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On Regular Fréchet-Lie Groups VII

The Group Generated by Pseudo-Differential Operators of Negative Order

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Introduction

Let $\mathscr{P}^m$ be the space of the pseudo-differential operators of order m defined on a compact manifold N without boundary. In this paper, we are mainly concerned with the case that m is a *non-positive integer*. We shall denote by $G\mathscr{P}^m$ the group of all invertible elements in $1+\mathscr{P}^m$. The purpose of this paper is to show the following:

THEOREM. *If m is a non-positive integer, then $G\mathscr{P}^m$ is a regular Fréchet-Lie group under a certain topology* (cf. §1). *Moreover, $G\mathscr{P}^m$ is $*$-closed and the $*$-operation is continuous with respect to the above topology, where $*$-operation is a mapping which assigns to an operator its adjoint one.*

In [7], the authors defined the concept of regular Fréchet-Lie groups and gave two fundamental theorems. Roughly speaking, a regular Fréchet-Lie group is a Lie group modeled on a Fréchet space, on which product integrals can be well-defined. As was seen in [7], [8], this concept is not only an amenable object among infinite dimensional groups, but also has a lot of concrete examples. Every strong ILB-Lie group defined in [3] or [4] is a regular Fréchet-Lie group (cf. [7]. §6). Moreover, we added in [9] another example which related to both general relativity and $G\mathscr{F}^0$, the group of all invertible Fourier-integral operators of order 0 defined on N. Now, by the above theorem, we have had one more concrete example of regular Fréchet-Lie groups. Moreover, the above result will play an important role in the next paper for the proof that $G\mathscr{F}^0$ is a regular Fréchet-Lie group, which will be done by the parallel

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manners as in [9].

In §1 we shall repeat several notations and several remarks. Also in §1 we shall give the precise statement of the above theorem, for the theorem is in fact proved for a slightly general class $G\mathscr{P}^m_{(m')}$, where m' is sufficiently small integer, or $-\infty$. The above theorem is obtained as a special case that $m'=-\infty$, i.e. $G\mathscr{P}^m=G\mathscr{P}^m_{(-\infty)}$.

Remark that $\mathscr{P}^0$ is an associative algebra and $\mathscr{P}^m$ $(m\leqq 0)$ is an ideal of $\mathscr{P}^0$. In §4, we shall show that $\mathscr{P}^0$ is a Fréchet-algebra under certain topology. This fact is also proved for a slightly general class $\mathscr{P}^0_{(m')}$, where m' is a sufficiently small integer or $-\infty$, and $\mathscr{P}^0=\mathscr{P}^0_{(-\infty)}$. The Fréchet algebra structure for $\mathscr{P}^0$ plays an essential role in the proof of our theorem. Thus, in §§2-3, we shall develop a general theory of Fréchet-algebras and a reduction of our theorem. By this, what we have really to do is reduced to establish several estimates of the norm of operators. This will be done in §4 by using similar computational methods as in [2].

§1. Notations, remarks and the precise statement of the theorem.

Let N be an n-dimensional closed C^∞ riemannian manifold. As usual, $C^k(N)$ means the space of all C-valued C^k functions on N. We denote also by $C^\infty_\Delta(N\times N)$, the space of all C-valued C^∞ functions on $N\times N$ $-$(diagonal set). For an element $K\in C^\infty_\Delta(N\times N)$, we denote by $K\circ$ the integral operator defined by

$$(1)\qquad (K\circ f)(x)=\int_N K(x, y)f(y)dy\ , \qquad f\in C^\infty(N)\ ,$$

where $dy=\sqrt{2\pi}^{\,-n}\times$volume element on N.

As a matter of course, (1) does not make sense in general. If K is contained in the dual space of $C^\infty(N\times N)$, then $K\circ$ is an operator of $C^\infty(N)$ into its dual space $C^{-\infty}(N)$. If $K(x, \cdot)\in C^{-\infty}(N)$ for every x, and $x\rightsquigarrow K(x, \cdot)$ is a C^∞ mapping of N into $C^{-\infty}(N)$, then $K\circ$ is an operator $C^\infty(N)$ into itself.

