\mu} P_i u_i \ .$$

We keep the assumption of Theorem 1.3, and denote by $\vec{u}$ tne column vector ${}^t(u_1, \cdots, u_\mu)$. Recall that the Gauss-Manin system H_F^λ is a left module over the ring

$$\mathscr{D}(S)[D_{t_0}^{-1}] = C[t_0, t, D_{t_0}, D_t][D_{t_0}^{-1}] \ . \tag{18}$$

Theorem 1.3 tells us that, as to the operation of t_0, D_{t_k} $(1\leqq k\leqq m)$, there exist $\mu\times\mu$ matrices A and $B^{(k)}$ $(1\leqq k\leqq m)$ with entries in $C[t][D_{t_0}^{-1}]$ such that

$$\begin{cases} t_0\vec{u} = A(t, D_{t_0})\vec{u} \\ D_{t_k} D_{t_0}^{-1} \vec{u} = B^{(k)}(t, D_{t_0})\vec{u} \qquad (1\leqq k\leqq m)\ . \end{cases} \tag{19}$$

The matrices A and $B^{(k)}$ $(1 \leqq k \leqq m)$ are uniquely determined since $u_1, \cdots u_\mu$ are independent over $C[t][D_{t_0}^{-1}]$. Since the vector field D_{t_0} operates on H_F^λ as an isomorphism, one may replace (19), if one prefers, by

$$\begin{cases} D_{t_0}^N t_0 \vec{u} = D_{t_0}^N A(t, D_{t_0}) \vec{u} \\ D_{t_0}^{N-1} D_{t_k} \vec{u} = D_{t_0}^N B^{(k)}(t_0, D_{t_0}) \vec{u} \qquad (1 \leqq k \leqq m) \end{cases} \tag{20}$$

with N large enough so that the operators should not involve $D_{t_0}^{-1}$.

PROPOSITION 1.5. *The differential system* (19) *gives a finite presentation of the Gauss-Manin system* H_F^λ *(resp.* $H_F^{(\lambda)}$*) over the ring* $\mathscr{D}(S)[D_{t_0}^{-1}]$ *(resp.* $\mathscr{D}(S)[D_{t_0}^{-1}](0)$*).*

PROOF. Let P be a row vector of size μ with entries in $\mathscr{D}(S)[D_{t_0}^{-1}]$. Then one can check that there exist a sequence of row vectors $Q_0, Q_1, \cdots, Q_m$ in $\mathscr{D}(S)[D_{t_0}^{-1}]^\mu$ and a row vector R in $C[t][D_{t_0}, D_{t_0}^{-1}]^\mu$ such that

$$P = Q_0(t_0 - A) + \sum_{k=1}^{m} Q_k(D_{t_k} D_{t_0}^{-1} - B^{(k)}) + R .$$

Assume $P\vec{u}=0$. Then we have $R\vec{u}=0$, which implies $R=0$ since $u_1, \cdots, u_\mu$ is free over $C[t][D_{t_0}, D_{t_0}^{-1}]$. Hence

$$P = Q_0(t_0 - A) + \sum_{k=1}^{m} Q_k(D_{t_k} D_{t_0}^{-1} - B^{(k)}) .$$

This shows that (19) gives a finite presentation of H_F^λ as a $\mathscr{D}(S)[D_{t_0}^{-1}]$-module. The "resp." part can be shown similarly. Q.E.D.

Here we include a remark on the Gauss-Manin system $H_{t_0+f}^\lambda$ (on the assumption (A.1)). Let $\varepsilon_1, \cdots, \varepsilon_\mu$ be the exponents of f and let $\omega_1, \cdots, \omega_\mu$ be weighted homogeneous n-forms, with $\omega_i \in \Omega_{X_0}^n(\varepsilon_i)$, whose residue classes form a basis of Ω_f. Then we define

$$w_i = \int \delta_{t_0+f}^{(\lambda)} \omega_i \in H_{t_0+f}^{(\lambda)} \qquad \text{for} \quad 1 \leqq i \leqq \mu . \tag{21}$$

Then the Gauss-Manin system for the column vector $\vec{w} = {}^t(w_1, \cdots, w_\mu)$ can be computed directly, so that we have

$$t_0 \vec{w} = -\Lambda D_{t_0}^{-1} \vec{w} , \tag{22}$$

where Λ is the diagonal matrix $\mathrm{diag}(\lambda - \varepsilon_1, \cdots, \lambda - \varepsilon_\mu)$.

We remark that our formation of the Gauss-Manin system H_F^λ is compatible with the "restriction of parameters t". Hence, as to the matrices of the presentation (19) for a general F, we have $A|_{t=0}=-\Delta D_{t_0}^{-1}$. In this sense, the differential system (19) gives a "deformation" of the system (22).

The matrices A and $B^{(k)}$ $(1\leqq k\leqq m)$ of the presentation (19) can be written in the form

$$\text{(23)}\qquad \begin{cases} A(t, D_{t_0})=\sum_{r=0}^{N} A_r(t)D_{t_0}^{-r} \\ B^{(k)}(t, D_{t_0})=\sum_{r=0}^{N} B_r^{(k)}(t)D_{t_0}^{-r} , \end{cases}$$

where $A_r(t)$, $B_r^{(k)}(t)$ are $\mu\times\mu$ matrices with entires in $\mathscr{R}(T)=C[t]$. To end this paragraph, we explain how the matrices A_0 and $B_0^{(k)}$ are computed. Let $e_1, \cdots, e_\mu$ be polynomials in $C[x]$ such that $\omega_i=e_i dx$ $(1\leqq i\leqq\mu)$ and set $\vec{e}={}^t(e_1, \cdots, e_\mu)$. Then, by chasing the proof of Lemma 1.4, one can see that the matrices in question are determined by the relations

$$\text{(24)}\qquad \begin{cases} -F_0\vec{e}\equiv A_0\vec{e} \\ \partial_{t_k}(F_0)\vec{e}\equiv B^{(k)}\vec{e} \end{cases} \quad \operatorname{mod}(\partial_x(F_0)) ,$$

in $\mathscr{R}(C)=\mathscr{R}(X)/(\partial_x(F_0))$. Recall that the affine ring $\mathscr{R}(C)$ of the critical variety has a structure of module over $\mathscr{R}(S)=C[t_0, t]$ induced by $\varphi: X\to S$. Then, the first equation of (24) involves that the $\mathscr{R}(S)$-module $\mathscr{R}(C)$ has a finite presentation

$$\text{(25)}\qquad 0\longleftarrow\mathscr{R}(C)\longleftarrow\mathscr{R}(S)^\mu\xleftarrow{t_0I-A_0(t)}\mathscr{R}(S)^\mu ,$$

where I stands for the identity matrix of size μ. So the discriminant set $D=\varphi(C)$ has a defining function

$$\text{(26)}\qquad \Delta(t_0, t):=\det(t_0I-A_0(t)) ,$$

which will be called the *discriminant* of φ. (See K. Saito [19].)

1.3. Generating function for H_F^λ.

This paragraph is an algebraic version of S. Ishiura [7]. Here, we review an explicit presentation of H_F^λ as a differential system involving a single unknown function $u=\int\delta_F^{(\lambda)}dx$, on an assumption of "non-degeneracy" of F.

Let l be a positive integer with $l\leqq n$ and $l\leqq m$. We use the notation

$$\text{(1)}\qquad \begin{cases} x=(x', x'') ; & x'=(x_1, \cdots, x_l) , & x''=(x_{l+1}, \cdots, x_n) \\ t=(t', t'') ; & t'=(t_1, \cdots, t_l) , & t''=(t_{l+1}, \cdots, t_m) . \end{cases}$$

Assume that

(B.1) *F is written in the form*

$$F(t_0, t, x) = t_0 + t_1 x_1 + \cdots + t_l x_l + G(t'', x') + x_{l+1}^2 + \cdots + x_n^2 \,.$$

where G is a polynomial in $C[t'', x']$.

On this assumption, we try to compute directly the differential system to be satisfied by the integral

$$u := \int \delta_F^{(\lambda)} dx \in H_F^{(\lambda)} \,. \tag{2}$$

By 1.1.(4), we have

$$\begin{cases} D_{t_i} D_{t_0}^{-1} \delta_F^{(\lambda)} = x_i \delta_F^{(\lambda)} & \text{for } 1 \leqq i \leqq l \\ D_{x_j} D_{t_0}^{-1} \delta_F^{(\lambda)} = 2x_j \delta_F^{(\lambda)} & \text{for } l+1 \leqq j \leqq n \,. \end{cases} \tag{3}$$

Hence, with the notation $D_{t'} D_{t_0}^{-1} = (D_{t_1} D_{t_0}^{-1}, \cdots, D_{t_l} D_{t_0}^{-1})$,

$$\begin{aligned} (t_0 + \lambda D_{t_0}^{-1}) \int \delta_F^{(\lambda)} dx &= -\int F_0(t, x) \delta_F^{(\lambda)} dx \\ &= -\sum_{i=1}^{l} t_i \int x_i \delta_F^{(\lambda)} dx - \int G(t'', x') \delta_F^{(\lambda)} dx - \sum_{j=l+1}^{n} \int x_j^2 \delta_F^{(\lambda)} dx \\ &= -\sum_{i=1}^{l} t_i D_{t_i} D_{t_0}^{-1} \int \delta_F^{(\lambda)} dx - G(t'', D_{t'} D_{t_0}^{-1}) \int \delta_F^{(\lambda)} dx \\ &\quad - \sum_{j=l+1}^{n} \frac{1}{2} \int x_j \cdot D_{x_j} D_{t_0}^{-1} \delta_F^{(\lambda)} dx \\ &= -\left(\sum_{i=1}^{l} t_i D_{t_i} D_{t_0}^{-1} + G(t'', D_{t'} D_{t_0}^{-1}) - \frac{n-l}{2} D_{t_0}^{-1} \right) \int \delta_F^{(\lambda)} dx \,. \end{aligned} \tag{4}$$

Thus we have

$$\left(t_0 + \sum_{i=1}^{l} t_i D_{t_i} D_{t_0}^{-1} + G(t'', D_{t'} D_{t_0}^{-1}) + \left(\lambda - \frac{n-l}{2} \right) D_{t_0}^{-1} \right) u = 0 \tag{5}$$

On the other hand, the presentation 1.1.(4) implies

$$\begin{aligned} D_{x_i} D_{t_0}^{-1} \delta_F^{(\lambda)} &= (t_i + G_{x_i}(t'', x')) \delta_F^{(\lambda)} \\ &= (t_i + G_{x_i}(t'', D_{t'} D_{t_0}^{-1})) \delta_F^{(\lambda)} \qquad \text{for } 1 \leqq i \leqq l \end{aligned} \tag{6}$$

where $G_{x_i} := \partial_{x_i} G$. Hence, we obtain

$$(t_i + G_{x_i}(t'', D_{t'} D_{t_0}^{-1})) u = 0 \qquad \text{for } 1 \leqq i \leqq l \,. \tag{7}$$

Similarly, we get

$$(8)\qquad (D_{t_j}-G_{t_j}(t'', D_{t'}D_{t_0}^{-1}))u=0 \qquad \text{for } l+1\leqq j\leqq m\,.$$

Summarizing, the integral (1) satisfies the system of differential equations

$$(9)\qquad \begin{cases} \left(t_0-\sum_{i=1}^{l} D_{t_i}D_{t_0}^{-1}G_{x_i}(t'', D_{t'}D_{t_0}^{-1})+G(t'', D_{t'}D_{t_0}^{-1})+\left(\lambda-\frac{n+l}{2}\right)D_{t_0}^{-1}\right)u=0 \\ (t_i+G_{x_i}(t'', D_{t'}D_{t_0}^{-1}))u=0 \qquad (1\leqq i\leqq l) \\ (D_{t_j}-G_{t_j}(t'', D_{t'}D_{t_0}^{-1}))u=0 \qquad (l+1\leqq j\leqq m)\,, \end{cases}$$

where the equation (5) is rewritten by using (7). Let $(\tau_0; \tau')=(\tau_0, \tau_1, \cdots, \tau_l)$ be the dual variables of $(t_0, t_1, \cdots, t_l)$. Then we find that the system (9) has the *generating function* $H(\tau_0, \tau', t'')$ in $C[\tau_0, \tau', t''][\tau_0^{-1}]$ defined by

$$(10)\qquad H(\tau_0, \tau', t'')=-\tau_0 G(t'', \tau'\tau_0^{-1})\,,$$

so that the system should have the form

$$(11)\qquad \begin{cases} \left(t_0-H_{\tau_0}(D_{t_0}, D_{t'}, t'')+\left(\lambda-\frac{n+l}{2}\right)D_{t_0}^{-1}\right)u=0 \\ (t_i-H_{\tau_i}(D_{t_0}, D_{t'}, t''))u=0 \qquad (1\leqq i\leqq l) \\ (D_{t_j}+H_{t_j}(D_{t_0}, D_{t'}, t''))u=0 \qquad (l+1\leqq j\leqq m)\,. \end{cases}$$

Now we propose to prove that the differential system (11) gives a finite presentation of H_F^λ, namely, the left $\mathscr{D}(S)[D_{t_0}^{-1}]$-module H_F^λ is isomorphic to the one defined to be the quotient module of $\mathscr{D}(S)[D_{t_0}^{-1}]$ modulo the left ideal generated by the operators appearing in (11).

The first thing we do is to erase the term $x_{l+1}^2+\cdots+x_n^2$ in F. Assume that F is written in the form

$$(12)\qquad F(t_0, t, x)=F'(t_0, t, x')+x_{l+1}^2+\cdots+x_n^2$$

with $F'(t_0, t, x')=t_0+F_0'(t, x')$. Then, one might expect the "equality"

$$(13)\qquad \int \delta_F^{(\lambda)}dx''=\text{const.}\cdot\delta_{F'}^{(\lambda-(n-l/2))}\,.$$

Let $M_{F'}^\lambda$ be the $\mathscr{D}(Z')[D_{t_0}^{-1}]$-module which is defined analogously to M_F^λ on the affine space $Z'=C^l\times S$ with coordinates (t_0, t, x').

PROPOSITION 1.6. *With the above notation, we have a natural isomorphism*

$$M_{F'}^{\lambda-(n-l)/2} \xrightarrow{\sim} M_F^\lambda \Big/ \sum_{j=l+1}^{n} D_{x_j} M_F^\lambda$$

which sends $\delta_{F'}^{(\lambda-(n-l)/2)}$ *to the modulo class of* $\delta_F^{(\lambda)}$.

PROOF. It is enough to consider the case where $l=n-1$. Note that

$$\int x_n^2 \delta_F^{(\lambda)} dx_n = \frac{1}{2} \int x_n D_{x_n} D_{t_0}^{-1} \delta_F^{(\lambda)} dx_n = -\frac{1}{2} D_{t_0}^{-1} \int \delta_F^{(\lambda)} dx_n ,$$

where the symbol $\int dx_n$ stands for $M_F^\lambda \to M_F^\lambda / D_{x_n} M_F^\lambda$. It is easy to show that $\int \delta_F^{(\lambda)} dx_n$ satisfies the equations of $\delta_{F'}^{(\lambda-(1/2))}$ corresponding to 1.1.(4). This means that there is a $\mathscr{D}(Z')[D_{t_0}^{-1}]$-homomorphism $M_{F'}^{\lambda-(1/2)} \to M_F^\lambda / D_{x_n} M_F^\lambda$ which sends $\delta_{F'}^{(\lambda-(1/2))}$ to $\delta_F^{(\lambda)} dx_n$. To prove that this homomorphism is bijective, it is enough to show that $M_F^\lambda / D_{x_n} M_F^\lambda$ is a free $C[t, x'][D_{t_0}, D_{t_0}^{-1}]$-module of rank one with basis $\int \delta_F^{(\lambda)} dx_n$, which can be checked by a direct computation. (Write each element of M_F^λ in the form $\sum_{k=0}^N a_k(t, x', D_{t_0}) x_n^k \delta_F^{(\lambda)}$ with $a_k(t, x', D_{t_0}) \in C[t, x'][D_{t_0}, D_{t_0}^{-1}]$.) Q.E.D.

PROPOSITION 1.7. *On the assumption* (B.1), *the Gauss-Manin system* H_F^λ *is a free module of rank one over the ring* $C[t'', D_{t'}][D_{t_0}, D_{t_0}^{-1}]$ *with basis* $\int \delta_F^{(\lambda)} dx$.

PROOF. By Proposition 1.6, we may assume that $l=n$, so that $x'=x=(x_1, \cdots, x_n)$. Set $A=C[t, x, D_{t_0}, D_{t_0}^{-1}]$. Then M_F^λ is a free A-module of rank one with basis $\delta_F^{(\lambda)}$. For each $d \in N$, let us denote by $A^{(d)}$ (resp. A_d) the $C[t'', x][D_{t_0}, D_{t_0}^{-1}]$-submodule of A consisting of all elements of degree $\leqq d$ (resp. homogeneous of degree d) in $t'=(t_1, \cdots, t_n)$, so that $A^{(0)}=A_0=C[t'', x][D_{t_0}, D_{t_0}^{-1}]$. Setting $K^\cdot = \Omega_{Z/S}(M_F^\lambda)^\cdot$, we define an increasing filtration $(F_r K^\cdot)_{r \in N}$ of $K^\cdot$ by

$$F_r K^\cdot = \Omega_{X_0}^\cdot \underset{C[x]}{\otimes} A^{(r-n+\cdot)} \delta_F^{(\lambda)} \qquad (r \in N) ,$$

so we have a convergent spectral sequence

$$E_1^{-r, r+p} = H^p(gr_r^F K^\cdot) \Longrightarrow H^p(K^\cdot) .$$

On the other hand, setting $\omega = \sum_{i=1}^n t_i D_{t_0} dx_i$, we consider the "graded" Koszul complex $L^\cdot = (\Omega_{X_0}^\cdot \otimes_{C[x]} A; \omega)$ of ω endowed with the graduation

$$(L_r^\cdot)_{r \in N} ; \qquad L_r^\cdot = (\Omega_{X_0}^\cdot \underset{C[x]}{\otimes} A_{r-n+\cdot}; \omega) \qquad (r \in N) .$$

Then it is easily checked that there is an natural isomorphism of complexes

$$gr_r^F(K^\cdot) \xrightarrow{\sim} L_r^\cdot$$

for each $r \in N$. Since $t_1 D_{t_0}, \cdots, t_n D_{t_0}$ form a regular sequence in A, one has

$$\begin{cases} H^p(L_r^\cdot)=0 & \text{if} \quad p \neq n \quad \text{or} \quad r \neq 0 \\ H^n(L_0^\cdot)=C[t'', x][D_{t_0}, D_{t_0}^{-1}] \,. \end{cases}$$

This means

$$\begin{cases} E_1^{-r,r+p}=0 & \text{except for} \quad (-r, r+p)=(0, n) \\ E_1^{0,n}=C[t'', x][D_{t_0}, D_{t_0}^{-1}] \,, \end{cases}$$

hence we have an isomorphism

$$H^n(K^\cdot) \xleftarrow{\sim} C[t'', x][D_{t_0}, D_{t_0}^{-1}] \,.$$

Thus we have proved that there is a natural isomorphism

$$C[t'', x][D_{t_0}, D_{t_0}^{-1}]\delta_F^{(\lambda)} \xrightarrow{\sim} H_F^\lambda \,.$$

Note here that

$$\int x_i \delta_F^{(\lambda)} dx = D_{t_i} D_{t_0}^{-1} \int \delta_F^{(\lambda)} dx \qquad (1 \leqq i \leqq n) \,,$$

and that the correspondence $x_i \mapsto D_{t_i} D_{t_0}^{-1} (1 \leqq i \leqq n)$ defines a ring isomorphism $C[t'', x][D_{t_0}, D_{t_0}^{-1}] \xrightarrow{\sim} C[t'', D_{t'}][D_{t_0}, D_{t_0}^{-1}]$. Hence we know that

$$C[t'', D_{t'}][D_{t_0}, D_{t_0}^{-1}] \int \delta_F^{(\lambda)} dx = H_F^\lambda$$

and $\int \delta_F^{(\lambda)} dx$ is a free basis over $C[t'', D_{t'}][D_{t_0}, D_{t_0}^{-1}]$. Q.E.D.