Let ρ be the distance function and r_0 the injectivity radius of N. If $\rho(x, y)<r_0$, then there exists uniquely $X\in T_x$ (the tangent space of N at x) such that $|X|=\rho(x, y)$ and $y=\cdot_x X$, where $\cdot_x X$ is an abbreviated notation of $\mathrm{Exp}_x X$ (cf. [5] p. 359). We denote by $\Delta(r_0/12)$ the points $(x, y)\in N\times N$ such that $\rho(x, y)<r_0/12$ and fix the cut off function $\nu_0\in C^\infty(N\times N)$ of the breadth $r_0/4$ (cf. [5] p. 358). If $\mathrm{supp}\, K$, support of K, is contained in $\Delta(r_0/12)$, $K\circ f$ can be written in the form

$$K\circ f(x)=\int_{T_x} K(x, \cdot_x X)\frac{dy}{dX}(x, \cdot_x X)(\nu_0 f)^{\cdot}(x; X)dX$$

where $(\nu_0 f)^{\cdot}(x; X)=\nu_0(x, \cdot_x X)f(x, \cdot_x X)$, (dy/dX) is the Jacobian between $y=\cdot_x X$ and X and $dX=\sqrt{2\pi}^{-n}\times$the volume element on T_x.

DEFINITION 1.1. The operator $K\circ$ in (1) is called a pseudo-differential operator of order m, if K can be written as follows:

$$K=K_1+K_2 , \tag{2}$$

where supp $K_1\subset\Delta(r_0/12)$, $K_2\in C^\infty(N\times N)$, and that $K_1(x, \cdot_x X)(dy/dX)(x, \cdot_x X)$ is a Fourier transform (as a distribution) of a function $A(x;\xi)\in\sum_C^m$, that is

$$K_1\frac{dy}{dX}(x, \cdot_x X)=\int_{T_x^*} A(x;\xi)e^{-i\langle\xi|X\rangle}d\xi , \tag{3}$$

where the definition $\sum_C^m$ is given by [5], p. 365, $\sum_C^m$ $(m\leqq 0)$ is a Fréchet space under the C^∞ topology by identifying this as a function space on $\bar{D}_N^*$ through the diffeomorphism $\tau: D_N^*\to T_N^*$ (cf. [5] (10)).

Let $\mathscr{P}^m$ be the linear space of all pseudo-differential operators of order m. In what follows, we shall define a series of topologies $T_{m'}$ on $\mathscr{P}^m$ so that $(\mathscr{P}^m, T_{m'})$ may be a topological algebra.

Consider in general a C^∞ riemannian manifold M. Let T_x be the tangent space at x, and let $(X^1, \cdots, X^n)$ be an orthonormal coordinate system on T_x. Then using the abbreviated notation, $\cdot_x(X^1, \cdots, X^n)$ can be regarded as a normal coordinate system at x (cf. [5] p. 359). For a C^∞ function f on M, we define

$$\partial_x^\alpha f=\left(\frac{\partial}{\partial X}\right)^\alpha\bigg|_{X=0} f(\cdot_x(X^1, \cdots, X^n)) , \tag{4}$$

where $\alpha=(\alpha_1, \cdots, \alpha_n)$. Obviously, this derivative depends on normal coordinate systems. However, the symmetric tensor $\sum_{|\alpha|=l}\partial_x^\alpha f(dX)^\alpha$ is independent of the choice of normal coordinates, where $(dX)^\alpha$ means the symmetric tensor product $(dX)^{\alpha_1}\cdots(dX)^{\alpha_n}$. We shall denote this by $\partial_x^l f$.

For $K\in C^\infty(N\times N)$, we define a norm $|K|_k$ by

$$|K|_k=\max_{i+j\leqq k}\max_{x,y\in N}|\partial_x^i\partial_y^j K(x, y)| . \tag{5}$$

Similarly, one can define a norm $|\ \ |_k$ for C^∞ functions on S^*N, the unit cosphere bundle on N (cf. [5], [6]).

For the cotangent bundle T^*N, we use a normal local trivialization at x, denoted by $\cdot_x(X^1, \cdots, X^n, \xi_1, \cdots, \xi_n)$ (cf. [5] p. 359). If A is in $\sum_C^m$, then, by definition of $\sum_C^m$, A can be written in the form

$$A(x;\xi)=(a_m(x;\hat{\xi})r^m+a_{m-1}(x;\hat{\xi})r^{m-1}+\cdots+a_{m'+1}(x;\hat{\xi})r^{m'+1})\phi(r)+A_{m'}(x;\xi) \tag{6}$$

for every $m'\leqq m$, where $a_j\in C^\infty(S^*N)$, $A_{m'}\in\sum_C^{m'}$, $r=|\xi|$ and ϕ is a fixed C^∞ function on $\boldsymbol{R}$, $\phi(r)\equiv 0$, for $r\leqq r_0/3$, $\phi(r)\equiv 1$ for $r\geqq r_0/2$.

For every $m'\leqq m$, we define a norm $|A|_{m',k}$ of $A\in\sum_C^m$ by

$$\begin{cases}|A|_{m',k}=\sum_{s=1}^{m-m'}|a_{m'+s}|_k+|A_{m'}|_{m',k}\\ |A_{m'}|_{m',k}=\max_{i+j\leqq k}\sup_{(x;\xi)\in T_N^*}(1+|\xi|)^{-m'+j}|\partial_x^i\partial_\xi^j A_{m'}|\,.\end{cases} \tag{7}$$

$C^\infty(S^*N)$ and $C^\infty(N\times N)$ are Fréchet spaces under the system of norms $\{|\ \ |_k;k\geqq 0\}$. However $\sum_C^m$ is not complete under the system of norms $\{|\ \ |_{m',k};k\geqq 0\}$ if m' is fixed. We shall denote this topology by $T_{m'}$, and denote by $\sum_{C,(m')}^m$ the completion of $(\sum_C^m, T_{m'})$. Obviously, $T_{m'-1}$ is stronger than $T_{m'}$, for

$$|A|_{m',k}\leqq|A|_{m'-1,k}\,,\qquad A\in\sum_C^m\,. \tag{8}$$

Hence $\{T_{m'};m'\leqq m\}$ forms an inverse system of topologies. The inverse limit topology of this system will be denoted by $T_{-\infty}$. Now, it is not hard to see that $\sum_C^m=\bigcap_{m'}\sum_{C,(m')}^m$ and $(\sum_C^m, T_{-\infty})$ is a Fréchet space. Moreover $T_{-\infty}$ is the C^∞ topology on $\bar{D}_N^*$ by identifying $\sum_C^m$ to the function space of $\bar{D}_N^*$ through the diffeomorphism $\tau\colon D_N^*\to T_N^*$.

Now, let $K\circ\in\mathscr{P}^m$. By definition (3), the inverted Fourier transform $A(x;\xi)$ of $K_1(dy/dX)$ is contained in $\sum_C^m$. We denote by $\mathscr{P}_{(m')}^m$ the totality of $K\circ$ such that the inverted Fourier transform $A(x;\xi)$ of $K_1(dy/dX)$ (cf. (2)) is contained in $\sum_{C,(m')}^m$. Obviously, $\mathscr{P}^m\subset\mathscr{P}_{(m')}^m$ for every $m'\leqq m$.

We define a norm $|K_1|_{m',k}$ by

$$|K_1|_{m',k}=|A|_{m',k}\,,\qquad m'\leqq m\,. \tag{9}$$

For every $m'\leqq m$, we define a norm $\|K\circ\|_{m',k}$ by

$$\|K\circ\|_{m',k}=\inf_{K=K_1+K_2}\{|K_1|_{m',k}+|K_2|_k\} \tag{10}$$

where the infimum is taken over all partitions of K in (2) and (3). $\mathscr{P}_{(m')}^m$, and hence $\mathscr{P}^m$ are topological vector spaces under the system of norms $\{\|\ \ \|_{m',k};k\geqq 0\}$.