By using Proposition 1.7, we can prove

PROPOSITION 1.8. *On the assumption* (B.1), *set*

$$H(\tau_0, \tau', t''):=-\tau_0 G(t'', \tau' \tau_0^{-1}) \,.$$

Then the $\mathscr{D}(S)[D_{t_0}^{-1}]$*-module* H_F^λ *has a finite presentation*

$$\begin{cases} \left(t_0 - H_{\tau_0}(D_{t_0}, D_{t'}, t'') + \left(\lambda - \frac{n+l}{2}\right) D_{t_0}^{-1}\right) u = 0 \\ (t_i - H_{\tau_i}(D_{t_0}, D_{t'}, t''))u = 0 & (1 \leqq i \leqq l) \\ (D_{t_j} + H_{t_j}(D_{t_0}, D_{t'}, , t''))u = 0 & (l+1 \leqq j \leqq m) \end{cases}$$

with respect to the integral

$$u = \int \delta_F^{(\lambda)} dx \in H_F^{(\lambda)} .$$

PROOF. We have already seen that the u satifies the above equations in H_F^λ. Note that each element $P \in \mathscr{D}(S)[D_{t_0}^{-1}]$ can be written in the form

$$P = Q_0\left(t_0 - H_{\tau_0} + \left(\lambda - \frac{n+l}{2}\right) D_{t_0}^{-1}\right) + \sum_{i=1}^{l} Q_i(t_i - H_{\tau_i}) + \sum_{j=l+1}^{m} Q_j(D_{t_j} + H_{t_j}) + R ,$$

with $Q_k \in \mathscr{D}(S)[D_{t_0}^{-1}]$ $(0 \leqq k \leqq m)$ and $R \in C[t'', D_{t'}][D_{t_0}, D_{t_0}^{-1}]$. If $Pu = 0$, then $Ru = 0$. Since u is a free basis of H_F^λ over $C[t'', D_{t'}][D_{t_0}, D_{t_0}^{-1}]$, this implies $R = 0$, which was to be proved. Q.E.D.

In the later arguments, we will be concerned with the case where $F = F(t_0, t, x)$ is weighted homogeneous in (t_0, t, x). Here we contain a remark on such cases. As in 1.2, let $\rho = (\rho_1, \cdots, \rho_n)$ be the weight of $x = (x_1, \cdots, x_n)$. Now let $\sigma = (\sigma_1, \cdots, \sigma_m)$ be an m-vector of rational numbers, which we do not assume to be positive. We make the assumption

(B.2) *$F = (t_0, t, x)$ is a weighted homogeneous polynomial of degree* 1 *with respect to the weight* $(1, \sigma, \rho)$ *of* (t_0, t, x).

Hereafter we assume that F satisfies the conditions (B.1) and (B.2). In this case, one necessarily has

$$\sigma_i = 1 - \rho_i \quad \text{for} \quad 1 \leqq i \leqq l \quad \text{and} \quad \rho_j = \frac{1}{2} \quad \text{for} \quad l+1 \leqq j \leqq m . \tag{14}$$

Moreover, one sees easily that the generating function $H(\tau_0, \tau', t'') = -\tau_0 G(t'', \tau' \tau_0^{-1})$ is weighted homogeneous of degree zero with respect to the weight $(-1, -\sigma', \sigma'')$ of (τ_0, τ', t''), where $\sigma' = (\sigma_1, \cdots, \sigma_l)$ and $\sigma'' = (\sigma_{l+1}, \cdots, \sigma_m)$. Hence we have an identity

$$-D_{t_0} H_{\tau_0} - \sum_{i=1}^{l} \sigma_i D_{t_i} H_{\tau_i} + \sum_{j=l+1}^{m} \sigma_j t_j H_{t_j} = 0 \tag{15}$$

in $C[t'', D_{t'}][D_{t_0}, D_{t_0}^{-1}]$. By (15), we obtain from Proposition 1.8

$$\left(D_{t_0}t_0+\sum_{i=1}^{l}\sigma_i D_{t_i}t_i+\sum_{j=l+1}^{m}\sigma_j t_j D_{t_j}+\left(\lambda-\frac{n+l}{2}\right)D_{t_0}^{-1}\right)u=0 . \tag{16}$$

We define the Euler vector field θ on the space S by

$$\theta:=t_0 D_{t_0}+\sum_{k=1}^{m}\sigma_k t_k D_{t_k} . \tag{17}$$

Then the equation (16) is rewritten in the form

$$(\theta+1+\lambda-\varepsilon_*)u=0 \quad \text{with} \quad \varepsilon_*=\sum_{i=1}^{n}\rho_i , \tag{18}$$

where we computed

$$\begin{aligned} 1+\sum_{i=1}^{l}\sigma_i+\lambda-\frac{n+l}{2} &= 1+\sum_{i=1}^{l}(1-\rho_i)+\lambda-\sum_{j=l+1}^{n}\rho_j-l \\ &= 1+\lambda-\sum_{i=1}^{n}\rho_i \end{aligned} \tag{19}$$

using (14). Thus we have

Proposition 1.9. *On the assumptions* (B.1) *and* (B.2), *the Gauss-Manin system H_F^λ has a finite presentation over $\mathscr{D}(S)[D_{t_0}^{-1}]$*

$$\begin{cases} (\theta+\lambda+1-\varepsilon_*)u=0 & \\ (t_i+G_{x_i}(t'', D_{t'}D_{t_0}^{-1}))u=0 & (1\leqq i\leqq l) \\ (D_{t_j}D_{t_0}^{-1}-G_{t_j}(t'', D_{t'}D_{t_0}^{-1}))u=0 & (l+1\leqq j\leqq m) , \end{cases}$$

where $\theta=t_0 D_{t_0}+\sum_{k=1}^{m}\sigma_k t_k D_{t_k}$ is the Euler vector field on S and $\varepsilon_*=\sum_{i=1}^{n}\rho_i$.

It is easy to show that the left ideal of $\mathscr{D}(S)[D_{t_0}^{-1}]$ generated by the operators appearing in Proposition 1.9 coincides with the one corresponding to the presentation of Proposition 1.8. (In fact, the operators other than $\theta+\lambda+1-\varepsilon_*$ are the same as before.)

Remark. Assume that F satisfies the condition (B.1). Then the "characteristic variety" of H_F^λ is given by the equations

$$\begin{cases} t_i-H_{\tau_i}(\tau_0, \tau', t'')=0 & (0\leqq i\leqq l) \\ \tau_j+H_{t_j}(\tau_0, \tau', t'')=0 & (l+1\leqq j\leqq m) \end{cases} \tag{20}$$

on the open set $\{\tau_0\neq 0\}$ of the cotangent bundle T^*S. In this sense, the polynomial $H(\tau_0, \tau', t'')$ is the generating function of the Lagrangean variety. (See S. Ishiura [7].)

§2. A reduction of the Gauss-Manin system.

2.1. Reduction at a point of infinity.

In §1.2, we have proved, on the assumptions (A.1) and (A.2), that the Gauss-Manin system H_F^λ has a finite presentation

$$\text{(1)}\qquad \begin{cases} t_0\vec{u}=A(t,\, D_{t_0})\vec{u} \\ D_{t_k}D_{t_0}^{-1}\vec{u}=B^{(k)}(t,\, D_{t_0})\vec{u} \qquad (1\leqq k\leqq m)\,, \end{cases}$$

where

$$\text{(2)}\qquad \begin{cases} A(t,\, D_{t_0})=\sum_{r=0}^{N} A_r(t)D_{t_0}^{-r} \\ B^{(k)}(t,\, D_{t_0})=\sum_{r=0}^{N} B_r^{(k)}(t)D_{t_0}^{-r} \end{cases} \qquad (A_r,\, B_r^{(k)}\in M(\mu;\, C[t]))$$

are $\mu\times\mu$ matrices with entries in $C[t][D_{t_0},\, D_{t_0}^{-1}]$. In "simple" examples, the integer N reduces to one, as we will see in §3, so that the system (1) has the form

$$\text{(3)}\qquad \begin{cases} t_0\vec{u}=(A_0(t)+A_1(t)D_{t_0}^{-1})\vec{u} \\ D_{t_k}D_{t_0}^{-1}\vec{u}=(B_0^{(k)}(t)+B_1^{(k)}(t)D_{t_0}^{-1})\vec{u} \qquad (1\leqq k\leqq m)\,. \end{cases}$$

Moreover, as was remarked in 1.2, the system (1) or (3) gives a "deformation" of the Gauss-Manin system $H_{t_0+f}^\lambda$ presented by

$$\text{(4)}\qquad t_0\vec{w}=-\Lambda D_{t_0}^{-1}\vec{w} \quad \text{with} \quad \vec{w}=\vec{u}|_{t=0}\,,$$

where Λ is a diagonal matrix.

Apart from the Gauss-Manin system, let T be an open polydisc in C^m with center at the origin and denote by $\mathcal{O}(T)$ the ring of holomorphic functions on T. In this paragraph, we start with a differential system on $S=C\times T$ in the form

$$\text{(H)}\qquad \begin{cases} D_{t_0}t_0\vec{u}=(A_0(t)D_{t_0}+A_1(t))\vec{u} \\ D_{t_k}\vec{u}=(B_0^{(k)}(t)D_{t_0}+B_1^{(k)}(t))\vec{u} \qquad (1\leqq k\leqq m)\,. \end{cases}$$

Here $\vec{u}={}^t(u_1,\, \cdots,\, u_\mu)$ is a column vector of μ unknown functions and A_r, $B_r^{(k)}$ are $\mu\times\mu$ matrices with entries in $\mathcal{O}(T)$. In the sequel, we use the notations $\mathrm{M}(\mu;\, A)$ and $\mathrm{GL}(\mu;\, A)$ to refer the ring of matrices and the group of invertible matrices of size μ with entries in a commutative ring A, respectively. Assume that the following compatibility condition is satisfied:

$$\text{(C.1)} \qquad \begin{cases} \text{i)} & [D_{t_0}t_0 - A_0 D_{t_0} - A_1,\ D_{t_k} - B_0^{(k)} D_{t_0} - B_1^{(k)}] = 0 \qquad (1 \leqq k \leqq m) \\ \text{ii)} & [D_{t_k} - B_0^{(k)} D_{t_0} - B_1^{(k)},\ D_{t_l} - B_0^{(l)} D_{t_0} - B_1^{(l)}] = 0 \qquad (1 \leqq k,\ l \leqq m)\ . \end{cases}$$

Note that the compatibility condition (C.1) is satisfied *a priori* if (H) comes from a Gauss-Manin system. Moreover, assume

$$\text{(C.2)} \qquad A_0|_{t=0} = 0$$

and set $\Lambda = -A_1|_{t=0}$ so that (H) gives a deformation of the system

$$(5) \qquad D_{t_0} t_0 \vec{w} = -\Lambda \vec{w} \quad \text{with} \quad \vec{w} = \vec{u}|_{t=0}$$

on C.

For the differential system (H), let us define its discriminant $\Delta = \Delta(t_0, t)$ to be $\det(t_0 I - A_0(t))$, which is a monic polynomial in t_0. It is immediately seen that the system (H) defines a meromorphic connection on S at most with poles along the discriminant set $D = \{\Delta = 0\}$. As being a deformation of (5), (H) has a unique fundamental system $\Phi = \Phi(t_0, t) \in \mathrm{GL}\,(\mu; \mathcal{O}(\widetilde{S \backslash D}))$ of many-valued holomorphic solutions on $S \backslash D$ such that

$$(6) \qquad \Phi|_{t=0} = t_0^{-\Lambda - I}$$

(putting Λ in the Jordan standard form, if necessary). Now we consider the compactification $\bar{S} = P^1 \times T$ of $S = C \times T$ with respect to the direction of t_0-axis. Since Δ is a *monic* polynomial in t_0, the discriminant set D is closed in $\bar{S}$ and does not intersect with the plane $\{t_0 = \infty\}$ at infinity. Furthermore, it can be directly checked that the system (H) has regular singularities along $\{t_0 = \infty\}$. (In fact, if one uses the local coordinates, say, $(z_0, t) = (z_0, t_1 \cdots, t_m)$ with $z_0 = 1/t_0$, then the system (H) can be rewritten in the form

$$(7) \qquad \begin{cases} z_0 D_{z_0} \vec{u} = \tilde{A}(z_0, t) \vec{u} \\ D_{t_k} \vec{u} = \tilde{B}^{(k)}(z_0, t) \vec{u} \qquad (1 \leqq k \leqq m) \end{cases}$$

where $\tilde{A}$, $\tilde{B}^{(k)}$ are holomorphic on $S \backslash (D \cup \{t_0 = 0\})$ and $\tilde{A}|_{z_0=0} = I + \Lambda$.) By this remark, we are convinced that there exist a unique invertible matrix $\Psi = \Psi(t_0, t) \in \mathrm{GL}\,(\mu; \mathcal{O}_{\bar{S}, (\infty, 0)})$ of holomorphic functions defined near $(t_0, t) = (\infty, 0)$ such that

$$(8) \qquad \Phi(t_0, t) = \Psi(t_0, t) t_1^{-\Lambda - I} \quad \text{and} \quad \Psi|_{t=0} = I$$

near $(\infty, 0)$. In that sense, the fundamental system Φ of many-valued holomorphic solutions of (H) on $S \backslash D$ has a power series expansion near the point $(t_0, t) = (\infty, 0)$ at infinity. Thus we are led to find an explicit procedure to determine such an expansion of the fundamental system of solutions.

For this purpose, we reformulate the above remarks from a different point of view. First, we propose

LEMMA 2.1. *On the compatibility condition* (C.1), *there is a unique invertible matrix* $P_0 = P_0(t) \in \mathrm{GL}(\mu; \mathcal{O}(T))$ *with the following properties*:

a) $P_0|_{t=0} = I$, $A_1 = -P_0 \Lambda P_0^{-1}$.

b) *By setting* $\vec{u} = P_0\vec{v}$, *the system* (H) *is transformed into a system*

$$\begin{cases} D_{t_0} t_0 \vec{v} = (\tilde{A}(t) D_{t_0} - \Lambda)\vec{v} \\ D_{t_k}\vec{v} = \tilde{B}^{(k)}(t) D_{t_0}\vec{v} \qquad (1 \leqq k \leqq m) \end{cases}$$

of $\vec{v}$, *where* $\tilde{A}$, $\tilde{B}^{(k)} \in \mathrm{M}(\mu; \mathcal{O}(T))$.

PROOF. It is directly checked that an invertible matrix $P_0 \in \mathrm{GL}(\mu; \mathcal{O}(T))$ has the property b) if and only if $A_1 = -P_0 \Lambda P_0^{-1}$ and P_0 satisfies the equations

$$(9) \qquad \partial_{t_k} U = B_1^{(k)} U \qquad (1 \leqq k \leqq m)\,,$$

where U is a $\mu \times \mu$ matrix of unknown functions. The condition (C.1.ii) implies

$$(10) \qquad \partial_{t_l} B_1^{(k)} - \partial_{t_k} B_1^{(l)} + [B_1^{(k)}, B_1^{(l)}] = 0 \qquad (1 \leqq k,\, l \leqq m)\,,$$

hence the above system (9) has a unique solution in $\mathrm{GL}(\mu; \mathcal{O}(T))$ with initial value I, which we take for P_0. On the other hand, (C.1.i) involves

$$(11) \qquad \partial_{t_k} A_1 + [A_1, B_1^{(k)}] = 0 \qquad (1 \leqq k \leqq m)\,.$$

Hence the matrices $A_1 P_0$ and $-P_0 \Lambda$ both satisfy the above equations (9). Since $A_1 P_0$ and $-P_0 \Lambda$ have the same initial value $-\Lambda$, one has $A_1 P_0 = -P_0 \Lambda$ everywhere on T. Q.E.D.

THEOREM 2.2. *Assume that the differential system* (H) *on* $S = C \times T$ *satisfies the conditions* (C.1) *and* (C.2). *Then, there exist a unique matrix*

$$P(t, D_{t_0}) = \sum_{r=0}^{\infty} P_r(t) D_{t_0}^r\,; \qquad P_r(t) \in \mathrm{M}(\mu; \mathcal{O}(T))$$

of formal differential operators of infinite order with the properties

a) $P|_{t=0} = I$, *i.e.*, $P_0|_{t=0} = I$ *and* $P_r|_{t=0} = 0$ $(r \geqq 1)$,

b) *By setting* $\vec{u} = P\vec{v}$, *the system* (H) *for* $\vec{u}$ *is transformed into the system*

$$D_{t_0} t_0 \vec{v} = -\Lambda \vec{v}\,, \qquad D_{t_k}\vec{v} = 0 \qquad (1 \leqq k \leqq m)$$

for $\vec{v}$. Namely, the matrix $P=P(t, \tau_0)$ satisfies the equations

$$\tau_0\partial_{\tau_0}P+P\Lambda+(A_0\tau_0+A_1)P=0\ ,\qquad \partial_{t_k}P=(B_0^{(k)}\tau_0+B_1^{(k)})P\qquad (1\leqq k\leqq m)\ .$$

Moreover, the operator $P=\sum_{r=0}^{\infty}P_rD_{t_0}^r$ satisfies the estimate

c) *For every compact set $K\subset T$, there is a positive number R_K such that*

$$\|P_r\|_K\leqq R_K^r/r!\qquad \textit{for}\quad r\geqq 1\ ,$$

where $\|\cdot\|_K$ stands for the supremum norm of a matrix of functions on K.

PROOF. By Lemma 2.1, we may assume that $B_1^{(k)}=0$ $(1\leqq k\leqq m)$ and $A_1=-\Lambda$. In this case, the property b) required of P is equivalent to the condition

$$\text{(12)}\qquad \begin{cases}\text{i)}\ \ rP_r-[\Lambda, P_r]+A_0P_{r-1}=0\\ \text{ii)}\ \ \partial_{t_k}P_r=B_0^{(k)}P_{r-1}\end{cases}\qquad (r\geqq 0;\ P_{-1}=0)\ .$$

On the other hand, the compatibility condition (C.1) says

$$\text{(13)}\qquad \begin{cases}\text{i)}\ \ [A_0, B_0^{(k)}]=0\ ,\qquad \partial_{t_k}A_0+B_0^{(k)}-[\Lambda, B_0^{(k)}]=0 & (1\leqq k\leqq m)\\ \text{ii)}\ \ [B_0^{(k)}, B_0^{(l)}]=0\ ,\qquad \partial_{t_l}B^{(k)}=\partial_{t_k}B^{(l)} & (1\leqq k,\ l\leqq m)\ .\end{cases}$$

Let us prove by the induction on r that one can find a unique sequence $(P_r)_{r=0}^{\infty}$ in $M(\mu\colon \mathcal{O}(T))$ satisfying (12.ii) with the condition a). For $r=0$, (12.ii) requires that P_0 is constant, hence $P_0=I$. For $r\geqq 1$, it is enough to show that the differential system (12.ii) for P_r is integrable. In fact, by the induction hypothesis, we have

$$\begin{aligned}&\partial_{t_l}(B_0^{(k)}P_{r-1})-\partial_{t_k}(B_0^{(l)}P_{r-1})\\ &\quad=(\partial_{t_l}B_0^{(k)}-\partial_{t_k}B_0^{(l)})P_{r-1}+B_0^{(k)}\partial_{t_l}P_{r-1}-B_0^{(l)}\partial_{t_k}P_{r-1}\\ &\quad=(\partial_{t_l}B_0^{(k)}-\partial_{t_k}B_0^{(l)})P_{r-1}+[B_0^{(k)}, B_0^{(l)}]P_{r-2}\ ,\end{aligned}$$

which vanish for $1\leqq k,\ l\leqq m$ by (13.ii). Next we will show the sequence $(P_r)_{r=0}^{\infty}$ determined as above satisfies the condition (12.i) by means of the induction on r. It is clear for $r=0$, since $P_0=I$. For $r\geqq 1$, one computes

$$\begin{aligned}&\partial_{t_k}(rP_r-[\Lambda, P_r]+A_0P_{r-1})\\ &\quad=rB_0^{(k)}P_{r-1}-[\Lambda, B_0^{(k)}P_{r-1}]+\partial_{t_k}A_0P_{r-1}+A_0B_0^{(k)}P_{r-2}\\ &\quad=B_0^{(k)}((r-1)P_{r-1}-[\Lambda, P_{r-1}]+A_0P_{r-2})\\ &\qquad+(\partial_{t_k}A_0+B_0^{(k)}-[\Lambda, B_0^{(k)}])P_{r-1}+[A_0, B_0^{(k)}]P_{r-2}\ ,\end{aligned}$$

which vanish for $1 \leqq k \leqq m$ by the induction hypothesis and (13.i). This shows $rP_r - [\Lambda, P_r] + A_0 P_{r-1}$ is constant on T. But, its initial value for $t=0$ is zero by the condition (C.2) and the property a), so we have (12.i), as desired. Lastly, we show the sequence $(P_r)_{r=0}^{\infty}$ satisfies the estimate c). Consider the linear mappings $r.\, \mathrm{id} - \mathrm{ad}\,(\Lambda)\colon \mathrm{M}\,(\mu; \boldsymbol{C}) \to \mathrm{M}\,(\mu; \boldsymbol{C})$ for $r \geqq 1$, where $\mathrm{ad}\,(\Lambda) = [\Lambda, \cdot]$. Since the sequence $(\mathrm{id} - (1/r)\,\mathrm{ad}\,(\Lambda))_{r=1}^{\infty}$ coverges to the identity mapping as $r \to +\infty$, there is a positive integer r_0 and a positive constant M such that i) $r.\, \mathrm{id} - \mathrm{ad}\,(\Lambda)$ is invertible for $r > r_0$ and ii) $\|(r.\, \mathrm{id} - \mathrm{ad}\,(\Lambda))^{-1}\| \leqq M/r$ for $r > r_0$. By (12.i) we have

$$P_r = (r.\, \mathrm{id} - \mathrm{ad}\,(\Lambda))^{-1}(A_0 P_{r-1}) \qquad \text{for} \quad r > r_0\,,$$

hence, for every compact set $K \subset T$,

$$\|P_r\|_K \leqq M \|A_0\|_K \|P_{r-1}\|_K / r \qquad \text{for} \quad r > r_0\,.$$

Putting $C_K = r_0!\,.\, \max\{\|P_s\|_K;\, 0 \leqq s \leqq r_0\}$ and $R_K = \max\{M\|A_0\|_K, 1\}$, we have

$$\|P_r\|_K \leqq C_K R_K^r / r! \qquad \text{for} \quad r \geqq 0\,,$$

which proves c). Q.E.D.

Note that the matrix P_0 of Theorem 2.2 coincides with P_0 of Lemma 2.1.

The estimate c) of Theorem 2.2 assures that the series

$$
\begin{aligned}
P(t, D_{t_0}) t_0^{-\Lambda - I} &= \sum_{r=0}^{\infty} P_r(t) D_{t_0}^r t_0^{-\Lambda - I} \\
&= \sum_{r=0}^{\infty} P_r(t)(-\Lambda - I) \cdots (-\Lambda - rI) t_0^{-\Lambda - (r+1)I}
\end{aligned}
\tag{14}
$$

converges in a neighborhood of the point $(t_0, t) = (\infty, 0)$. Moreover, the series (14) is nothing but the expansion of the fundamental system Φ of many-valued solutions of (H) near the point $(\infty, 0)$ at infinity. It is in this form that we are to construct an explicit expansion of the solutions of the Gauss-Manin system H_F^λ. Thus, our problem is equivalent to finding the unique matrix $P(t, D_{t_0})$ of operators satisfying the conditions a) and b) of Theorem 2.2. (See also Remark at the end of 2.2.)

2.2. Effect of the generating function for H_F^λ.

Now we return to the Gauss-Manin system H_F^λ. In §1, we showed that the Gauss-Manin system H_F^λ has two types of finite presentation under certain conditions. One, appeared in Proposition 1.5, has the form 2.1.(1). In this direction, we explained in the previous paragraph that the fundamental system of many-valued holomorphic solutions has a power series expansion at a point of infinity so far as the presentation reduces to the form 2.1(3). This presentation, however, does not fit for the explicit computation of the solutions of an individual Gauss-Manin system, for matrices A, $B^{(k)}$ in 2.1(1) cannot be easily computed. On the other hand, the presentation given in Proposition 1.8 has an advantage in that one can know its explicit form directly from the polynomial F. For this reason, we will make use of the latter presentation in concrete computations and then reconstruct the expansion of the form 2.1.(14) from the solutions computed in that manner. Let us explain this procedure in more details.

In what follows, we will use freely the notation of §1. Let $\mathscr{F}$ be a $\mathscr{D}(S)[D_{t_0}^{-1}]$-module, which we regard as a space of "functions". Then, by a "solution of H_F^λ in $\mathscr{F}$", we mean a $\mathscr{D}(S)[D_{t_0}^{-1}]$-homomorphism ϕ: $H_F^\lambda \to \mathscr{F}$. If H_F^λ has a finite presentation of Proposition 1.5

$$(1)\qquad \begin{cases} t_0\vec{u}=A(t, D_{t_0})\vec{u} \\ D_{t_k}D_{t_0}^{-1}\vec{u}=B^{(k)}(t, D_{t_0})\vec{u} \qquad (1\leqq k\leqq m)\,, \end{cases}$$

then, by setting $\phi_i=\phi(u_i)$, a solution ϕ: $H_F^\lambda\to\mathscr{F}$ corresponds to a column vector $\vec{\phi}={}^t(\phi_1, \cdots, \phi_\mu)$ of μ elements of $\mathscr{F}$ satisfying the equations (1), and *vice versa*. On the other hand, if H_F^λ has a finite presentation of Proposition 1.8

$$(2)\qquad \begin{cases} \left(\left(t_0-H_{\tau_0}(D_{t_0}, D_{t'}, t'')+\left(\lambda-\frac{n+l}{2}\right)D_{t_0}^{-1}\right)u=0 \\ (t_i-H_{\tau_i}(D_{t_0}, D_{t'}, t''))u=0 \qquad (1\leqq i\leqq l) \\ (D_{t_j}+H_{t_j}(D_{t_0}, D_{t'}, t''))u=0 \qquad (l+1\leqq j\leqq m)\,, \end{cases}$$

then a solution ϕ: $H_F^\lambda\to\mathscr{F}$ corresponds to an element $\phi(u)$ of $\mathscr{F}$ satisfying the equations (2), and *vice versa*.

Now suppose that the conditions (A.1), (A.2) and (B.1) of §1 are satisfied. In this case, the polynomial $f=f(x)$ can be written in the form $f(x)=g(x')+x_{l+1}^2+\cdots+x_n^2$, where $g=g(x')\in C[x']$, $x'=(x_1, \cdots, x_l)$, so that $C[x']/(\partial_{x'}g)\xrightarrow{\sim}C[x]/(\partial_x f)$. Hence we see that one can find a sequence $e_1, \cdots, e_\mu$ of weighted homogeneous polynomials in $C[x']$ such that the residue classes of $\omega_1=e_1dx, \cdots, \omega_\mu=e_\mu dx$ form a C-basis of Ω_f. By using such a basis, we define

$$(3)\qquad u_i=\int e_i\delta_F^{(\lambda)}dx\in H_F^{(\lambda)} \qquad (1\leqq i\leqq \mu)\,.$$

As a convention, we always take 1 for e_1 so that $\omega_1=dx$ and

$$u := u_1 = \int \delta_F^{(\lambda)} dx \, . \tag{4}$$

We denote by ε_i the exponent of f corresponding to $\omega_i = e_i dx$. (Then, ε_1 is nothing but the minimal exponent $\varepsilon_* = \sum_{i=1}^n \rho_i$.) Recall that, on the assumption (B.1), one has $D_{t_i} D_{t_0}^{-1} \delta_F^{(\lambda)} = x_i \delta_F^{(\lambda)}$ for $1 \leqq i \leqq l$. Accordingly for any polynomial $a = a(x')$ in $C[x']$, one has an equality in $M_F^{(\lambda)}$

$$a(x') \delta_F^{(\lambda)} = a(D_{t'} D_{t_0}^{-1}) \delta_F^{(\lambda)} \, . \tag{5}$$

Hence we see that the above basis $\vec{u} = {}^t(u_1, \cdots, u_\mu)$ is recovered from u by the formulas

$$u_i = e_i(D_{t'} D_{t_0}^{-1}) u \qquad (1 \leqq i \leqq \mu) \, . \tag{6}$$

Moreover, if an element ϕ in $\mathscr{F}$ is a solution of (2), then the corresponding solution $\vec{\phi} = {}^t(\phi_1, \cdots, \phi_\mu)$ of (1) is obtained by the formulas

$$\phi_i = e_i(D_{t'} D_{t_0}^{-1}) \phi \qquad (1 \leqq i \leqq \mu) \, . \tag{7}$$

Having these in mind, we set to investigate the solutions of (2). On our assumptions, the Gauss-Manin system H_F^λ is a deformation of the system $H_{t_0+f}^\lambda$ given by

$$t_0 \vec{w} = -\Lambda D_{t_0}^{-1} \vec{w} \quad \text{with} \quad \vec{w} = \vec{u}|_{t=0} \, , \tag{8}$$

where $\Lambda = \operatorname{diag}(\lambda - \varepsilon_1, \cdots, \lambda - \varepsilon_\mu)$ is the diagonal matrix whose (i, i)-component is $\lambda - \varepsilon_i$. As we did in §1, we will use the symbols $\delta_{t_0}^{(\kappa)}$ $(\kappa \in C)$ representing the derivatives of the "delta function" δ_{t_0} in describing the solutions of $H_{t_0+f}^\lambda$ or H_F^λ. With these symbols, the fundamental system of solutions of (8) can be given by

$$\delta_{t_0}^{(\Lambda)} := \operatorname{diag}(\delta_{t_0}^{(\lambda - \varepsilon_1)}, \cdots, \delta_{t_0}^{(\lambda - \varepsilon_\mu)}) \, . \tag{9}$$

If $\lambda - \varepsilon_i \notin \boldsymbol{Z}$ for $1 \leqq i \leqq \mu$, we may well take

$$t_0^{-\Lambda - I} = \operatorname{diag}(t_0^{\varepsilon_1 - \lambda - 1}, \cdots, t_0^{\varepsilon_\mu - \lambda - 1}) \tag{10}$$

for the fundamental system of solutions of (8) as the operation of $D_{t_0}^{-1}$ can be justified. As to H_F^λ, Theorem 2.2 implies that the *fundamental system of solutions* of (1) can be expanded into a series

$$\sum_{r=0}^{\infty} P_r(t) \delta_{t_0}^{(\Lambda + rI)} \, , \tag{11}$$

where $P_r(t) \in \mathrm{M}(\mu; C[[t]])$ for $r \in \boldsymbol{N}$, on the condition that the presentation (1) should reduce to the form 2.1.(3). By taking the first row of (11), we can see that the system (2) has μ independent solutions in the form

$$\sum_{r=0}^{\infty} a_r(t)\delta_{t_0}^{(\kappa+r)}, \tag{12}$$

where $\kappa \in \boldsymbol{C}$ and $a_r(t) \in \boldsymbol{C}[[t]]$ $(r \in \boldsymbol{N})$, in that case. Here we used the notation $\boldsymbol{C}[[t]]$ to refer the ring of formal power series in $t=(t_1, \cdots, t_m)$.

Taking general cases into account, we will try to find solutions of (2) in a larger space of "functions" than that of all series in the form (12). First, we denote by $\mathscr{A}=\boldsymbol{C}[[t]]\ [[D_{t_0}, D_{t_0}^{-1}]]$ the $\boldsymbol{C}[[t]]$-module consisting of all formal operators

$$P(t, D_{t_0})=\sum_{r \in \boldsymbol{Z}} a_r(t)D_{t_0}^r \tag{13}$$

where $a_r(t) \in \boldsymbol{C}[[t]]$ for $r \in \boldsymbol{Z}$. Note that this module $\mathscr{A}$ does *not* have a natural ring structure, while the module $\boldsymbol{C}[[t]]((D_{t_0})):=\boldsymbol{C}[[t]][[D_{t_0}]][D_{t_0}^{-1}]$ does. For each complex number κ, we define a $\mathscr{D}(S)[D_{t_0}^{-1}]$-module, denoted by $\mathscr{A}\delta_{t_0}^{(\kappa)}$, as follows. $\mathscr{A}\delta_{t_0}^{(\kappa)}$ is the space of all formal series

$$\phi(t_0, t)=\sum_{r \in \boldsymbol{Z}} a_r(t)D_{t_0}^r\delta_{t_0}^{(\kappa)}=\sum_{r \in \boldsymbol{Z}} a_r(t)\delta_{t_0}^{(\kappa+r)}, \tag{14}$$

where $a_r(t) \in \boldsymbol{C}[[t]]$ for $r \in \boldsymbol{Z}$, so that $\mathscr{A} \xrightarrow{\sim} \mathscr{A}\delta_{t_0}^{(\kappa)}$. Moreover, as in §1, we define the operation of t_0 on $\delta_{t_0}^{(\kappa)}$ by

$$t_0\delta_{t_0}^{(\kappa)}=-\kappa D_{t_0}^{-1}\delta_{t_0}^{(\kappa)}=-\kappa\delta_{t_0}^{(\kappa-1)} \qquad (\kappa \in \boldsymbol{C}). \tag{15}$$

By this, one can specialize the $\mathscr{D}(S)[D_{t_0}^{-1}]$-module structure of $\mathscr{A}\delta_{t_0}^{(\kappa)}$. If κ is not an integer, the space $\mathscr{A}\delta_{t_0}^{(\kappa)}$ is identified with the space $\boldsymbol{C}[[t]][[t_0, t_0^{-1}]]t_0^{-\kappa-1}$ of formal power series

$$\psi(t_0, t)=\sum_{r \in \boldsymbol{Z}} b_r(t)t_0^{-\kappa-1-r} \qquad (b_r(t) \in \boldsymbol{C}[[t]]), \tag{16}$$

by the correspondence

$$\delta_{t_0}^{(\kappa)} \longmapsto \frac{\Gamma(\kappa+1)}{2\pi\sqrt{-1}} \cdot (-t_0)^{-\kappa-1} \qquad (\kappa \in \boldsymbol{C}). \tag{17}$$

Now assume that the conditions (B.1) and (B.2) in 1.3 are satisfied. In this case, we may take the "weighted homogeneous operators" into consideration. Let $\sigma=(\sigma_1, \cdots, \sigma_m)$ be a weight of $t=(t_1, \cdots, t_m)$, fixed so that (B.2) should be verified. Then, for a rational number ε, an operator $P=\sum_{r \in \boldsymbol{Z}} a_r(t)D_{t_0}^r$ in $\mathscr{A}$ is called weighted homogeneous of degree ε if each $a_r(t)$ is weighted homogeneous of σ-degree $\varepsilon+r$ for $r \in \boldsymbol{Z}$. Symbolically, it is equivalent to saying that $[\theta, P]=\varepsilon P$, where θ is the Euler vector field $t_0D_{t_0}+\sum_{k=1}^{m}\sigma_k t_k D_{t_k}$. For each $\varepsilon\in\boldsymbol{Q}$, we set

$$\mathscr{A}(\varepsilon):=\{P\in\mathscr{A}: [\theta, P]=\varepsilon P\}\,. \tag{18}$$

Then, noting that

$$\theta\delta_{t_0}^{(\kappa)}=-(\kappa+1)\delta_{t_0}^{(\kappa)} \qquad (\kappa\in\boldsymbol{C})\,, \tag{19}$$

one can easily show by Proposition 1.9

LEMMA 2.3. *On the assumptions* (B.1) *and* (B.2), *an element* ϕ *in* $\mathscr{A}\delta_{t_0}^{(\kappa)}$ *is a solution of* (2) *if and only if it can be written in the form*

$$\phi(t_0, t)=P(t, D_{t_0})\delta_{t_0}^{(\kappa)}$$

with a weighted homogeneous operator in $\mathscr{A}(\varepsilon_*+\kappa-\lambda)$ *such that*

$$\begin{cases}(t_i+G_{x_i}(t'', \partial_{t'}D_{t_0}^{-1}))P=0 & (1\leqq i\leqq l)\\ \partial_{t_j}P=G_{t_j}(t'', \partial_{t'}D_{t_0}^{-1})P & (l+1\leqq j\leqq m)\,,\end{cases}$$

where $\partial_{t_j}=[D_{t_j}, \cdot]$.

Our problem is thus reduced to finding the weighted homogeneous operators satisfying the equations of Lemma 2.3.

Let ε be a rational number. Then, each operator P in $\mathscr{A}(\varepsilon)$ has a unique expression

$$P(t, D_{t_0})=\sum_{\langle\sigma,\alpha\rangle-r=\varepsilon} a(\alpha)\frac{t^\alpha}{\alpha!}D_{t_0}^r \qquad (a(\alpha)\in\boldsymbol{C})\,, \tag{20}$$

where $r\in\boldsymbol{Z}$ and $\alpha\in\boldsymbol{N}^m$. (For a multi-index $\alpha=(\alpha_1, \cdots, \alpha_m)\in\boldsymbol{N}^m$, $t^\alpha/\alpha!=t_1^{\alpha_1}\cdots t_m^{\alpha_m}/\alpha_1!\cdots\alpha_m!$, as usual.) In view of (20), we define

$$\boldsymbol{N}^m(\varepsilon):=\{\alpha\in\boldsymbol{N}^m; \langle\sigma, \alpha\rangle\equiv\varepsilon \bmod \boldsymbol{Z}\}\,. \tag{21}$$

(Note that $\boldsymbol{N}^m(\varepsilon)=\boldsymbol{N}^m(\varepsilon')$ if $\varepsilon\equiv\varepsilon' \bmod \boldsymbol{Z}$.) Then, via the expression (20), there is a one-to-one correspondence between the weighted homogeneous operators in $\mathscr{A}(\varepsilon)$ and the functions $a: \boldsymbol{N}^m\to\boldsymbol{C}$ with $\operatorname{supp}(a)\subset\boldsymbol{N}^m(\varepsilon)$. By this correspondence, one can naturally translate operations on P into those on a. It is convenient for our purpose to introduce the operators Δ_k, T_k $(1\leqq k\leqq m)$ as follows: For a function $a: \boldsymbol{N}^m\to\boldsymbol{C}$, we define the functions $\Delta_k a$, $T_k a: \boldsymbol{N}^m\to\boldsymbol{C}$ by

$$\begin{aligned}&(\Delta_k a)(\alpha)=\alpha_k a(\alpha-1_k)\\ &(T_k a)(\alpha)=a(\alpha+1_k) \qquad \text{for} \quad \alpha=(\alpha_1, \cdots, \alpha_m)\in\boldsymbol{N}^m\,,\end{aligned} \tag{22}$$

where $\mathbf{1}_k=(0, \cdots, \overset{k}{1}, \cdots, 0)$. Then it is easy to check

LEMMA 2.4. *Let P be the operator of* (20). *Then one has*

$$\begin{cases} t_kP=\displaystyle\sum_{\langle\sigma,\alpha\rangle-r=\varepsilon+\sigma_k}(\varDelta_k a)(\alpha)\frac{t^\alpha}{\alpha!}D_{t_0}^r\in\mathscr{A}(\varepsilon+\sigma_k) \\ \partial_{t_k}D_{t_0}^{-1}P=\displaystyle\sum_{\langle\sigma,\alpha\rangle-r=\varepsilon+1-\sigma_k}(T_k a)(\alpha)\frac{t^\alpha}{\alpha!}D_{t_0}^r\in\mathscr{A}(\varepsilon+1-\sigma_k)\ . \end{cases}$$

By Lemma 2.4, it follows, for example, that

$$(\partial_{t_k}D_{t_0}^{-1})^{\nu'}P=\sum_{\langle\sigma,\alpha\rangle-r=\varepsilon+\langle\rho',\nu'\rangle}a(\alpha+\nu')\frac{t^\alpha}{\alpha!}D_{t_0}^r\ , \tag{23}$$

for each multi-index $\nu'=(\nu_1, \cdots, \nu_l)\in\boldsymbol{N}^l$. Using Lemma 2.4, one can easily translate the differential system for P of Lemma 2.3 into a difference system for the corresponding $a\colon\boldsymbol{N}^m\to\boldsymbol{C}$. But, before going furthur, we propose in the next paragraph to reduce a general Gauss-Manin system to the one associated with a special type of deformation by an operator of exponential type.