LEMMA 1.1. *$\mathscr{P}_{(m')}^m$ is a Fréchet space for every $m'\leqq m$. Moreover, $\mathscr{P}^m=\bigcap_{m'}\mathscr{P}_{(m')}^m$ and hence $\mathscr{P}^m$ is a Fréchet space under the inverse limit* *topology.*

PROOF. For the first statement, we have only to prove the completeness of $\mathscr{P}^m_{(m')}$. Let $\{K^l\circ\}$ be a Cauchy sequence in $\mathscr{P}^m_{(m')}$. It is enough to see that there exist subsequence $\{K^{l_n}\circ\}$ and $K\circ\in\mathscr{P}^m_{(m')}$ such that $K^{l_n}\circ$ converges to K in $\mathscr{P}^m_{(m')}$. Set $\varepsilon_n=2^{-(n+1)}$, $\delta_n=2^{-(n+1)}$ and $k_n=2^n$. Since $\{K^l\circ\}$ is a Cauchy sequence in $\mathscr{P}^m_{(m')}$, there exists a subsequence $\{K^{l_n}\circ\}$ such that $l_n<l_{n+1}$, $n=0, 1, 2 \cdots$, and for any $p\geqq n$

$$\|K^{l_p}\circ-K^{l_n}\circ\|_{m',k_n}<\varepsilon_n \,. \tag{11}$$

By definition of $\|\ \ \|_{m',k}$, there exist $\{L_1^{l_{n+1},l_n}\}$, $\{L_2^{l_{n+1},l_n}\}$ such that $K^{l_{n+1}}-K^{l_n}=L_1^{l_{n+1},l_n}+L_2^{l_{n+1},l_n}$, where $\operatorname{supp} L_1^{l_{n+1},l_n}\subset\Delta(r_0/12)$, $L_2^{l_{n+1},l_n}\in C^\infty(N\times N)$, and

$$\|K^{l_{n+1}}\circ-K^{l_n}\circ\|_{m',k_n}+\delta_n>|L_1^{l_{n+1},l_n}|_{m',k_n}+|L_2^{l_{n+1},l_n}|_{k_n} \,.$$

So we get

$$|L_1^{l_{n+1},l_n}|_{m',k_n}<1/2^n \,, \qquad |L_2^{l_{n+1},l_n}|_{k_n}<1/2^n \,.$$

Hence it is easy to see that

$$\sum_{n=0}^{\infty} L_1^{l_{n+1},l_n} \quad \text{and} \quad \sum_{n=0}^{\infty} L_2^{l_{n+1},l_n}$$

converge in $\sum^m_{C,(m')}$ and C^∞ topology respectively. Put

$$K_1=\sum_{n=0}^{\infty} L_1^{l_{n+1},l_n} \,, \qquad K_2=\sum_{n=0}^{\infty} L_2^{l_{n+1},l_n} \quad \text{and} \quad K=K_1+K_2 \,,$$

then since $\operatorname{supp} K_1\subset\Delta(r_0/12)$, $K_2\in C^\infty(N\times N)$ and $K\circ\in\mathscr{P}^m_{(m')}$, we get that $K^{l_n}\circ$ converges to $K\circ$ in $\mathscr{P}^m_{(m')}$.