REMARK. It is easy to show the following:
i) $\sigma_k\geqq 0$ for $1\leqq k\leqq m\Rightarrow\mathscr{A}(\varepsilon)\subset\boldsymbol{C}[[t]]((D_{t_0}))$ for $\varepsilon\in\boldsymbol{Q}$.
ii) $\sigma_k>0$ for $1\leqq k\leqq m\Rightarrow\mathscr{A}(\varepsilon)\subset\boldsymbol{C}[t]((D_{t_0}))$ for $\varepsilon\in\boldsymbol{Q}$.

REMARK. Let $P=(P_{ij})$ be a matrix in $\mathrm{M}(\mu;\boldsymbol{C}[[t]][[D_{t_0}]])$ with $P|_{t=0}=I$ and let $\varLambda$ be a diagonal matrix $\mathrm{diag}\,(\lambda_1, \cdots, \lambda_\mu)$ in $\mathrm{M}(\mu;\boldsymbol{C})$. Suppose that the matrix $\varPhi=P\delta_{t_0}^{(\varLambda)}=(P_{ij}\delta_{t_0}^{(\lambda_j)})$ satisfies the differential system

$$\begin{cases} D_{t_0}t_0\varPhi=(A_0D_{t_0}+A_1)\varPhi \\ D_{t_k}\varPhi=(B_0^{(k)}D_{t_0}+B_1^{(k)})\varPhi \qquad (1\leqq k\leqq m)\ , \end{cases} \tag{24}$$

where A_i, $B_i^{(k)}$ are matrices satisfying the conditions (C.1) and (C.2). Then, one can show easily

$$\begin{cases} \{[P, D_{t_0}t]+P\varLambda+(A_0D_{t_0}+A_1)P\}\delta_{t_0}^{(\varLambda)}=0 \\ \{\partial_{t_k}(P)-(B_0^{(k)}D_{t_0}+B_1^{(k)})P\}\delta_{t_0}^{(\varLambda)}=0 \qquad (1\leqq k\leqq m)\ . \end{cases} \tag{25}$$

Note that $\varLambda$ is assumed to be diagonal and each $\delta_{t_0}^{(\lambda_j)}$ is free over $\boldsymbol{C}[[t]][[D_{t_0}]]$. Hence, one necessarily has

$$\begin{cases} \tau_0\partial_{\tau_0}P+P\varLambda+(A_0\tau_0+A_1)P=0 \\ \partial_{t_k}P=(B_0^{(k)}\tau_0+B_1^{(k)})P\ . \end{cases} \tag{26}$$

By this remark, we see that, if one can find a matrix P with $P|_{t=0}=I$ satisfying (24), then it must coincide with the one constructed in Theorem 2.2. In view of this, a matrix $\Phi=P\delta_{t_0}^{(\Lambda)}$ will be called the *fundamental system of solutions* if it satisfies (24) and $\Phi|_{t=0}=\delta_{t_0}^{(\Lambda)}$ in the case where Λ is diagonal.

2.3. A reduction of H_F^λ by an operator of exponential type.

Throughout this paragraph, we confine ourselves to the case where the condition (B.1) is satisfied. Consider a polynomial

$$(1)\qquad f(x)=g(x')+\sum_{j=l+1}^{n} x_j^2 \qquad (x'=(x_1, \cdots, x_l))$$

in $C[x]$, where $g\in C[x']$. Then, the "smallest" deformation of f satisfying (B.1) is given by

$$(2)\qquad \mathring{F}(t_0, t, x):=t_0+\sum_{i=1}^{l} t_i x_i+f(x)$$

which does not depend on the parameters $t''=(t_{l+1}, \cdots, t_m)$. As to the Gauss-Manin system $H_{\mathring{F}}^\lambda$ on S, let us denote its canonical generator by

$$(3)\qquad v:=\int \delta_{\mathring{F}}^{(\lambda)}dx \in H_{\mathring{F}}^{(\lambda)} .$$

Then, with the generating function

$$(4)\qquad \mathring{H}(\tau_0, \tau'):=-\tau_0 g(\tau'\tau_0^{-1}) ,$$

$H_{\mathring{F}}^\lambda$ is presented by the differential system

$$(5)\qquad \begin{cases} \left(t_0-\mathring{H}_{\tau_0}(D_{t_0}, D_{t'})+\left(\lambda-\dfrac{n+l}{2}\right)D_{t_0}^{-1}\right)v=0 & \\ (t_i-\mathring{H}_{\tau_i}(D_{t_0}, D_{t'}))v=0 & (1\leqq i\leqq l) \\ D_{t_j}v=0 & (l+1\leqq j\leqq m) \end{cases}$$

(Proposition 1.8). Let

$$(6)\qquad F(t_0, t, x)=t_0+\sum_{i=1}^{l} t_i x_i+G(t'', x')+\sum_{j=l+1}^{n} x_j^2$$

be a general deformation of f satisfying the condition (B.1), where $G|_{t''=0}=g$. Recall that the generating function H for H_F^λ is defined as

$$(7)\qquad H(\tau_0, \tau', t'')=-\tau_0 G(t'', \tau'\tau_0^{-1}) ,$$

so that one has $H|_{t''=0}=\mathring{H}$. In this paragraph, we will show that, on the condition (B.1), the general Gauss-Manin system H_F^λ can be reduced to the above $H_{\mathring{F}}^\lambda$ by an operator of exponential type. To begin with, we explain in a heuristic manner what it means. Define a polynomial $E=E(t'', x')$ to be $G(t'', x')-g(x')$. Then, one has

$$F(t_0, t, x)=t_0+E(t'', x')+\sum_{i=1}^{l} t_i x_i+f(x) . \tag{8}$$

Here we consider the "formal" Taylor expansion of $\delta_F^{(\lambda)}$

$$\delta_F^{(\lambda)}=\sum_{d=0}^{\infty} \frac{1}{d!} E(t'', x')^d \delta_{\mathring{F}}^{(\lambda+d)}=\exp(D_{t_0} E(t'', x')) \cdot \delta_{\mathring{F}}^{(\lambda)} . \tag{9}$$

Nothing that $D_{t_i} D_{t_0}^{-1} \delta_{\mathring{F}}^{(\lambda)}=x_i \delta_{\mathring{F}}^{(\lambda)}$ $(1 \leqq i \leqq l)$, we rewrite (9) in the form

$$\delta_F^{(\lambda)}=\sum_{d=0}^{\infty} \frac{1}{d!} E(t'', D_{t'} D_{t_0}^{-1}) \delta_{\mathring{F}}^{(\lambda+d)}=\exp(D_{t_0} E(t'', D_{t'} D_{t_0}^{-1})) \cdot \delta_{\mathring{F}}^{(\lambda)} . \tag{10}$$

Then, by integrating both sides, one might have

$$\int \delta_F^{(\lambda)} dx=\exp(D_{t_0} E(t'', D_{t'} D_{t_0}^{-1})) \int \delta_{\mathring{F}}^{(\lambda)} dx . \tag{11}$$

This "equality" suggests that the Gauss-Manin system H_F^λ could be regarded as an "evolution" of $H_{\mathring{F}}^\lambda$ by the operator $\exp(D_{t_0} E(t'', D_{t'} D_{t_0}^{-1}))$.

In view of (11), we define a rational function $K=K(\tau_0, \tau', t'')$ in $\boldsymbol{C}[\tau_0, \tau', t''][\tau_0^{-1}]$ by

$$\begin{aligned} K(\tau_0, \tau', t''): &=-\tau_0 E(t'', \tau' \tau_0^{-1}) \\ &=H(\tau_0, \tau', t'')-\mathring{H}(\tau_0, \tau') . \end{aligned} \tag{12}$$

It should be noted here that $K|_{t''=0}=0$ since $\mathring{H}|_{t''=0}=\mathring{H}$. Then, for the operator $K=K(D_{t_0}, D_{t'}, t'')$, we define a ring automorphism $\operatorname{Ad}(e^K)$ of $\mathscr{D}(S)[D_{t_0}^{-1}]$ by

$$\operatorname{Ad}(e^K) P=\sum_{d=0}^{\infty} \frac{1}{d!} \operatorname{ad}(K)^d P \qquad \text{for} \quad P \in \mathscr{D}(S)[D_{t_0}^{-1}] , \tag{13}$$

where $\operatorname{ad}(K)=[K, \cdot]$.

LEMMA 2.5. *For any operator $K=K(D_{t_0}, D_{t'}, t'')$ in $\boldsymbol{C}[t'', D_{t'}][D_{t_0}, D_{t_0}^{-1}]$, $\operatorname{Ad}(e^K)$ is a well-defined ring automorphism of $\mathscr{D}(S)[D_{t_0}^{-1}]$. Moreover, its inverse is given by $\operatorname{Ad}(e^{-K})$.*

PROOF. Let us show that the right side of (13) is eventually a finite sum. Then, the other part of Lemma can be proved by the Leibniz rule in a standard manner. First note that $\operatorname{ad}(K)=[K, \cdot]$ is a derivation of $\mathscr{D}(S)[D_{t_0}^{-1}]$ and that linear over the ring $C[t'', D_{t'}][D_{t_0}, D_{t_0}^{-1}]$. On the other hand, one computes

$$(14)\qquad \begin{cases} \operatorname{ad}(K)t_i = K_{\tau_i}(D_{t_0}, D_{t'}, t''), & \operatorname{ad}(K)^2 t_i = 0 \qquad (0 \leqq i \leqq l) \\ \operatorname{ad}(K)D_{t_j} = -K_{t_j}(D_{t_0}, D_{t'}, t''), & \operatorname{ad}(K)^2 D_{t_j} = 0 \qquad (l+1 \leqq j \leqq m), \end{cases}$$

where $K_{\tau_i} = \partial_{\tau_i} K$ and $K_{t_j} = \partial_{t_j} K$. Note that any operator P in $\mathscr{D}(S)[D_{t_0}^{-1}]$ has an expression

$$P = \sum_{k+|\alpha'|+|\beta''| \leqq N} a_{k,\alpha',\beta''} t_0^k t'^{\alpha'} D_{t''}^{\beta''} \qquad (a_{k,\alpha',\beta''} \in C[t'', D_{t'}][D_{t_0}, D_{t_0}^{-1}]),$$

where $k \in N$, $\alpha' \in N^l$, $\beta'' \in N^{m-l}$. It can be shown inductively that $\operatorname{ad}(K)^d P = 0$ for $d > N$ if P can be written as above. Q.E.D.

PROPOSITION 2.6. *Let $K = K(D_{t_0}, D_{t'}, t'')$ be the operator defined by* (12). *Then the ring automorphism* $\operatorname{Ad}(e^K)$ *of* $\mathscr{D}(S)[D_{t_0}^{-1}]$ *transforms the* $\mathscr{D}(S)[D_{t_0}^{-1}]$*-module* H_F^λ *into* $H_{\mathring{F}}^\lambda$.

PROOF. By (14), one computes

$$\operatorname{Ad}(e^K)(t_i - H_{\tau_i}) = (t_i + K_{\tau_i}) - H_{\tau_i} = t_i - \mathring{H}_{\tau_i} \qquad (0 \leqq i \leqq l),$$
$$\operatorname{Ad}(e^K)(D_{t_j} + H_{t_j}) = (D_{t_j} - K_{t_j}) + H_{t_j} = D_{t_j} \qquad (l+1 \leqq j \leqq m).$$

Hence, one sees directly that the presentation of H_F^λ in Proposition 1.8 is transformed by $\operatorname{Ad}(e^K)$ into the presentation (5) of $H_{\mathring{F}}^\lambda$. Q.E.D.

Let us denote by $e^K H_F^\lambda$ the $\mathscr{D}(S)[D_{t_0}^{-1}]$-module defined by "twisting" the action of $\mathscr{D}(S)[D_{t_0}^{-1}]$ through $\operatorname{Ad}(e^K)$. Then Proposition 2.6 says that there is a unique $\mathscr{D}(S)[D_{t_0}^{-1}]$-isomorphism

$$(15)\qquad H_{\mathring{F}}^\lambda \xrightarrow{\sim} e^K H_F^\lambda \qquad (\text{resp. } H_F^\lambda \xrightarrow{\sim} e^{-K} H_{\mathring{F}}^\lambda)$$

which sends v to $e^K u$ (resp. u to $e^{-K} v$).

Now we propose to apply Proposition 2.6 to solving the Gauss-Manin system H_F^λ in the space of formal series $\mathscr{A}\delta_{t_0}^{(\kappa)}$ introduced in 2.2. For this purpose, we will show the operator e^K (or e^{-K}) *actually* operates on the space $\mathscr{A}\delta_{t_0}^{(\kappa)}$. We consider the operator e^K to be the infinite sum

$$(16)\qquad e^K = \sum_{d=0}^{\infty} \frac{1}{d!} K(D_{t_0}, D_{t'}, t'')^d,$$

which is convergent in the ring $C[D_{t'}, D_{t_0}, D_{t_0}^{-1}][[t'']]$ since $K|_{t''=0} = 0$ as remarked before.

Lemma 2.7. *The space* $\mathscr{A}\delta_{t_0}^{(\kappa)}=C[[t]][[D_{t_0}, D_{t_0}^{-1}]]\delta_{t_0}^{(\kappa)}$ *has a natural structure of left module over the ring* $C[D_{t'}, D_{t_0}, D_{t_0}^{-1}][[t'']]$.

Proof. Leaving the operation of t_0 out of consideration, we have only to specialize the $C[D_{t'}, D_{t_0}, D_{t_0}^{-1}][[t'']]$-module structure of $\mathscr{A}=C[[t]][[D_{t_0}, D_{t_0}^{-1}]]$. Note that each element P of $\mathscr{A}$ can be written uniquely as

$$P=\sum_{\alpha\in N^{m-l}} t''^{\alpha}P_{\alpha}(t', D_{t_0}) \qquad (P_{\alpha}\in C[[t']][[D_{t_0}, D_{t_0}^{-1}]]) .$$

On the other hand, any operator L in $C[D_{t'}, D_{t_0}, D_{t_0}^{-1}][[t'']]$ has a unique expression

$$L=\sum_{\beta\in N^{m-l}} t''^{\beta}L_{\beta}(D_{t'}, D_{t_0}) \qquad (L_{\beta}\in C[D_{t'}, D_{t_0}, D_{t_0}^{-1}]) .$$

So, the operation of L on P can be defined by

$$L\cdot P=\sum_{\gamma\in N^{m-l}} t''^{\gamma}Q_{\gamma}(t', D_{t_0}) ,$$

where

$$Q_{\gamma}=\sum_{\beta+\alpha=\gamma} L_{\beta}(\partial_{t'}, D_{t_0})P_{\alpha}(t', D_{t_0})$$

which has sense since the right side is a finite sum. The other part of Lemma is easily proved. Q.E.D.

By Lemma 2.7, we know that e^{K} and e^{-K} actually operate on the space $\mathscr{A}\partial_{t_0}^{(\kappa)}$, so that one gives the inverse operator of the other since $e^{K}\cdot e^{-K}=1$ in $C[D_{t'}, D_{t_0}, D_{t_0}^{-1}][[t'']]$. The next step is to show

Lemma 2.8. *Let* ϕ *be an element of* $\mathscr{A}\delta_{t_0}^{(\kappa)}$ *and* P *an operator in* $\mathscr{D}(S)[D_{t_0}^{-1}]$. *Then, we have*

$$e^{K}Pe^{-K}\phi=\mathrm{Ad}\,(e^{K})(P)\phi \tag{17}$$

where $\mathrm{Ad}\,(e^{K})(P)$ *is the operator defined by* (13).

Proof. Note first that either side of (17) is linear over $C[t'', D_{t'}][D_{t_0}, D_{t_0}^{-1}]$ and multiplicative with respect to P. So, it is enough to show the equality (17) in the case where $P=t_i$ $(0\leqq i\leqq l)$ or $P=D_{t_i}$ $(l+1\leqq j\leqq m)$. Let P be either t_i or D_{t_j}. Then P acts continuously on $\mathscr{A}\delta_{t_0}^{(\kappa)}$ with respect to the adic topology defined by the ideal $(t'')=(t_{l+1}, \cdots, t_m)$. Since the infinite sum

$$e^{-K}\phi=\sum_{d=0}^{\infty}\frac{1}{d!}(-K)^d\phi$$

is convergent in this topology, one may compute

$$Pe^{-K}\phi=\sum_{d=0}^{\infty}\frac{1}{d!}((-K)^dP+d(-K)^{d-1}[K, P])\phi$$
$$=e^{-K}(P+[K, P])\phi=e^{-K}\mathrm{Ad}\,(e^{-K})(P)\phi\ ,$$

by using the property that $\mathrm{ad}\,(K)^dP=0$ for $d>1$. This gives the desired equality. Q.E.D.

By Lemma 2.8 combined with Proposition 2.6, we have

PROPOSITION 2.9. *An element ϕ of $\mathscr{A}\delta_{t_0}^{(\kappa)}$ ($\kappa\in\boldsymbol{C}$) is a solution of the equations* 2.2.(2) *of H_F^λ if and only if $\psi=e^K\phi$ satisfies the equations* 2.3.(5) *of $H_{\mathring{F}}^\lambda$.*

REMARK. In the case where both (B.1) and (B.2) are satisfied, the operator $K=K(D_{t_0}, D_{t'}, t'')$ commutes with the operator θ since $K=K(\tau_0, \tau', t'')$ is weighted homogeneous of degree zero with respect to the weight $(-1, -\sigma', \sigma'')$ of (τ_0, τ', t''). Hence one has $\mathrm{Ad}\,(e^K)\theta=\theta$. This shows that the operator e^K defines an isomorphism $\mathscr{A}(\varepsilon)\delta_{t_0}^{(\kappa)}\xrightarrow{\sim}\mathscr{A}(\varepsilon)\delta_{t_0}^{(\kappa)}$ for any rational number ε.

2.4. How to determine the solutions of H_F^λ in $\mathscr{A}\delta_{t_0}^{(\kappa)}$.

In this paragraph, we consider the case where (B.1) and (B.2) are satisfied. For simplicity, we assume that $l=n$, so that $x'=x=(x_1, \cdots, x_n)$ and $t'=(t_1, \cdots, t_l)$. (The term $\sum_{j=l+1}^n x_j^2$ is not essential as Proposition 1.6 shows.)

Let $f(x)$ be a weighted homogeneous polynomial of degree 1 with respect to the weight $\rho=(\rho_1, \cdots, \rho_n)$ of $x=(x_1, \cdots, x_n)$. First, we consider the Gauss-Manin system H_F^λ associated with the deformation

$$F(t_0, t, x)=t_0+\sum_{i=1}^n t_ix_i+f(x)\ . \tag{1}$$

(For the moment, we write $t=(t_1, \cdots, t_n)$.) In this case, the differential system that the integral $u=\int\delta_F^{(\lambda)}dx$ should verify is given simply by

$$\begin{cases}\theta u=(\varepsilon_*-\lambda-1)u\\ (t_i+f_{x_i}(D_tD_{t_0}^{-1}))u=0 \qquad (1\leqq i\leqq n)\ ,\end{cases} \tag{2}$$

where $\theta=t_0D_{t_0}+\sum_{i=1}^n \sigma_i t_i D_{t_i}$ $(\sigma_i=1-\rho_i)$ and $\varepsilon_*=\sum_{i=1}^n \rho_i$. We consider the solutions of (2) in the form

$$\phi(t_0, t)=\sum_{r\in \boldsymbol{Z}} a_r(t)D_{t_0}^r\delta_{t_0}^{(\kappa)} \qquad (a_r \in \boldsymbol{C}[[t]]) \tag{3}$$

where $\kappa\in \boldsymbol{C}$. Then, we know by Lemma 2.3 combined with Lemma 2.4

LEMMA 2.10. *ϕ satisfies the equations* (2) *if and only if it has an expression*

$$\phi(t_0, t)=\sum_{\langle\sigma,\alpha\rangle-r=\varepsilon} c(\alpha)\frac{t^\alpha}{\alpha!}D_{t_0}^r\delta_{t_0}^{(\kappa)} \tag{4}$$

where $\varepsilon=\varepsilon_*+\kappa-\lambda$ and $c\colon \boldsymbol{N}^n\to \boldsymbol{C}$ is a function satisfying

$$(\Delta_i+f_{x_i}(T))c=0 \qquad (1\leqq i\leqq n)\ . \tag{5}$$

(For the definition of Δ_i and $T=(T_1, \cdots, T_n)$, see 2.2.(22).)

As to the difference system (5), we have a very helpful

PROPOSITION 2.11. *Let γ be an n-chain (possibly infinite) on the universal covering of $(\boldsymbol{C}^*)^n (\boldsymbol{C}^*=\boldsymbol{C}\backslash\{0\})$ and consider the integral*

$$C(\xi)=\int_\gamma e^{f(x)}x^\xi dx\ ,$$

where $\xi=(\xi_1, \cdots, \xi_n)$ and $x^\xi dx=x_1^{\xi_1}\cdots x_n^{\xi_n}dx_1\wedge\cdots\wedge dx_n$. Assume that the integral $C(\xi)$ is convergent in a domain of $\boldsymbol{C}^n$ and allows the integration by parts. Then the function $C(\xi)$ satisfies the difference system (5) *as far as it makes sense.*

PROOF. It suffices to note that

$$(\Delta_i C)(\xi)=\xi_i C(\xi-1_i)=\int_\gamma e^f\xi_i x^{\xi-1_i}dx=-\int_\gamma f_{x_i}e^f x^\xi dx$$
$$(T_i C)(\xi)=C(\xi+1_i)=\int_\gamma e^f x^{\xi+1_i}dx=\int_\gamma x_i e^f x^\xi dx\ .$$

for $1\leqq i\leqq n$. Q.E.D.

Any function $C(\xi)$ obtained as above can be taken for the coefficients $c(\alpha)$ of (4), indeed. But, unfortunately, the author does not know how one can obtain *all* the solutions of the difference system (5). In §3, we will determine all the independent solutions of (5), in individual cases, by the aid of the integral $C(\xi)$ as above.

Now, let

$$(6)\qquad F(t_0, t, x)=t_0+\sum_{i=1}^{n} t_i x_i+G(t'', x)$$

be a general deformation of f with the condition $G|_{t''=0}=f$. In 2.3, we showed that the Gauss-Manin system H_F^λ can be reduced to $H_{\mathring{F}}^\lambda$ associated with

$$(7)\qquad \mathring{F}(t_0, t', x)=t_0+\sum_{i=1}^{n} t_i x_i+f(x)$$

by the operator $e^{K(D_{t_0}, D_{t'}, t'')}$ of exponential type, where $K(\tau_0, \tau', t'')=-\tau_0(G(t'', \tau'\tau_0^{-1})-f(\tau'\tau_0^{-1}))$. Suppose that the solutions of $H_{\mathring{F}}^\lambda$ in $\mathscr{A}\delta_{t_0}^{(\kappa)}$ are determined by any means. Then, we know by Proposition 2.9 that the solutions of H_F^λ in $\mathscr{A}\delta_{t_0}^{(\kappa)}$ are obtained from those of $H_{\mathring{F}}^\lambda$ by applying the operator e^{-K}. Let us elaborate on the operation of e^{-K}, so that one can apply this idea smoothly to concrete examples.

Here we assume that the polynomial F is given in the form

$$(8)\qquad F(t_0, t, x)=t_0+\sum_{k=1}^{m} t_k x^{\nu(k)}+f(x)\ ,$$

where $\nu(k)=(\nu_1(k), \cdots, \nu_m(k))$ $(1\leqq k\leqq m)$ are multi-indices in N^m. As the condition (B.1) requires, we assume that

$$(9)\qquad \nu(i)=1_i \qquad \text{for}\quad 1\leqq i\leqq n\ .$$

Moreover, the condition (B.2) is satisfied in this case with respect to the weight $\sigma=(\sigma_1, \cdots, \sigma_m)$ of $t=(t_1, \cdots, t_m)$ defined by

$$(10)\qquad \sigma_k=1-\langle\rho, \nu(k)\rangle=1-\sum_{i=1}^{n}\rho_i\nu_i(k) \qquad (1\leqq k\leqq m)\ .$$

Let ε be a rational number and let

$$(11)\qquad \psi(t_0, t')=\sum_{\langle\sigma', \beta\rangle-r=\varepsilon} c(\beta)\frac{t'^\beta}{\beta!}D_{t_0}^r\delta_{t_0}^{(\kappa)} \qquad (\beta\in N^n)$$

be a series in $\mathscr{A}\delta_{t_0}^{(\kappa)}$ such that $\theta\psi=(\varepsilon-\kappa-1)\psi$ and $D_{t_j}\psi=0$ for $n+1\leqq j\leqq m$. Then, we have

PROPOSITION 2.12. *On the above assumptions, the series $\phi=e^{-K}\psi$ is given by*

$$\phi(t_0, t)=\sum_{\langle\sigma, \alpha\rangle-r=\varepsilon} c(l(\alpha))\frac{t^\alpha}{\alpha!}D_{t_0}^r\delta_{t_0}^{(\kappa)} \qquad (\alpha\in N^m)\ .$$

Here $l(\alpha)=(l_1(\alpha),\cdots,l_n(\alpha))$ *is a multi-index in* N^n *defined by*

$$l_i(\alpha)=\sum_{k=1}^{m}\nu_i(k)\alpha_k \qquad (1\leqq i\leqq n)$$

for $\alpha=(\alpha_1,\cdots,\alpha_m)\in N^m$.

PROOF. Write ψ and ϕ in the form $\psi=Q\delta_{t_0}^{(\kappa)}$ and $\phi=P\delta_{t_0}^{(\kappa)}$ with $Q,P\in\mathscr{A}(\varepsilon)$. In our case, the operator $K=K(D_{t_0},D_{t'},t'')$ is defined by

$$K(\tau_0,\tau',t'')=-\tau_0E(t'',\tau'\tau_0^{-1})\ ;\qquad E(t'',x)=\sum_{k=n+1}^{m}t_kx^{\nu(k)}\ .$$

Note here that we have an expansion of the exponential function

$$e^{E(t'',x)}=\sum_{d=0}^{\infty}\frac{1}{d!}\Big(\sum_{k=n+1}^{m}t_kx^{\nu(k)}\Big)^d=\sum_{\gamma\in N^{m-n}}\frac{t''^{\gamma}}{\gamma!}x^{l(\gamma)}\ ,$$

where $\gamma=(\gamma_{n+1},\cdots,\gamma_m)$ in N^{m-n} are identified with $(0',\gamma)=(0,\cdots,0,\gamma_{n+1},\cdots,\gamma_m)$ in N^m. Hence we have

$$e^{-K}Q=\sum_{\gamma}\frac{t''^{\gamma}}{\gamma!}(\partial_{t'}D_{t_0}^{-1})^{l(\gamma)}D_{t_0}^{|\gamma|}Q\ .$$

By Lemma 2.4, one has

$$(\partial_{t'}D_{t_0}^{-1})^{l(\gamma)}Q=\sum_{\langle\sigma',\beta\rangle-r=\varepsilon+\langle\rho,l(\gamma)\rangle}c(\beta+l(\gamma))\frac{t'^{\beta}}{\beta!}D_{t_0}^{r}\ .$$

Hence,

$$\begin{aligned}e^{-K}Q&=\sum_{\langle\sigma',\beta\rangle-r=\varepsilon+\langle\rho,l(\gamma)\rangle}c(\beta+l(\gamma))\frac{t'^{\beta}}{\beta!}\cdot\frac{t''^{\gamma}}{\gamma!}D_{t_0}^{r+|\gamma|}\\&=\sum_{\langle\sigma',\beta\rangle+|\gamma|-\langle\rho,l(\gamma)\rangle-r=\varepsilon}c(\beta+l(\gamma))\frac{t'^{\beta}}{\beta!}\cdot\frac{t''^{\gamma}}{\gamma!}D_{t_0}^{r}\ .\end{aligned}$$

Here we compute

$$|\gamma|-\langle\rho,l(\gamma)\rangle=\sum_{j=n+1}^{m}(\gamma_j-\langle\rho,\nu(j)\rangle\gamma_j)=\langle\sigma'',\gamma\rangle\ .$$

Nothing that $\beta+l(\gamma)=l(\beta+\gamma)$ and $\langle\sigma',\beta\rangle+\langle\sigma'',\gamma\rangle=\langle\sigma,\beta+\gamma\rangle$, we conclude

$$P=e^{-K}Q=\sum_{\langle\sigma,\alpha\rangle-r=\varepsilon}c(l(\alpha))\frac{t^{\alpha}}{\alpha!}D_{t_0}^{r}\qquad(r\in Z,\ \alpha\in N^m)\ ,$$

as desired. Q.E.D.

REMARK. The linear mapping $l: N^m \to N^n$ is defined by the relation

$$(t_1 x^{\nu(1)})^{\alpha_1} \cdots (t_m x^{\nu(m)})^{\alpha_m} = t^\alpha x^{l(\alpha)} . \tag{12}$$

Hence one sees that $|\alpha| = \langle \sigma, \alpha \rangle + \langle \rho, l(\alpha) \rangle$. This implies $|\alpha| - \langle \sigma, \alpha \rangle = |l(\alpha)| - \langle \sigma, l(\alpha) \rangle$, so that one has an equality $l(N^m(\varepsilon)) = N^n(\varepsilon)$ for any rational number ε.

§3. Expansion of the solutions of a Gauss-Manin system H_F^λ.

3.0. Conventions.

In the previous paragraph, we suggested how we try to determine the solutions of a Gauss-Manin system. In this section, we consider the following two types of weighted homogeneous polynomials with an isolated critical point:

$$(1) \quad \begin{array}{lll} \text{(I)} & f(x) = x_1^{p_1} + x_2^{p_2} + \cdots + x_n^{p_n} & (p_i \geqq 2 \text{ for } 1 \leqq i \leqq n) \\ \text{(II)} & f(x) = x_1^{p_1} + x_1 x_2^{p_2} + x_3^{p_3} + \cdots + x_n^{p_n} & (p_2 \geqq 1;\ p_i \geqq 2 \text{ for } i \neq 2) . \end{array}$$

Let f be one of these polynomials. Then, one can choose a subset N of N^n so that the residue classes of monomials $x = x_1^{\nu_1} \cdots x_n^{\nu_n}$ ($\nu = (\nu_1, \cdots, \nu_n) \in N$) form a C-basis of the ring $C[x]/(\partial_x f)$. We take such an N to be the set of multi-indices $\nu = (\nu_1, \cdots, \nu_n) \in N^n$ such that

$$(2) \quad \begin{array}{lll} \text{(I)} & & 0 \leqq \nu_i \leqq p_i - 2 \quad \text{for} \quad 1 \leqq i \leqq n , \\ \text{(II)} & \text{a)} & 0 \leqq \nu_1 \leqq p_1 - 1;\ 0 \leqq \nu_i \leqq p_i - 2 \quad \text{for} \quad 2 \leqq i \leqq n \\ \text{or} & \text{b)} & (\nu_1, \nu_2) = (0, p_2 - 1);\ 0 \leqq \nu_i \leqq p_i - 2 \quad \text{for} \quad 3 \leqq i \leqq n , \end{array}$$

according to the case (I) or (II). So, the Milnor number of f is given by

$$(3) \quad \begin{array}{ll} \text{(I)} & \mu = \mu(f) = \# N = \prod_{i=1}^{n} (p_i - 1) \\ \text{(II)} & \mu = \mu(f) = \# N = \{p_1(p_2 - 1) + 1\} \prod_{i=3}^{n} (p_i - 1) . \end{array}$$

The polynomial f is weighted homogeneous of degree 1 with respect to the weight $\rho = (\rho_1, \cdots, \rho_n)$ of $x = (x_1, \cdots, x_n)$ defined by

$$(4) \quad \begin{array}{ll} \text{(I)} & \rho_i = \dfrac{1}{p_i} \quad \text{for} \quad 1 \leqq i \leqq n , \\ \text{(II)} & \rho_2 = \left(1 - \dfrac{1}{p_1}\right) \dfrac{1}{p_2}; \qquad \rho_i = \dfrac{1}{p_i} \quad \text{for} \quad i \neq 2 . \end{array}$$

The exponent ε_ν corresponding to the n-form $\omega_\nu = x^\nu dx$ is computed to be $\varepsilon_\nu = \sum_{i=1}^n \rho_i(\nu_i+1) = \varepsilon_0 + \langle \rho, \nu \rangle$, where $\varepsilon_0 = \sum_{i=1}^n \rho_i$ is the minimal exponent.

In the subsequent paragraphs, we will determine explicitly the expansion of solutions of H_F^λ in the case where

i) $F = t_0 + t_1 x_1 + \cdots + t_n x_n + f$ or

ii) F is a versal deformation of a simple singularity.

3.1. Case where $F = t_0 + t_1 x_1 + \cdots + t_n x_n + f$.

Let $f(x)$ be one of the polynomials in 3.0.(1). Here we consider the Gauss-Manin system H_F^λ associated with the deformation

$$(1) \qquad \mathrm{F}(t_0, t, x) = t_0 + t_1 x_1 + \cdots + t_n x_n + f(x) \qquad (t = (t_1, \cdots, t_n)) .$$

(The weight $\sigma = (\sigma_1, \cdots, \sigma_n)$ of $t = (t_1, \cdots, t_n)$ is given by $\sigma_i = 1 - \rho_i$ $(1 \leqq i \leqq n)$.) With the notation of 3.0, we define

$$(2) \qquad u_\nu = \int x^\nu \delta_F^{(\lambda)} dx \in H_F^{(\lambda)} \qquad (\nu \in N) ,$$

so that $u_\nu (\nu \in N)$ form a free basis of H_F^λ over $C[t][D_{t_0}, D_{t_0}^{-1}]$. (Theorem 1.3.) We denote by $\vec{u}$ the column vector ${}^t(u_\nu)_{\nu \in N}$, and by Λ the diagonal matrix of size μ whose (ν, ν)-component is $\lambda - \varepsilon_\nu$.

PROPOSITION 3.1. *On the assumptions above, the Gauss-Manin system H_F^λ has a finite presentation*

$$(3) \qquad \begin{cases} t_0 \vec{u} = (A_0(t) + A_1(t) D_{t_0}^{-1}) \vec{u} \\ D_{t_k} D_{t_0}^{-1} \vec{u} = (B_0^{(k)}(t) + B_1^{(k)}(t) D_{t_0}^{-1}) \vec{u} \qquad (1 \leqq k \leqq n) , \end{cases}$$

where $A_r, B_r^{(k)} \in \mathrm{M}(\mu; C[t])$ and $A_1|_{t=0} = -\Lambda$. Moreover,

a) *if $f = x_1^{p_1} + x_2^{p_2} + \cdots + x_n^{p_n}$, then $B_1^{(k)} = 0$ for $1 \leqq k \leqq n$ and $A_1 = -\Lambda$,*

b) *if $f = x_1^p + x_1 x_2^{p_2} + \cdots + x_n^{p_n}$, then $B_1^{(k)} = 0$ for $k \neq 1$.*

Proposition 3.1 can be proved by chasing explicity the proof of Theorem 1.2. So we omit its proof.

By Theorem 2.2, we know that there is a unique fundamental system of solutions of (3) in the form

$$(4) \qquad \Phi(t_0, t) = P(t, D_{t_0}) \delta_{t_0}^{(\Lambda)} \qquad (\text{or } P(t, D_{t_0}) t_0^{-\Lambda - I}) ,$$

where

$$(5) \qquad P \in \mathrm{M}(\mu: \mathcal{O}(T)[[D_{t_0}]]) \quad \text{and} \quad P|_{t=0} = I .$$

We propose to determine the explicit form of P.

For this purpose, we construct μ independent solutions of the differential system

$$
\text{(6)} \qquad \begin{cases} \theta u=(\varepsilon_0-\lambda-1)u \\ (t_i+f_{x_i}(D_t D_{t_0}^{-1}))u=0 \qquad (1\leqq i\leqq n) \end{cases}
$$

that the integral $u=u_0=\int \delta_F^{(\lambda)}dx$ should verify (Proposition 1.9). We try to find the solutions in the form

$$
\text{(7)} \qquad \phi(t_0, t)=\sum_{\langle\sigma,\alpha\rangle-r=\varepsilon} c(\alpha)\frac{t^\alpha}{\alpha!}D_{t_0}^r\delta_{t_0}^{(\kappa)} \qquad (\varepsilon=\varepsilon_0+\kappa-\lambda)\ .
$$

Here, the coefficients $c(\alpha)$ are determined as a solution c of the following difference system: according as f belongs to (I) or (II),

$$
\text{(8.I)} \qquad \alpha_i c(\alpha-\mathbf{1}_i)+p_i c(\alpha+(p_i-1)\mathbf{1}_i)=0 \qquad (1\leqq i\leqq n)
$$

$$
\text{(8.II)} \qquad \begin{cases} \alpha_1 c(\alpha-\mathbf{1}_1)+p_1 c(\alpha+(p_1-1)\mathbf{1}_1)+c(\alpha+p_2\mathbf{1}_2)=0 \\ \alpha_2 c(\alpha-\mathbf{1}_2)+p_2 c(\alpha+\mathbf{1}_1+(p_2-1)\mathbf{1}_2)=0 \\ \alpha_i c(\alpha-\mathbf{1}_i)+p_i c(\alpha+(p_i-1)\mathbf{1}_i)=0 \qquad (3\leqq i\leqq n)\ , \end{cases}
$$

where $\alpha=(\alpha_1, \cdots, \alpha_n)\in \boldsymbol{N}^n$. (See Lemma 2.10.)

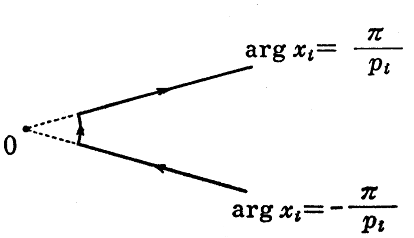
We propose here to determine the solutions of these difference systems.

CASE (I). We define a "lattice" L of $\boldsymbol{C}^n$ by

$$
\text{(9)} \qquad L=\sum_{i=1}^n \boldsymbol{Z}p_i\mathbf{1}_i\ . \qquad (\mathbf{1}_i=(0, \cdots, \overset{i}{1}, \cdots, 0))\ .
$$

If one regards (8.I) as a difference system for a meromorphic function $C(\xi)$ on $\boldsymbol{C}^n$, then one sees that, for any couple of non-zero solutions C_1, C_2 of (8.I), the ratio C_1/C_2 is periodic with respect to the lattice L. In this sense, the lattice L is *adapted* to the difference system (8.I). First, we compute an integral as indicated in Proposition 2.11. Let γ_i be a path in the x_i-plane as figured in

(10)



for $1\leqq i\leqq n$ and set $\gamma=\gamma_1\times\cdots\times\gamma_n$. Then we have

$$(11)\qquad \begin{aligned} C(\xi)&=\int_\gamma e^{f(x)}x^\xi dx=\prod_{i=1}^n\int_{\gamma_i}e^{x_i^{p_i}}x_i^{\xi_i}dx_i\\ &=\prod_{i=1}^n\frac{2\pi\sqrt{-1}}{p_i}\Gamma\Big(1-\frac{\xi_i+1}{p_i}\Big)^{-1},\end{aligned}$$

which gives a solution of (8.I). For each $\nu\in N$, we set

$$(12)\qquad \begin{aligned} L(\nu)&=(\nu+L)\cap N^n\\ &=\{\alpha=(\alpha_1,\cdots,\alpha_n)\in N^n;\ \alpha_i\equiv\nu_i \bmod p_i\ (1\leqq i\leqq n)\}.\end{aligned}$$

Then, we define a function $c_\nu\colon N^n\to C$ *with support in* $L(\nu)$ by

$$(13)\qquad c_\nu(\alpha)=\begin{cases}\dfrac{C(\alpha)}{C(\nu)}=\displaystyle\prod_{i=1}^n(-1)^{(\alpha_i-\nu_i)/p_i}\Big(\frac{\nu_i+1}{p_i}\,;\,\frac{\alpha_i-\nu_i}{p_i}\Big) & \text{for}\quad \alpha\in L(\nu)\\ 0 & \text{for}\quad \alpha\notin L(\nu),\end{cases}$$

where $(\xi;k):=\Gamma(\xi+k)/\Gamma(\xi)$. One sees immediately

LEMMA 3.2.I. *For each $\nu\in N$, define the function $c_\nu\colon N^n\to C$ with support in $L(\nu)$ as follows: for each $\alpha=(\alpha_1,\cdots,\alpha_n)\in L(\nu)$*

$$c_\nu(\alpha)=\prod_{i=1}^n(-1)^{k_i}\Big(\frac{\nu_i+1}{p_i};\,k_i\Big)\quad \textit{where}\quad k_i=\frac{\alpha_i-\nu_i}{p_i}.$$

Then, $c_\nu(\nu\in N)$ give the fundamental system of solutions of (8.I) *normalized by the condition*

$$c_\nu(\tilde\nu)=\delta_{\nu,\tilde\nu}\qquad \textit{for}\quad \nu,\tilde\nu\in N,$$

where $\delta_{\nu,\tilde\nu}$ is the Kronecker delta.