For the second statement, remark at first $\mathscr{P}^m\subset\bigcap_{m'}\mathscr{P}^m_{(m')}$, hence we have only to show the converse. Let $K\circ\in\bigcap_{m'}\mathscr{P}^m_{(m')}$. Then, for every m' there is a partition $K=K_1^{(m')}+K_2^{(m')}$ such that $K_2^{(m')}\in C^\infty(N\times N)$ and

$$K_1^{(m')}\frac{dy}{dX}(x, \cdot_x X)=\int_{T_x^*} A^{(m')}(x;\xi)e^{-i\langle\xi|X\rangle}d\xi \,, \qquad A^{(m')}\in \textstyle\sum^m_{C,(m')} \,.$$

By the system $\{A^{(m')}, m'\leqq m\}$, we can determine an asymptotic series $a_m+a_{m-1}+\cdots$, and there is an element $A\in\sum^m_C$ such that $A\sim a_m+a_{m-1}+\cdots$, and the support of the Fourier transform of A is contained in $\Delta(r_0/12)$. Define $K_1(x, y)$ by (3). Then, it is easy to see that $(K-K_1)\circ$ is an operator of order $-\infty$ and hence $K-K_1\in C^\infty(N\times N)$. It follows $K\circ\in\mathscr{P}^m$. $\square$

We shall identify $\mathscr{P}^m$ with $\cap_{m'}\mathscr{P}^m_{(m')}$ and denote it sometimes by $\mathscr{P}^m_{(-\infty)}$. In §4 we shall prove the following:

PROPOSITION 1.2. (a) *For every m, m' such that $0\geqq m\geqq m'>-\infty$, $\mathscr{P}^m_{(m')}$ is an associative Fréchet algebra.*

(b) *For every i, j such that $m\geqq i$, $j\geqq m'$, $\mathscr{P}^i_{(m')}\mathscr{P}^j_{(m')}\subset\mathscr{P}^{\max\{i+j,m'\}}_{(m')}$ and each $\mathscr{P}^i_{(m')}$ is a closed subspace of $\mathscr{P}^m_{(m')}$.*

(c) *Suppose $m'\leqq -n-1$. There is a positive constant C_k such that if $K\circ$, $L\circ\in\mathscr{P}^{m'}_{(m')}$, then*

$$\|K\circ L\circ\|_{m',k}\leqq C_k\|K\circ\|_{m',k}\|L\circ\|_{m',\delta(k)}\ ,$$

where $\delta(k)$ is a function of k such that $\delta(k)\geqq k$.

(d) *$*$-operation is continuous with respect to $\mathscr{P}^m_{(m')}$, where $*$-operation is a mapping which assigns to an operator its adjoint one.*

We shall define on $1+\mathscr{P}^m_{(m')}$ the same topology as $\mathscr{P}^m_{(m')}$ through the natural identification, and for $G\mathscr{P}^m_{(m')}$, we shall use the relative topology in $1+\mathscr{P}^m_{(m')}$.

Although most of the results stated in the above proposition are rather well-known for experts, our main theorem can be obtained by a fairy general method in Fréchet algebras. So, we shall show at first in the next two sections how Proposition 1.2 yields our main theorem.

Remark finally that

$$\mathscr{P}^m_{(m')}/\mathscr{P}^{m'}_{(m')}\cong \underbrace{C^\infty(S^*N)\oplus\cdots\oplus C^\infty(S^*N)}_{(m-m')}\ ,$$

and that

$$G\mathscr{P}^m_{(m')}/G\mathscr{P}^{m'}_{(m')}\cong G(\mathscr{P}^m_{(m')}/\mathscr{P}^{m'}_{(m')})$$

the group of all invertible elements in $1+\mathscr{P}^m_{(m')}/\mathscr{P}^{m'}_{(m')}$ (cf. Lemma 2.2 in §2). Especially, for every $m'\leqq -1$, $G\mathscr{P}^0_{(m')}/G\mathscr{P}^{-1}_{(m')}$ is naturally isomorphic to the multiplicative group of the nonvanishing C^∞ functions on S^*N. Hence, this is a regular Fréchet-Lie group (cf. Proposition 6.6 in [8]). Moreover, the result in [6] §4 shows that for every $m\leqq 0$, $G\mathscr{P}^m_{(m')}$ is an open subset of $1+\mathscr{P}^m_{(m')}$ and if K is sufficiently close to 0, then $(1-K)^{-1}$ is given by $\sum_{i=0}^\infty K^i$.

§2. Filtered algebras.

Let $\mathfrak{a}^0$ be an (associative) Fréchet algebra with unit 1 and with a series of closed ideals $\mathfrak{a}^0\supset\mathfrak{a}^{-1}\supset\mathfrak{a}^{-2}\supset\cdots\supset\mathfrak{a}^{m'}$ $(0\geqq m'>-\infty)$ satisfying the following:

(i) $\mathfrak{a}^i \cdot \mathfrak{a}^j \subset \mathfrak{a}^{\max\{i+j, m'\}}$

(ii) There are Fréchet subspaces $\boldsymbol{F}_0, \boldsymbol{F}_{-1}, \cdots, \boldsymbol{F}_{m'+1}$ such that $\mathfrak{a}^m = \boldsymbol{F}_m \oplus \boldsymbol{F}_{m-1} \oplus \cdots \oplus \boldsymbol{F}_{m'+1} \oplus \mathfrak{a}^{m'}$ for every m, $0 \geqq m \geqq m'$.

For every m with $0 \geqq m \geqq m'$, we denote by $G\mathfrak{a}^m$ the group of all invertible elements in $1+\mathfrak{a}^m$. For such an m, $\mathfrak{a}^0/\mathfrak{a}^m$ is also a Fréchet algebra with unit. Thus, $G(\mathfrak{a}^0/\mathfrak{a}^m)$ is the group of all invertible elements in $\mathfrak{a}^0/\mathfrak{a}^m$. The goal of this section is to show the following:

THEOREM 2.1. *If $G\mathfrak{a}^{m'}$ is an open subset of $1+\mathfrak{a}^{m'}$ and a regular Fréchet-Lie group under the relative topology, then so is $G\mathfrak{a}^m$ for each m, $m' \leqq m \leqq -1$. If moreover $G(\mathfrak{a}^0/\mathfrak{a}^{-1})$ is an open connected subset of $\mathfrak{a}^0/\mathfrak{a}^{-1}$ and a regular Fréchet-Lie group under the relative topology, then so is $G\mathfrak{a}^0$.*

The above theorem will be proved by induction in several lemmas bellow. If $m=-1$, nothing is to be proved for the first statement. Thus, we assume $m<-1$, and assume that $G\mathfrak{a}^{m'}$ is an open subset of $1+\mathfrak{a}^{m'}$ and a regular Fréchet-Lie group under the relative topology. We have only to show that $G\mathfrak{a}^{m'+1}$ is an open subset of $1+\mathfrak{a}^{m'+1}$ and a regular Fréchet-Lie group whenever $m'+1 \leqq -1$.

Recall that $\mathfrak{a}^{m'+1} = \boldsymbol{F}_{m'+1} \oplus \mathfrak{a}^{m'}$. First of all, we have the following:

LEMMA 2.2. *$G\mathfrak{a}^{m'+1}$ is an open subset of $1+\mathfrak{a}^{m'+1}$, and $G\mathfrak{a}^{m'+1}/G\mathfrak{a}^{m'}$ is isomorphic to the additive group $\boldsymbol{F}_{m'+1}$, hence it is isomorphic to $G(\mathfrak{a}^{m'+1}/\mathfrak{a}^{m'})$.*

PROOF. Let $a, b \in \mathfrak{a}^{m'+1}$ be sufficiently close to 0. Then $ab \in \mathfrak{a}^{m'}$ for $m'<-1$, and $1-ab$ is invertible by virture of the assumption that $G\mathfrak{a}^{m'}$ is an open subset of $1+\mathfrak{a}^{m'}$. Since $(1+a)(1-a)=1-a^2$ is invertible, so is $1+a$, hence $G\mathfrak{a}^{m'+1}$ is an open subset of $1+\mathfrak{a}^{m'+1}$. Recall also that