CASE (II). In this case, we also have a lattice L adapted to (8.II):

$$(14)\qquad L=Zp_1\mathbf{1}_1+Z(p_2\mathbf{1}_2+\mathbf{1}_1)+\sum_{i=3}^n Zp_i\mathbf{1}_i.$$

For each $\nu\in N$, we set

$$(15)\qquad L(\nu)=(\nu+L)\cap N^n.$$

Note that each element $\alpha=(\alpha_1,\cdots,\alpha_n)$ of $L(\nu)$ has an expression

$$(16)\qquad \begin{cases}\alpha_1=\nu_1+k_1p_1+k_2\\ \alpha_i=\nu_i+k_ip_i\qquad (2\leqq i\leqq n)\end{cases}$$

where $k_1 \in \boldsymbol{Z}$, $k_i \in \boldsymbol{N}$ $(2 \leqq i \leqq n)$ and $k_1 p_1 + k_2 \geqq -\nu_1$.

LEMMA 3.2.II. *For each* $\nu \in N$, *define the function* $c_\nu: \boldsymbol{N}^n \to \boldsymbol{C}$ *with support in* $L(\nu)$ *as follows: for each* $\alpha = (\alpha_1, \cdots, \alpha_n) \in L(\nu)$,

$$c_\nu(\alpha) = (-1)^{k_1}\left(\frac{\nu_1+1}{p_1} - \frac{\nu_2+1}{p_1 p_2};\ k_1\right) \prod_{i=2}^{n} (-1)^{k_i}\left(\frac{\nu_i+1}{p_i};\ k_i\right),$$

where

$$k_1 = \frac{\alpha_1 - \nu_1}{p_1} - \frac{\alpha_2 - \nu_2}{p_1 p_2}, \qquad k_i = \frac{\alpha_i - \nu_i}{p_i} \qquad (2 \leqq i \leqq n),$$

and $(\xi; k) = \Gamma(\xi + k)/\Gamma(\xi)$. *Then,* $c_\nu (\nu \in N)$ *give the fundamental system of solutions of* (8.II) *normalized by the condition*

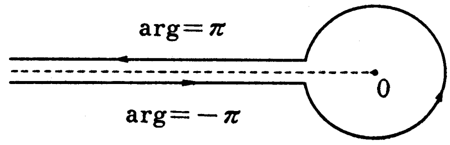
$$c_\nu(\tilde{\nu}) = \delta_{\nu,\tilde{\nu}} \qquad \textit{for} \quad \nu, \tilde{\nu} \in N.$$

Note here that, though k_1 may take negative values, the definition above makes sense since $(\nu_1+1)/p_1 - (\nu_2+1)/p_1 p_2 < 1$. Then, Lemma can be checked directly. In this case, the integral of Proposition 2.11 may be taken as follows. First, we take two continuous functions $r, \theta: \boldsymbol{R} \to \boldsymbol{R}$ so that the path

$$\gamma_0: \boldsymbol{R} \longrightarrow \boldsymbol{C}\,; \qquad \gamma_0(s) = r(s) e^{\sqrt{-1}\theta(s)} \qquad (s \in \boldsymbol{R}) \tag{17}$$

should correspond to the figure:

(18)



Then, we define an n-chain $\gamma: \boldsymbol{R}^n \to (\widetilde{\boldsymbol{C}^*})^n$ by

$$\begin{cases} x_1 = r(s_1) e^{\sqrt{-1}\theta(s_1)/p_1} \\ x_2 = r(s_2) e^{\sqrt{-1}\theta(s_2)/p_2 - \sqrt{-1}\theta(s_1)/p_1 p_2} \qquad (s = (s_1, \cdots, s_n) \in \boldsymbol{R}^n.) \\ x_i = r(s_i) s^{\sqrt{-1}\theta(s_i)/p_i} \qquad (3 \leqq i \leqq n)\,. \end{cases} \tag{19}$$

Then, by a direct computation, one obtains

$$\begin{aligned} C(\xi) &= \int_\gamma e^{f(x)} x^\xi dx \\ &= \frac{2\pi\sqrt{-1}}{p_1} \Gamma\left(1 - \frac{\xi_1+1}{p_1} + \frac{\xi_2+1}{p_1 p_2}\right)^{-1} \prod_{i=2}^{n} \frac{2\pi\sqrt{-1}}{p_i} \Gamma\left(1 - \frac{\xi_i+1}{p_i}\right)^{-1}. \end{aligned} \tag{20}$$

In fact, it was by means of this integral $C(\xi)$ that the author found the solutions $c_\nu: N^n \to C$ $(\nu \in N)$ above. But, note here that, for any index $\nu=(\nu_1, \cdots, \nu_n) \in N$ such that $(\nu_1, \nu_2)=(0, p_2-1)$, the value $C(\nu)$ is equal to zero. So one needs to multiply $C(\xi)$ by a periodic factor in obtaining the function c_ν corresponding to such an index ν by a formula analogous to (13).

Now that we know all the independent solutions of the difference system (8), we can determine the solutions of the differential system (6) or (3) as explained in §2.

PROPOSITION 3.3. *Assume that the polynomial f belongs to the type* (I) *or* (II) *and let $c_\nu: N^n \to C$ $(\nu \in N)$ be the functions of Lemma* 3.2. *For each $\nu \in N$, define an operator $P_\nu = P_\nu(t, D_{t_0})$ in $C[t][[D_{t_0}]]$ by*

$$P_\nu(D_{t_0}) = \sum_{\langle\sigma,\alpha\rangle - r = -\langle\rho,\nu\rangle} c_\nu(\alpha) \frac{t^\alpha}{\alpha!} D_{t_0}^r .$$

Then, the series

$$\phi_\nu(t_0, t) = P_\nu(t, D_{t_0}) \delta_{t_0}^{(\lambda-\varepsilon_\nu)} \qquad (\nu \in N)$$

give μ independent solutions of the Gauss-Manin system (6) *associated with the integral $u=\int \delta_F^{(\lambda)} dx$.*

COROLLARY. *Assume $\lambda \notin Z$ and $\lambda - \varepsilon_\nu \notin Z$ $(\nu \in N)$. Then the Gauss-Manin system* (6) *associated with the integral $u=\int F^{-\lambda-1} dx$ has μ independent many-valued holomorphic solutions ϕ_ν on $S \backslash D$, which are expanded in the form*

$$\phi_\nu(t_0, t) = P_\nu(t, D_{t_0}) t_0^{\varepsilon_\nu - \lambda - 1} \qquad (\nu \in N)$$

near the point $(t_0, t) = (\infty, 0)$.

Let us show that, for each $\nu \in N$, the operator P_ν defined as above belongs to $C[t][[D_{t_0}]]$. The other part of Proposition 3.3 is a consequence of Lemma 2.10. (Note here that $-\langle\rho, \nu\rangle = \varepsilon_0 - \varepsilon_\nu$ and $\operatorname{supp} c_\nu \subset L(\nu) \subset N(-\langle\rho, \nu\rangle)$.) In either case (I) or (II), set

$$\bar{N} = \{\nu = (\nu_1, \cdots, \nu_n) \in N^n;\ 0 \leqq \nu_i \leqq p_i - 1 (1 \leqq i \leqq n)\} . \tag{21}$$

Then, for any $\alpha \in N^n$, one can find a unique index in $\bar{N}$, which we denote by $[\alpha]$, such that $[\alpha] \equiv \alpha \bmod L$. Then one can show by an elementary computation in each case

Lemma 3.4. a) *For each $\nu \in \bar{N}$, set $L(\nu)=(\nu+L)\cap N^n$. Then, one has* $\min\{\langle\sigma, \alpha\rangle; \alpha \in L(\nu)\}=\langle\sigma, \nu\rangle=|\nu|-\langle\rho, \nu\rangle$.

b) *Set $\bar{N}-\bar{N}=\{\nu-\tilde{\nu}; \nu, \tilde{\nu} \in \bar{N}\}$. Then, for each $\alpha \in \bar{N}-\bar{N}$, one has $|[\alpha]| \geqq \langle\rho, [\alpha]-\alpha\rangle$, or equivalently, $\langle\rho, \alpha\rangle \geqq -\langle\sigma, [\alpha]\rangle$.*

For each $\nu \in N$, Lemma 3.4.a) implies that $r=\langle\sigma, \alpha\rangle+\langle\rho, \nu\rangle$ takes minimal value $|\nu|$ as α ranges in $L(\nu)$. This shows that P_ν belongs to $C[t][[D_{t_0}]]D_{t_0}^{|\nu|}$ ($|\nu|=\nu_1+\cdots+\nu_n$).

From these solutions ϕ_ν ($\nu \in N$), we can construct the fundamental system of solutions $\Phi=P\delta_{t_0}^{(\Lambda)}$ in (4) by the method given in 2.2. Recall that one has equalities

$$u_{\tilde{\nu}}=\int x^{\tilde{\nu}}\delta_F^{(\lambda)}dx=(D_tD_{t_0}^{-1})^{\tilde{\nu}}\int \delta_F^{(\lambda)}dx=(D_tD_{t_0}^{-1})^{\tilde{\nu}}u \qquad (\tilde{\nu} \in N) \tag{22}$$

in H_F^λ. This shows that the column vectors

$$\vec{\phi}_\nu={}^t((D_tD_{t_0}^{-1})^{\tilde{\nu}}\phi_\nu)_{\tilde{\nu} \in N} \qquad (\nu \in N) \tag{23}$$

give μ independent solutions of the differential system (3) of Proposition 3.1. For each couple $(\tilde{\nu}, \nu) \in N\times N$, the series $(D_tD_{t_0}^{-1})^{\tilde{\nu}}\phi_\nu=(\partial_tD_{t_0}^{-1})^{\tilde{\nu}}P_\nu\delta_{t_0}^{(\lambda-\varepsilon_\nu)}$ is determined by

$$(\partial_tD_{t_0}^{-1})^{\tilde{\nu}}P_\nu=\sum_{\langle\sigma,\alpha\rangle-r=\langle\rho,\tilde{\nu}-\nu\rangle} c_\nu(\alpha+\tilde{\nu})\frac{t^\alpha}{\alpha!}D_{t_0}^r \tag{24}$$

(Lemma 2.4). Let us show that this operator belongs to $C[t][[D_{t_0}]]$. Note that $c_\nu(\alpha+\tilde{\nu})=0$ unless $\alpha \in L([\nu, \tilde{\nu}])$. If $\alpha \in L([\nu-\tilde{\nu}])$, then one has

$$\begin{aligned}\langle\sigma, \alpha\rangle-\langle\rho, \tilde{\nu}-\nu\rangle &\geqq \langle\sigma, \alpha\rangle-\langle\sigma, [\nu-\tilde{\nu}]\rangle && \text{(Lemma 3.4.b))}\\ &\geqq 0 && \text{(Lemma 3.4.a))}\,.\end{aligned} \tag{25}$$

This implies that $(\partial_tD_{t_0}^{-1})^{\tilde{\nu}}P_\nu \in C[t][[D_{t_0}]]$ for $\tilde{\nu}, \nu \in N$. Remark that

$$(\partial_tD_{t_0}^{-1})^{\tilde{\nu}}P_\nu|_{t=0}=c_\nu(\tilde{\nu})D_{t_0}^{\langle\rho,\nu-\tilde{\nu}\rangle}=\delta_{\nu,\tilde{\nu}} \qquad \text{for} \quad \nu, \tilde{\nu} \in N\,. \tag{26}$$

We define a matrix $P=(P_{\tilde{\nu},\nu})_{\tilde{\nu},\nu \in N}$ by $P_{\tilde{\nu},\nu}=(\partial_tD_{t_0}^{-1})^{\tilde{\nu}}P_\nu$, so that $P|_{t=0}=I$. Then, by the construction of P, the matrix $\Phi=P\delta_{t_0}^{(\Lambda)}$ satisfies the equations (3). As remarked in 2.2, we can conclude from this that the matrix P coincides with the one constructed as in Theorem 2.2. Summarizing,

Theorem 3.5. *Let $f(x)$ be a polynomial of type either*

(I) $f(x)=x_1^{p_1}+x_2^{p_2}+\cdots+x_n^{p_n}$ ($p_i \geqq 2$ *for* $1 \leqq i \leqq n$) *or*

(II) $f(x)=x_1^{p_1}+x_1x_2^{p_2}+x_3^{p_3}+\cdots+x_n^{p_n}$ ($p_2 \geqq 1$, $p_i \geqq 2$ *for* $i \neq 2$)

and set $F(t_0, t, x)=t_0+t_1x_1+\cdots+t_nx_n+f(x)$. According to the type of f, *let* $c_\nu\colon N^n \to C$ $(\nu \in N)$ *be the functions defined in Lemma* 3.2.I *or Lemma* 3.2.II *and define a matrix* $P=(P_{\tilde{\nu},\nu})_{\tilde{\nu},\nu\in N}$ *of operators* $C[t][[D_{t_0}]]$ *by*

$$P_{\tilde{\nu},\nu}(t, D_{t_0}) = \sum_{\langle\sigma,\alpha\rangle - r = \langle\rho,\tilde{\nu}-\nu\rangle} c_\nu(\alpha+\tilde{\nu}) \frac{t^\alpha}{\alpha!} D_{t_0}^r \qquad (r \in N,\ \alpha \in N^n)$$

for $\tilde{\nu}, \nu \in N$. *Consider the Gauss-Manin system associated with the column vector*

$$\vec{u} = {}^t(u_\nu)_{\nu\in N}\ , \qquad \textit{where} \quad u_\nu = \int x^\nu \delta_F^{(\lambda)} dx \in H_F^\lambda \qquad (\nu \in N)\ .$$

Then, the matrix $\Phi(t_0, t) = P(t, D_{t_0})\delta_{t_0}^{(\Lambda)}$ *gives its fundamental system of solutions, normalized by the condition* $\Phi|_{t=0} = \delta_{t_0}^{(\Lambda)}$. *Here* Λ *is the diagonal matrix* diag $(\lambda - \varepsilon_\nu;\ \nu \in N)$.

COROLLARY. *Assume that* $\lambda \notin Z$ *and* $\lambda - \varepsilon_\nu \notin Z$, *and consider the Gauss-Manin system associated with the column vector*

$$\vec{u} = {}^t(u_\nu)_{\nu\in N}\ , \qquad \textit{where} \quad u_\nu = \int x^\nu F^{-\lambda-1} dx \qquad (\nu \in N)\ .$$

Let $\Phi(t_0, t)$ *be its fundamental system if many-valued holomorphic solutions on* $S\backslash D$ *with* $\Phi|_{t=0} = t_0^{-\Lambda - I}$. *Then, the matrix* Φ *can be developed into the power series*

$$\Phi(t_0, t) = P(t, D_{t_0}) t_0^{-\Lambda - I}$$

near the point $(t_0, t) = (\infty, 0)$ *by the matrix of operators* P *above.*

3.2. Representation by hypergeometric series.

In this paragraph, we confine ourselves to the Gauss-Manin system H_F^λ associated with the deformation

$$F(t_0, t, x) = t_0 + t_1 x_1 + \cdots + t_n x_n + x_1^{p_1} + \cdots + x_n^{p_n} \tag{1}$$

of type (I), where $p_i \geqq 2$ for $1 \leqq i \leqq n$. Here we consider H_F^λ as associated with the integral

$$u = \int F^{-\lambda-1} dx\ , \tag{2}$$

assuming that $\lambda \notin Z$. In this case, the differential system to be satisfied by u is presented by

$$\begin{cases} \theta u = (\varepsilon_0 - \lambda - 1)u \\ (t_i + p_i(D_{t_i} D_{t_0}^{-1})^{p_i - 1})u = 0 \qquad (1 \leqq i \leqq n)\ , \end{cases} \tag{3}$$

where

$$\theta = t_0 D_{t_0} + \sum_{i=1} \frac{p_i - 1}{p_i} t_i D_{t_i} \quad \text{and} \quad \varepsilon_0 = \sum_{i=1}^{n} \frac{1}{p_i} . \tag{4}$$

Hereafter, we assume that $\lambda \notin \boldsymbol{Z}$ and $\lambda - \varepsilon_\nu \notin \boldsymbol{Z}$ ($\nu \in N$). If λ is *generic* in this sense, we know that (3) has μ independent many-valued holomorphic solutions ϕ_ν on $S \backslash D$. Moreover, we gave their explicit expansions at the point of infinity (Corollary to Proposition 3.3). In this paragraph, we will show, from a different point of view, that they can be represented by certain hypergeometric series. In what follows we use the notation $\vartheta_t = tD_t$, recalling that

$$t^k D_t^k = \vartheta_t(\vartheta_t - 1) \cdots (\vartheta_t - k + 1) = [\vartheta_t; k] \qquad \text{for} \quad k \in \boldsymbol{N} , \tag{5}$$

where $[\xi; k] := \xi(\xi - 1) \cdots (\xi - k + 1)$.

In view of (3), consider the system of differential equations

$$\begin{cases} \text{i)} & \theta u = -(\kappa_0 + 1)u \qquad \text{with} \quad \kappa_0 = \lambda - \varepsilon_0 \\ \text{ii)} & (t_i^{p_i} D_{t_i}^{p_i - 1} + p_i D_{t_i}^{p_i - 1})u = 0 \qquad (1 \leqq i \leqq n) . \end{cases} \tag{6}$$

By multiplying (6.ii) by $t_i^{p_i - 1}$, one can rewrite them in the form

$$(t_i^{p_i} t_0^{-p_i + 1} [\vartheta_{t_0}; p_i - 1] + p_i [\vartheta_{t_i}; p_i - 1])u = 0 \qquad (1 \leqq i \leqq n) . \tag{7}$$

Here we make the transformation $u(t_0, t) = t_0^{-(\kappa_0 + 1)} v(t_0, t)$, so that one has the equations for v

$$\begin{cases} \text{i)} & \theta v = 0 \\ \text{ii)} & (t_i^{p_i} t_0^{-p_i + 1} [\vartheta_{t_0} - \kappa_0 - 1; p_i - 1] + p_i [\vartheta_{t_i}; p_i - 1])v = 0 \qquad (1 \leqq i \leqq n) . \end{cases} \tag{8}$$

Since v must be weighted homogeneous of degree zero, one might anticipate that it would be a function depending on the variables·

$$z = (z_1, \cdots, z_n) , \qquad z_i = (-1)^{p_i} t_i^{p_i} / p_i^{p_i} t_0^{p_i - 1} \qquad (1 \leqq i \leqq n) . \tag{9}$$

(The constants $(-1)^{p_i}/p_i^{p_i}$ are put for convenience.) In these variables $z = (z_1, \cdots, z_n)$, one has

$$\begin{cases} \vartheta_{t_0} = -\langle p - 1, \vartheta_z \rangle := -\sum_{i=1}^{n} (p_i - 1)\vartheta_{z_i} \\ \vartheta_{t_i} = p_i \vartheta_{z_i} \qquad (1 \leqq i \leqq n) , \end{cases} \tag{10}$$

so that $\theta = 0$. Setting $w(z) = v(t_0, t)$, one obtain from (9) a system of *hypergeometric equations* for w:

$$\left\{[\langle p-1, \vartheta_z\rangle+\kappa_0; p_i-1]z_i-\prod_{k=1}^{p_i-2}\left(\vartheta_{z_i}-\frac{k}{p_i}\right)\right\}w=0 . \tag{11}$$

The differential system (11) can be solved easily, so that one sees that (11) has $\mu=\prod_{i=1}^{n}(p_i-1)$ independent solutions in power series

$$\sum_{\alpha\in N^n} a(\alpha)z^{\zeta+\alpha} \qquad (\zeta\in C) . \tag{12}$$

To write down such solutions of (11) in an explicit manner, we introduce the following hypergeometric series: for each $\nu=(\nu_1, \cdots, \nu_n)$ with $0\leqq\nu_i\leqq p_i-2$, we define

$$G_\nu(z)=\sum_{\alpha\in N^n}\frac{(1+\kappa_0-\langle\sigma, \nu\rangle; \langle p-1, \alpha\rangle)}{\prod_{i=1}^{n}\prod_{k=0}^{p_i-2}\left(1+\frac{\nu_i-k}{p_i}; \alpha_i\right)}z^\alpha , \tag{13}$$

where $\sigma=(\sigma_1, \cdots, \sigma_n)$, $\sigma_i=1-(1/p_i)$, $(\xi; k)=\xi(\xi+1)\cdots(\xi+k-1)$. It can be checked directly that the power series $G_\nu(z)$ are convergent in a neighborhood of the origin $z=0$. Then it is easy to show

PROPOSITION 3.6. *The many-valued functions $z^{\nu/p}G_\nu(z)$ $(\nu\in N)$ give μ independent solutions of the hypergeometric equations* (11), *where $z^{\nu/p}=z_1^{\nu_1/p_1}\cdots z_n^{\nu_n/p_n}$.*

By comparing these $z^{\nu/p}G_\nu(z)$ with the expression of Proposition 3.3, we have

THEOREM 3.7. *On the condition $\lambda\notin Z$ and $\lambda-\varepsilon_\nu\notin Z$ $(\nu\in N)$, consider the Gauss-Manin system associated with the integral*

$$u=\int F^{-\lambda-1}dx \quad \textit{where} \quad F=t_0+t_1x_1+\cdots+t_nx_n+x_1^{p_1}+\cdots+x_n^{p_n} .$$

Then, the many-valued holomorphic solutions ϕ_ν of Corollary to Proposition 3.3 are expressed in the form

$$\phi_\nu(t_0, t)=\frac{(-1)^{|\nu|}}{\nu!}(\lambda-\varepsilon_\nu+1; |\nu|)t^\nu t_0^{\varepsilon_\nu-\lambda-1-|\nu|}G_\nu(z) ,$$

in terms of the hypergeometric series G_ν defined by (13), *where*

$$z=(z_1, \cdots, z_n) ; \qquad z_i=(-1)^{p_i}t_i^{p_i}/p_i^{p_i}t_0^{p_i-1} .$$

We include here some examples of such hypergeometric expression of the solutions of Gauss-Manin systems. Examples below are restricted to the case where $f=x_1^p+\cdots+x_n^p$ and expressed in the variables $z=$

$(z_1, \cdots, z_n)$ multiplied by suitable constants to fit for classical hypergeometric series.

Let us consider the case where $f=x_1^p+\cdots+x_n^p$ $(p\geqq 2)$. Then, the Milnor number μ equals $(p-1)^n$ and the set N of indices is defined by

$$N=\{\nu=(\nu_1, \cdots, \nu_n)\in \boldsymbol{N}^n;\ 0\leqq \nu_i\leqq p-2\ \ (1\leqq i\leqq n)\}\ . \tag{14}$$

To describe the solutions of the Gauss-Manin system for

$$u=\int (t_0+t_1x_1+\cdots+t_nx_n+x_1^p+\cdots+x_n^p)^{-\lambda-1}dx \qquad (\lambda\text{: generic})\ , \tag{15}$$

we use the variables $z=(z_1, \cdots, z_n)$ defined by

$$z_i=(-1)^p(p-1)^{p-1}t_i^p/p^pt_0^{p-1} \qquad (1\leqq i\leqq n)\ . \tag{16}$$

Then the hypergeometric series G_ν $(\nu\in N)$ above are rewritten in the form

$$G_\nu(z)=\sum_{\alpha\in N^n} z^\alpha \prod_{k=0}^{p-2}\frac{(a_k, |\alpha|)}{\prod_{i=1}^{n}(b_{ki}, \alpha_i)}\ , \tag{17}$$

where

$$a_k=\frac{1}{p-1}\left(\lambda+1-\frac{n}{p}+k\right)+\frac{|\nu|}{p}\ , \qquad b_{ki}=\frac{\nu_i+2+k}{p}\ . \tag{18}$$

The solutions ϕ_ν $(\nu\in N)$ are given by

$$\phi_\nu=\frac{(-1)^{|\nu|}}{\nu!}\left(\lambda+1-\frac{n+|\nu|}{p};\ |\nu|\right)t^\nu t_0^{(n/p)-\lambda-1-(p-1/p)|\nu|}G_\nu(z)\ . \tag{19}$$

We denote by Δ the discriminant of H_F^λ. (See 1.2) For hypergeometric functions, see Appell-Kampé de Fériet [2].

EXAMPLE 0. Case where $F=t_0+t_1x_1+\cdots+t_nx_n+x_1^2+\cdots+x_n^2$. ($\mu=1$, $\Delta=t_0-(1/4)(t_1^2+\cdots+t_n^2)$.) The solution ϕ is a power function

$$\phi=\left(t_0-\frac{1}{4}(t_1^2+\cdots+t_n^2)\right)^{(n/2)-\lambda-1}\ . \tag{20}$$

EXAMPLE 1. Case where $F=t_0+t_1x+x^3$. ($\mu=2$, $\Delta=t_0^2+(4/27)t_1^3$.) The solutions ϕ_0, ϕ_1 are given by

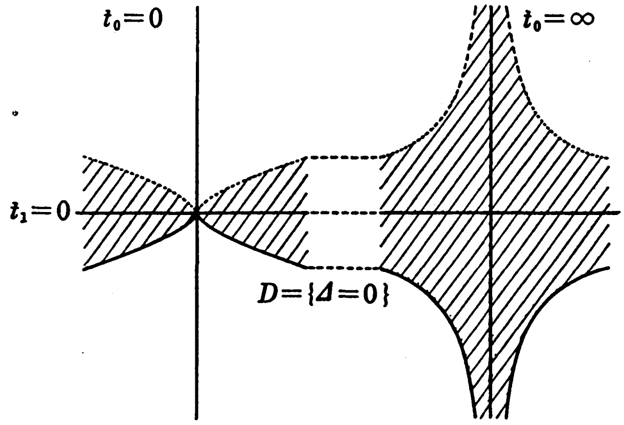
$$\phi_0=t_0^{-\lambda-(2/3)}F\left(\frac{\lambda}{2}+\frac{1}{3}, \frac{\lambda}{2}+\frac{5}{6}; \frac{2}{3}; -\frac{4t_1^3}{27t_0^2}\right) \tag{21}$$

$$\phi_1=-\left(\lambda+\frac{1}{3}\right)t_1t_0^{-\lambda-(4/3)}F\left(\frac{\lambda}{2}+\frac{2}{3},\frac{\lambda}{2}+\frac{7}{6};\frac{4}{3};-\frac{4t_1^3}{27t_0^2}\right).$$

Here $F(a, b; c; z)$ denotes the Gauss hypergeometric function

$$F(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a;k)(b;k)}{(c;k)}\frac{z^k}{k!}. \tag{22}$$

By $z=-4t_1^3/27t_0^2$, one sees that the series on the right side of (21) converge in the region $|-4t_1^3/27t_0^2|<1$.

(23) 

Moreover, the discriminant set $D=\{\Delta=0\}$ corresponds to the singularity at $z=1$ of Gauss' hypergeometric functions. (By the transformation formulas for Gauss' hypergeometric functions, one can also obtain some other types of power series expansion for ϕ_0 and ϕ_1.)

EXAMPLE 2. Case where $F=t_0+t_1x+x^p$ $(p\geqq 3)$. $(\mu=p-1,\ \Delta=t_0^{p-1}-(-1)^p(p-1)^{p-1}t_1^p/p^p.)$ In this case, the $p-1$ solutions ϕ_ν $(0\leqq\nu\leqq p-2)$ are expressed by generalized hypergeometric functions

$${}_{p-1}F_{p-2}(a_1,\cdots,a_{p-1};b_1,\cdots,b_{p-2};z)=\sum_{k=0}^{\infty}\frac{(a_1;k)\cdots(a_{p-1};k)}{(b_1;k)\cdots(b_{p-2};k)}\frac{z^k}{k!}. \tag{24}$$

For example,

$$\phi_0=t_0^{-\lambda-(p-1/p)}{}_{p-1}F_{p-2}\left(a_1,\cdots,a_{p-1},b_1,\cdots,b_{p-2};(-1)^p\frac{(p-1)^{p-1}t_1^p}{p^pt_0^{p-1}}\right), \tag{25}$$

where $a_i=\lambda/(p-1)-1/p(p-1)+i/(p-1)$, $b_i=(i+1)/p$. Note that $\Delta t_0^{1-p}=1-z$.

EXAMPLE 3. Case where $F=t_0+t_1x_1+t_2x_2+x_1^3+x_2^3$. $(\mu=4.)$ In this example, the discriminant Δ is given by

(26) $$\Delta = t_0^4 + \frac{8}{27}(t_1^3 + t_2^3)t_0^2 + \left(\frac{4}{27}\right)^2 (t_1^3 - t_2^3)^2 \ .$$

The four solutions ϕ_ν $(\nu=(0, 0), (0, 1), (1, 0), (1, 1))$ are expressed by Appell's hypergeometric functions F_4:

(27) $$F_4(a, b; c_1, c_2; z_1, z_2) = \sum_{k_1, k_2 \geq 0} \frac{(a; k_1+k_2)(b; k_1+k_2)}{(c_1; k_1)(c_2; k_2)} \cdot \frac{z_1^{k_1}}{k_1!} \cdot \frac{z_2^{k_2}}{k_2!} \ .$$

For example,

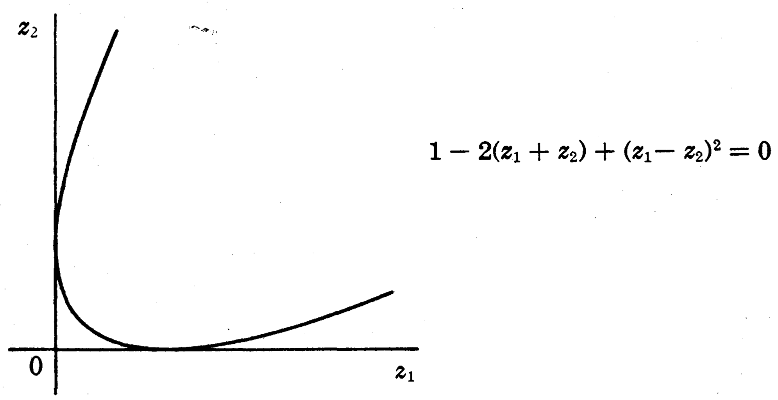
(28) $$\phi_0 = t_0^{-\lambda-(1/3)} F_4\left(\frac{\lambda}{2}+\frac{1}{6}, \frac{\lambda}{2}+\frac{2}{3}; \frac{2}{3}, \frac{2}{3}; z_1, z_2\right) ,$$

where $z_i = -4t_i^3/27t_0^2$ for $i=1, 2$. In terms of these variables (z_1, z_2), one has

(29) $$\Delta t_0^{-4} = 1 - 2(z_1+z_2) + (z_1-z_2)^2 \ .$$

Note that the right side defines an irreducible component of the singular locus of Appell's F_4. (See, T. Kimura [10].)

(30)



EXAMPLE 4. Case where $F = t_0 + t_1x_1 + \cdots + t_nx_n + x_1^3 + \cdots + x_n^3$ $(\mu = 2^n)$. In this case, the 2^n solutions ϕ_ν $(\nu = (\nu_1, \cdots, \nu_n), \nu_i = 0, 1)$ are expressed by Lauricella's hypergeometric functions F_C:

(31) $$F_C(a, b; c_1, \cdots, c_n; z_1, \cdots, z_n) = \sum_{k_1, \cdots, k_n \geq 0} \frac{(a; k_1+\cdots+k_n)(b; k_1+\cdots+k_n)}{(c_1; k_1)\cdots(c_n; k_n)} \cdot \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_n^{k_n}}{k_n!} \ .$$

In fact, ϕ_ν are given by

(32) $$\phi_\nu = (-1)^{|\nu|}\left(\lambda+1-\frac{n}{3}-\frac{|\nu|}{3}; |\nu|\right) t_0^{(n/3)-\lambda-1-(3/2)|\nu|} t_1^{\nu_1} \cdots t_n^{\nu_n}$$

$$\times F_C\Big(\frac{1}{2}+\frac{\lambda}{2}-\frac{n}{6}+\frac{|\nu|}{3},\ 1+\frac{\lambda}{2}-\frac{n}{6}+\frac{|\nu|}{3}\ ;$$
$$\frac{2}{3}(\nu_1+1),\ \cdots,\ \frac{2}{3}(\nu_n+1);\ z_1,\ \cdots,\ z_n\Big)$$

where $|\nu|=\nu_1+\cdots+\nu_n$ and $z_i=-4t_i^3/27t_0^2$ $(1\leqq i\leqq n)$.

3.3. Case where F is a versal deformation of a simple singularity.

We pay our attention, in this paragraph, to so-called simple singularities and their versal deformations. Let f be one of the following canonical forms of simple singularities:

$$(1)\qquad \begin{array}{lll} A_l; & f(x)=x_1^{l+1}+x_2^2+\cdots+x_n^2 & (l\geqq 1)\\ D_l: & f(x)=x_1^{l-1}+x_1x_2^2+x_3^2+\cdots+x_n^2 & (l\geqq 4)\\ E_6: & f(x)=x_1^4+x_2^3+x_3^2+\cdots+x_n^2 & \\ E_7: & f(x)=x_1^3x_2+x_2^3+x_3^2+\cdots+x_n^2 & \\ E_8: & f(x)=x_1^5+x_2^3+x_3^2+\cdots+x_n^2\ . & \end{array}$$

Noting that f belongs to the type (I) or (II) in each case, we consider a versal deformation F of f in the form

$$(2)\qquad F(t_0,\ t,\ x)=\sum_{\nu\in N} t_\nu x^\nu+f(x)\ ,\qquad t=(t_\nu)_{\nu\in N\backslash\{0\}}\ ,$$

where N is the set of indices defined as in 3.0. We define the weight σ_ν of t_ν to be $1-\langle\rho,\nu\rangle$, so that F should be weighted homogeneous of degree 1. F is a *versal* deformation of f in the sense that $\partial_{t_\nu}F|_{t=0}$ $(\nu\in N)$ form a C-basis of Ω_f. Moreover, it should be noted that F satisfies the condition (A.2) of 1.2, so that the weights σ_ν of t_ν are all positive. It is known that, in a sense, this property characterizes the polynomials listed in (1). (See K. Saito [15].)

Now we consider the Gauss-Manin system H_F^λ. Define

$$(3)\qquad u_\nu=\int x^\nu\delta_F^{(\lambda)}dx\in H_F^{(\lambda)}\qquad(\lambda\in C)$$

for each $\nu\in N$ and denote by $\vec{u}$ the column vector ${}^t(u_\nu)_{\nu\in N}$. Note here that we have

$$(4)\qquad u_\nu=D_{t_\nu}D_{t_0}^{-1}u\ ,\qquad u=u_0=\int\delta_F^{(\lambda)}dx\ ,$$

for each $\nu\in N$, since F is given by (2).

PROPOSITION 3.8. *On the above assumptions, the Gauss-Manin system*

H_F^λ has a finite presentation

$$\begin{cases} t_0\vec{u}=(A_0(t)+A_1(t)D_{t_0}^{-1})\vec{u} \\ D_{t_\nu}D_{t_0}^{-1}\vec{u}=(B_0^{(\nu)}(t)+B_1^{(\nu)}(t)D_{t_0}^{-1})\vec{u} \qquad (\nu \in N\backslash\{0\}) \end{cases} \tag{5}$$

where $A_r(t)$, $B_r^{(\nu)}(t) \in \mathrm{M}\,(\mu: C[t])$.

PROOF. By Theorem 2.2, we know that H_F^λ has a finite presentation

$$\begin{cases} t_0\vec{u}=\sum_{r=0}^{R} A_r(t)D_{t_0}^{-r}\vec{u} \\ D_{t_\nu}D_{t_0}^{-1}\vec{u}=\sum_{r=0}^{R} B_r^{(\nu)}D_{t_0}^{-r}\vec{u} \ , \end{cases}$$

for some A_r, $B_r^{(\nu)} \in \mathrm{M}\,(\mu: C[t])$. Recall that $\theta u=(\varepsilon_0-\lambda-1)u$, where $\theta=\sum_{\nu\in N}\sigma_\nu t_\nu D_{t_\nu}$. Hence, by $u_\nu=D_{t_\nu}D_{t_0}^{-1}u$, one sees

$$\theta(u_\nu)=(\varepsilon_\nu-\lambda-1)u_\nu \qquad (\varepsilon_\nu=\varepsilon_0+\langle\rho,\nu\rangle) \ ,$$

so that $\theta\vec{u}=(-\Lambda-I)\vec{u}$. Applying the Euler operator θ to the above presentation,

$$-\Lambda t_0\vec{u}=\sum_{r=0}^{R}\{\theta(A_r)+A_r((r-1)I-\Lambda)\}D_{t_0}^{-r}\vec{u} \ .$$

Hence

$$-\Lambda\sum_{r=0}^{R}A_r(t)D_{t_0}^{-r}\vec{u}=\sum_{r=0}^{R}\{\theta(A_r)+A_r((r-1)I-\Lambda)\}D_{t_0}^{-r}\vec{u} \ .$$

Since $(u_\nu)_{\nu\in N}$ are free over $C[t][D_{t_0}, D_{t_0}^{-1}]$, one knows

$$\theta(A_r)=-[\Lambda, A_r]-(r-1)A_r \ .$$

Setting $A_r=(a_{\tilde{\nu}\nu}^r)_{\tilde{\nu},\nu\in N}$, one obtains

$$\theta(a_{\tilde{\nu}\nu}^r)=(\varepsilon_{\tilde{\nu}}-\varepsilon_\nu+1-r)a_{\tilde{\nu}\nu}^r=(\langle\rho, \tilde{\nu}-\nu\rangle+1-r)a_{\tilde{\nu}\nu}^r$$

for $\tilde{\nu}$, $\nu\in N$. Recall that $\langle\rho, \nu\rangle<1$ for $\nu\in N$, on our assumptions. Hence, $\langle\rho, \tilde{\nu}-\nu\rangle<1$ for $\tilde{\nu}$, $\nu\in N$. This shows that $a_{\tilde{\nu},\nu}^r=0$ unless $r\leqq 1$, namely, $A_r=0$ for $r\geqq 2$. Similary one can show $B_r^{(\nu)}=0$ for $r\geqq 2$. Q.E.D.

By Proposition 3.8 combined with Theorem 2.2, we know that there is a unique matrix $P(t, D_{t_0})$ of operators in $\mathrm{M}\,(\mu: C[[t]][[D_{t_0}]])$ with $P|_{t=0}=I$ such that the matrix $\Phi=P\delta_{t_0}^{(\Lambda)}$ gives the fundamental system of solutions of (3).

For the versal deformation F of (2), we consider the following deformation $\mathring{F}$:

$$(6)\qquad \begin{cases} \mathring{F}(t_0, t_1, x)=t_0+t_1x_1+f(x) & \text{if } f \text{ is of type } A_l\,, \\ \mathring{F}(t_0, t_1, t_2, x)=t_0+t_1x_1+t_2x_2+f(x) & \text{otherwise}\,. \end{cases}$$

As explained in 2.3, the Gauss-Manin system H_F^λ ($\lambda \in C$) can be reduced to $H_{\mathring{F}}^\lambda$ by an operator e^K of exponential type. (See 2.2.) So, we can determine the matrix P for H_F^λ from that for $H_{\mathring{F}}^\lambda$, say Q, which we have already given in Theorem 3.5.

Let $Q_\nu \in C[t][[D_{t_0}]]$ ($\nu \in N$) be the operators of Proposition 3.3 for $H_{\mathring{F}}^\lambda$, so that $\psi_\nu=Q_\nu\delta_{t_0}^{(\lambda-\epsilon_\nu)}$ give μ independent solutions of the Gauss-Manin system associated with the integral

$$(7)\qquad v=\int \delta_{\mathring{F}}^{(\lambda)}dx \in H_{\mathring{F}}^{(\lambda)}\,.$$

Then, we know by Proposition 2.9 that $\phi_\nu=e^{-K}\psi_\nu$ are solutions of the Gauss-Manin system associated with the integral

$$(8)\qquad u=\int \delta_F^{(\lambda)}dx \in H_F^{(\lambda)}\,.$$

Furthermore, we know how the series $\phi_\nu=e^{-K}\psi_\nu$ (or equivalently, the $P_\nu=e^{-K}Q_\nu$) can be determined explicitly. (Proposition 2.12.)

We begin with elaborating on the functions c_ν ($\nu \in N$) given in Lemma 3.2.

CASE (A_l) 1) $N=\{0, 1, \cdots, l-1\}$.

2) $L(\nu)=\{\alpha \in N;\ \alpha\equiv\nu \bmod (l+1)\}=\{\nu+k(l+1);\ k\geqq 0\}$.

3) $c_\nu(\alpha)=\begin{cases} (-1)^k\left(\dfrac{\nu+1}{l+1}; k\right) & \text{if } \alpha \in L(\nu), \\ 0 & \text{otherwise.} \end{cases}$

CASE (D_l) 1) $N=\{(\nu, 0);\ 0\leqq\nu\leqq l-2\}\cup\{(0, 1)\}$.

2) $L(\nu, 0)=\left\{(\alpha_1, \alpha_2) \in N^2;\ \alpha_2\equiv 0 \bmod 2,\ \alpha_1\equiv\nu+\dfrac{\alpha_1}{2} \bmod (l-1)\right\}$

$$=\left\{(\nu+k_1(l-1)+k_2,\ 2k_2);\ k_2\geqq 0,\ k_1\geqq -\frac{\nu+k_2}{l-2}\right\}.$$

$$L(0, 1)=\left\{(\alpha_1, \alpha_2) \in N^2;\ \alpha_2\equiv 1 \bmod 2,\ \alpha_2\equiv\frac{\alpha_1-1}{2} \bmod (l-1)\right\}$$

$$=\left\{(k_1(l-1)+k_2,\ 2k_2+1);\ k_2\geqq 0,\ k_1\geqq -\frac{k_2}{l-1}\right\}.$$

$$3)\quad c_{\nu 0}(\alpha)=\begin{cases}(-1)^{k_1+k_2}\left(\frac{\nu+1}{l-1}-\frac{1}{2(l-1)};\,k_1\right)\left(\frac{1}{2};\,k_2\right) & \text{if } \alpha\in L(\nu,0)\\ 0 & \text{otherwise}\end{cases}$$

$$c_{01}(\alpha)=\begin{cases}(-1)^{k_2}k_2!/(-k_1)! & \text{if } \alpha\in L(0,1) \text{ and } k_1\leqq 0,\\ 0 & \text{otherwise.}\end{cases}$$

CASE (E_6) 1) $N=\{(\nu_1,\nu_2);\,\nu_1=0,1,2;\,\nu_2=0,1\}$.

2) $L(\nu_1,\nu_2)=\{(\alpha_1,\alpha_2)\in N^2;\,\alpha_1\equiv\nu_1 \bmod 4,\,\alpha_2\equiv\nu_2 \bmod 3\}$
$=\{(\nu_1+4k_1,\,\nu_2+3k_2);\,k_1,k_2\geqq 0\}$.

$$3)\quad c_\nu(\alpha)=\begin{cases}(-1)^{k_1+k_2}\left(\frac{\nu_1+1}{4};\,k_1\right)\left(\frac{\nu_2+1}{3};\,k_2\right) & \text{if } \alpha\in L(\nu),\\ 0 & \text{otherwise.}\end{cases}$$

CASE (E_7) 1) $N=\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0)\}$.

2) $L(\nu_1,\nu_2)=\left\{(\alpha_1,\alpha_2)\in N^2;\,\alpha_1\equiv\nu_1 \bmod 3,\,\alpha_2\equiv\nu_2+\frac{\alpha_1-\nu_1}{3} \bmod 3\right\}$

$=\left\{(\nu_1+3k_1,\,\nu_2+3k_2+k_1);\,k_1\geqq 0,\,k_2\geqq -\frac{\nu_2+k_1}{3}\right\}$.

$$3)\quad c_\nu(\alpha)=\begin{cases}(-1)^{k_1+k_2}\left(\frac{\nu_1+1}{3};\,k_1\right)\left(\frac{\nu_2+1}{3}-\frac{\nu_1+1}{9};\,k_2\right) & \text{for } \alpha\in L(\nu),\\ 0 & \text{otherwise.}\end{cases}$$

CASE (E_8) 1) $N=\{(\nu_1,\nu_2);\,\nu_1=0,1,2,3;\,\nu_2=0,1\}$.

2) $L(\nu_1,\nu_2)=\{(\alpha_1,\alpha_2)\in N^2;\,\alpha_1\equiv\nu_1 \bmod 5,\,\alpha_2\equiv\nu_2 \bmod 3\}$
$=\{(\nu_1+5k_1,\,\nu_2+3k_2);\,k_1,k_2\geqq 0\}$.

$$3)\quad c_\nu(\alpha)=\begin{cases}(-1)^{k_1+k_2}\left(\frac{\nu_1+1}{5};\,k_1\right)\left(\frac{\nu_2+1}{3};\,k_2\right) & \text{if } \alpha\in L(\nu),\\ 0 & \text{otherwise.}\end{cases}$$

Remark that, in the cases other than D_l with even l, $\langle\rho,\tilde{\nu}\rangle=\langle\rho,\nu\rangle$ implies $\tilde{\nu}=\nu$ for $\nu,\tilde{\nu}\in N$. Hence, for each index $\nu\in N$, $L(\nu)$ coincides with the set of indices α such that $\langle\sigma,\alpha\rangle\equiv\langle\sigma,\nu\rangle \bmod \boldsymbol{Z}$. As in Lemma 2.12, introduce the linear mapping $l(\alpha)$ as follow. Set $N^*=N\backslash\{0\}$. For any index $\alpha=(\alpha_\nu)_{\nu\in N^*}$, define

$$(9)\quad\begin{cases}(A_l) & l(\alpha):=\sum_{\nu=0}^{l-1}\nu\alpha_\nu\in N\\ (D_l,E_l) & l(\alpha):=(l_1(\alpha),l_2(\alpha))\in N^2 \quad\text{where } l_i(\alpha)=\sum_{\nu\in N^*}\nu_i\alpha_\nu\,,\end{cases}$$

according to the type of f. Then, Proposition 2.12 implies

PROPOSITION 3.9. *On the assumptions above, consider the Gauss-Manin system associated with the integral* (8). *In each case, define an* *operator*

$$P_\nu(t, D_{t_0}) = \sum_{\langle\sigma,\alpha\rangle - r = \sigma_\nu - 1} c_\nu(l(\alpha)) \frac{t^\alpha}{\alpha!} D_{t_0}^r \begin{pmatrix} r \in \boldsymbol{Z} \\ \alpha \in \boldsymbol{N}^{\mu-1} \end{pmatrix}$$

for each $\nu \in N$. *Then* $P_\nu(t, D_{t_0})$ *are operators in* $\boldsymbol{C}[t][[D_{t_0}]]$. *Moreover, the series*

$$\phi_\nu(t_0, t) = P_\nu(t, D_{t_0})\delta_{t_0}^{(\lambda-\varepsilon_\nu)} \qquad (\nu \in N)$$

give μ *independent solutions of the Gauss-Manin system in question.*

In defining P_ν, we wrote $\sigma_\nu - 1$ in place of $-\langle\rho, \nu\rangle$. Since $0 < \sigma_\nu \leqq 1$ for $\nu \in N$, one has $\langle\sigma, \alpha\rangle + 1 - \sigma_\nu \geqq 0$ for $\alpha \in \boldsymbol{N}^{\mu-1}$, $\nu \in N$. This shows that $P_\nu \in \boldsymbol{C}[[t]][[D_{t_0}]]$. Since $\sigma_\nu > 0$, one has

(10) $$\#\{\alpha \in \boldsymbol{N}^{\mu-1}; \langle\sigma, \alpha\rangle = r + \sigma_\nu - 1\} < +\infty \qquad \text{for} \quad \nu \in N,\ r \in \boldsymbol{N},$$

which implies that $P_\nu \in \boldsymbol{C}[t][[D_{t_0}]]$.

Next we define the matrix $P = (P_{\tilde{\nu}\nu})_{\tilde{\nu},\nu \in N}$ of operators by

(11) $$P_{\tilde{\nu}\nu}(t, D_{t_0}) = (\partial_t D_{t_0}^{-1})^{\tilde{\nu}} P_\nu \qquad \text{for} \quad \tilde{\nu}, \nu \in N.$$

Then, by the same argument as we proved Theorem 3.5, we have

THEOREM 3.10. *Let* f *be a canonical form of simple singularity listed in* (1) *and let* F *be the versal deformation* $\sum_{\nu \in N} t_\nu x^\nu + f$. *With the functions* $c_\nu(\nu \in N)$ *and* $l(\alpha)$ *defined before Proposition* 3.9, *we define*

$$P_{\tilde{\nu}\nu}(t, D_{t_0}) = \sum_{\langle\sigma,\alpha\rangle - r = \sigma_\nu - \sigma_{\tilde{\nu}}} c_\nu(l(\alpha) + \tilde{\nu}) \frac{t^\alpha}{\alpha!} D_{t_0}^r \quad \begin{pmatrix} r \in \boldsymbol{N} \\ \alpha \in \boldsymbol{N}^{\mu-1} \end{pmatrix}$$

for $\tilde{\nu}, \nu \in N$ *and set* $P = (P_{\tilde{\nu}\nu})_{\tilde{\nu},\nu \in N}$. *Then the matrix*

$$\Phi(t, D_{t_0}) = P(t, D_{t_0})\delta_{t_0}^{(\Lambda)}$$

give the fundamental system of solutions of the Gauss-Manin system (5) *for* $\vec{u} = {}^t(u_\nu)_{\nu \in N}$.

In defining $P_{\tilde{\nu}\nu}$, we used $\sigma_\nu - \sigma_{\tilde{\nu}}$ in place of $\langle\rho, \tilde{\nu} - \nu\rangle$. Note that $\langle\sigma, \alpha\rangle - \sigma_\nu + \sigma_{\tilde{\nu}} > -1$ for $\tilde{\nu}, \nu \in N$, $\alpha \in \boldsymbol{N}^{\mu-1}$, which follows from the inequalities $0 < \sigma_\nu \leqq 1$ for $\nu \in N$. Hence one sees $P_{\tilde{\nu}\nu} = (\partial_t D_{t_0}^{-1})^{\tilde{\nu}} P_\nu$.

In the case where $\lambda \notin \boldsymbol{Z}$ and $\lambda - \varepsilon_\nu \notin \boldsymbol{Z}$ $(\nu \in N)$, one can also take $\Phi(t, D_{t_0}) = P(t, D_{t_0}) t_0^{-\Lambda - I}$ for the fundamental system of solutions of the Gauss-Manin system (5), which gives an explicit expansion, near the point $(t_0, t) = (\infty, 0)$, of the many-valued holomorphic solutions on $S \backslash D$.

By virtue of these explicit computations, we can obtain an interesting result concerning the *flat coordinate systems*. The notion of flat coordinate system is introduced by K. Saito- T. Yano- J. Sekiguchi [20] as a canonical generator system of the ring of invariants of a finite reflection group. In the sequel, our attention will be restricted to the effect of flat coordinate systems on the Gauss-Manin systems.

Consider the finite presentation (5) of Proposition 3.8. Then, by Lemma 2.1, we know that there is a unique matrix $P_0 \in \mathrm{M}(\mu: \mathcal{O}(T))$ such that

$$P_0|_{t=0} = I \quad \text{and} \quad \partial_{t_\nu} P_0 = B_1^{(\nu)} P_0 \qquad (\nu \in N\backslash\{0\}) . \tag{12}$$

Noting that $D_{t_\nu} D_{t_0}^{-1} u = u_\nu$ ($\nu \in N$), one can show that the matrices $B_1^{(\nu)}$ ($\nu \in N$) satisfy the condition

$$B_{1,\nu_2\nu}^{(\nu_1)} = B_{1,\nu_1\nu}^{(\nu_2)} \qquad \text{for} \quad \nu_1, \nu_2, \nu \in N , \tag{13}$$

where we define $B_1^{(0)}$ to be 0. By (13) combined with (12), one can prove

$$\partial_{t_{\nu_1}} P_{0,\nu_2\nu} = \partial_{t_{\nu_2}} P_{0,\nu_1\nu} \qquad \text{for} \quad \nu_1, \nu_2, \nu \in N . \tag{14}$$

This shows that one can find a unique coordinate system $(s_\nu)_{\nu \in N}$ defined in a neighborhood of the origin of S such that

$$s_\nu(0) = 0 \; ; \qquad \frac{\partial s_\nu}{\partial t_{\tilde{\nu}}} = P_{0,\tilde{\nu}\nu} \qquad (\tilde{\nu}, \nu \in N) , \tag{15}$$

which we call the *flat coordinate system* for H_F^ν. (From (13), one can show that $D_{t_0} = D_{s_0}$.) Let us define a new basis $(v_\nu)_{\nu \in N}$ by

$$v_\nu = D_{s_\nu} D_{s_0}^{-1} u = \int \partial_{s_\nu}(F) \delta_F^{(\nu)} dx \qquad (\nu \in N) . \tag{16}$$

Then the column vector $\vec{v} = {}^t(v_\nu)_{\nu \in N}$ is connected with $\vec{u} = {}^t(u_\nu)_{\nu \in N}$ by the relation

$$\vec{u} = P_0 \vec{v} . \tag{17}$$

Thus, if one make use of this basis $(v_\nu)_{\nu \in N}$, the finite presentation corresponding to (5) reduces to the form

$$\begin{cases} s_0 \vec{v} = (\widetilde{A}(s) - \Lambda D_{s_0}^{-1}) \vec{v} \\ D_{s_\nu} D_{s_0}^{-1} \vec{v} = \widetilde{B}^{(\nu)}(s) \vec{v} \qquad (\nu \in N), \; s = (s_\nu)_{\nu \in N^*} , \end{cases} \tag{18}$$

as Lemma 2.1 shows. (Strictly speaking, the above arguments should be delivered in the holomorphic category. For the details, see K. Saito

[17], S. Ishiura- M. Noumi [9].)

On the assumption of Theorem 3.10, we have already determined the matrix $P(t, D_{t_0})$ for our Gauss-Manin system H_F^λ. Moreover, the matrix P_0 above is nothing but the coefficient P_0 in the expansion

$$P(t, D_{t_0}) = \sum_{r=0}^{\infty} P_r(t) D_{t_0}^r , \tag{19}$$

as remarked in 2.1. In terms of the functions $c_\nu(\nu \in N)$ and $l(\alpha)$, we can write down the entries of $P_0 = (P_{0,\tilde{\nu}\nu})_{\tilde{\nu},\nu\in N}$ as

$$P_{0,\tilde{\nu}\nu} = \sum_{\langle\sigma,\alpha\rangle=\sigma_\nu-\sigma_{\tilde{\nu}}} c_\nu(l(\alpha)+\tilde{\nu})\frac{t^\alpha}{\alpha!} \qquad (\nu, \tilde{\nu} \in N) , \tag{20}$$

which are weighted homogeneous polynomials in $t=(t_\nu)_{\nu\in N\setminus\{0\}}$.

THEOREM 3.11. *On the assumption of Theorem* 3.10, *the flat coordinate system* $(s_\nu)_{\nu\in N}$ *for* H_F^λ *is determined by the formulas*

$$s_0 = t_0 + \sum_{\langle\sigma,\alpha\rangle=1} c_0(l(\alpha))\frac{t^\alpha}{\alpha!}$$

and

$$s_\nu = \sum_{\langle\sigma,\alpha\rangle=\sigma_\nu} c_\nu(l(\alpha))\frac{t^\alpha}{\alpha!} \qquad \textit{for} \quad \nu \in N\setminus\{0\} .$$

PROOF. Since $P_{0,\tilde{\nu}\nu}$ is weighted homogeneous of degree $\sigma_\nu - \sigma_{\tilde{\nu}}$ for $\tilde{\nu}, \nu \in N$, the function s_ν determined by the equations (15) must be weighted homogeneous of degree σ_ν. Hence, one has

$$\sigma_\nu s_\nu = t_0\frac{\partial s_\nu}{\partial t_0} + \sum_{\tilde{\nu}\in N} \sigma_{\tilde{\nu}} t_{\tilde{\nu}} \frac{\partial s_\nu}{\partial t_{\tilde{\nu}}} = t_0\delta_{\nu 0} + \sum_{\tilde{\nu}\in N} \sigma_{\tilde{\nu}} t_{\tilde{\nu}} P_{0,\tilde{\nu}\nu} .$$

Here the second term can be computed as

$$\begin{aligned}\sum_{\tilde{\nu}\in N} \sigma_{\tilde{\nu}} t_{\tilde{\nu}} P_{0,\tilde{\nu}} &= \sum_{\tilde{\nu}\in N} \sum_{\langle\sigma,\alpha\rangle=\sigma_\nu-\sigma_{\tilde{\nu}}} \sigma_{\tilde{\nu}} c_\nu(l(\alpha)+\tilde{\nu}) t_{\tilde{\nu}} \frac{t^\alpha}{\alpha!} \\ &= \sum_{\tilde{\nu}\in N} \sum_{\langle\sigma,\alpha\rangle=\sigma_\nu-\sigma_{\tilde{\nu}}} \sigma_{\tilde{\nu}}(\alpha_{\tilde{\nu}}+1) c_\nu(l(\alpha)+\tilde{\nu}) \frac{t^{\alpha+1_{\tilde{\nu}}}}{(\alpha+1_{\tilde{\nu}})!} \cdots (*) .\end{aligned}$$

Noting that $c_\nu(l(\beta-1_{\tilde{\nu}})+\tilde{\nu}) = c_\nu(l(\beta))$ and $\langle\sigma, \beta-1_{\tilde{\nu}}\rangle = \langle\sigma, \beta\rangle - \sigma_{\tilde{\nu}}$, one has

$$(*) = \sum_{\tilde{\nu}\in N} \sum_{\langle\sigma,\beta\rangle=\sigma_\nu} \sigma_{\tilde{\nu}}\beta_{\tilde{\nu}} c_\nu(l(\beta)) \frac{t^\beta}{\beta!} = \sigma_\nu \sum_{\langle\sigma,\beta\rangle=\sigma_\nu} c_\nu(l(\beta)) \frac{t^\beta}{\beta!}$$

Hence we have

$$s_\nu = t_0\delta_{\nu,0} + \sum_{\langle\sigma,\alpha\rangle=\sigma_\nu} c_\nu(l(\alpha))\frac{t^\alpha}{\alpha!} \qquad \text{Q.E.D.}$$

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Present Address:
DEPARTMENT OF MATHEMATICS
SOPHIA UNIVERSITY
KIOI-CHO, CHIYODA-KU, TOKYO 102