$$(1+a)(1+b) = 1+a+b+ab = (1+a+b)(1+(1+a+b)^{-1}ab) .$$

Since $1+(1+a+b)^{-1}ab \in G\mathfrak{a}^{m'}$, we see that

$$(1+a)(1+b) \equiv 1+a+b \pmod{G\mathfrak{a}^{m'}} .$$

Therefore $G\mathfrak{a}^{m'+1}/G\mathfrak{a}^{m'}$ is locally isomorphic to $\boldsymbol{F}_{m'+1}$, but this implies $G\mathfrak{a}^{m'+1}/G\mathfrak{a}^{m'} \cong \boldsymbol{F}_{m'+1}$. Obviously, $\boldsymbol{F}_{m'+1} \cong G(\mathfrak{a}^{m'+1}/\mathfrak{a}^{m'})$. □

By the above result, we have an exact sequence

$$\{1\} \longrightarrow G\mathfrak{a}^{m'} \longrightarrow G\mathfrak{a}^{m'+1} \longrightarrow \boldsymbol{F}_{m'+1} \longrightarrow \{0\} .$$

Note that $\boldsymbol{F}_{m'+1}$ is a regular Fréchet-Lie group (cf. [7] §3, Lemma 3.9). Hence in what follows we shall apply Theorem 5.4 in [8] §5.

Define a mapping $\gamma: \boldsymbol{F}_{m'+1} \to 1+\mathfrak{a}^{m'+1}$ by $\gamma(a)=1+a$. If a is sufficiently close to 0, then $\gamma(a) \in G\mathfrak{a}^{m'+1}$. If $b \in \boldsymbol{F}_{m'+1}$ is sufficiently close to 0, then $\gamma(a+b)$ is invertible and hence

$$r_\gamma(a, b)=\gamma(a+b)^{-1}\gamma(a)\gamma(b)=1+\gamma(a+b)^{-1}ab \in G\mathfrak{a}^{m'} .$$

For the above $a \in \boldsymbol{F}_{m'+1}$ and $1+c \in G\mathfrak{a}^{m'}$, we set $\alpha_\gamma(a, 1+c)=\gamma(a)^{-1}(1+c)\gamma(a)$.

LEMMA 2.3. *r_γ and α_γ are C^∞ mappings.*

PROOF. Note that $\gamma(a+b)^{-1}=(1-(a+b))(1-(a+b)^2)^{-1}$. Since $G\mathfrak{a}^{m'}$ is an FL-group, and the multiplication in $\mathfrak{a}^{m'+1}$ is continuously bilinear, we see that $\gamma(a+b)^{-1}$ is C^∞ with respect to a, b. Hence r_γ is C^∞. Note again that $\gamma(a)^{-1}=(1-a)(1-a^2)^{-1}$. Hence, by the same reason as above, α_γ is C^∞. □

PROOF OF THEOREM 2.1. Assume γ and α_γ are defined for $a, b \in V$, where V is a neighborhood of 0 of $\boldsymbol{F}_{m'+1}$. The conditions (Ext. 2), (Ext. 3) in the definition in [8] are obtained by Lemma 2.3. Moreover, it is easily seen that $\pi^{-1}(V) \cong V \times N$. By Theorem 5.4 of [8], we have only to show that $G\mathfrak{a}^{m'+1}$ is generated by $\pi^{-1}(V)$, where π is the natural projection of $G\mathfrak{a}^{m'+1}$ onto $\boldsymbol{F}_{m'+1}$. However, $\pi^{-1}(V) \supset G\mathfrak{a}^{m'}$, and $\pi^{-1}(V) \supset 1+V$. Thus, $\pi^{-1}(V)$ generates $G\mathfrak{a}^{m'+1}$. Thus, we get the first statement of Theorem 2.1.

Now, it is not hard to see that $G\mathfrak{a}^0/G\mathfrak{a}^{-1}$ is naturally isomorphic to an open subgroup $\boldsymbol{F}_{0*}=G(\mathfrak{a}^0/\mathfrak{a}^{-1})$. Note that $\boldsymbol{F}_{0*}$ is naturally imbedded in $\boldsymbol{F}_0$, hence in $\mathfrak{a}^0$ by using the splitting $\mathfrak{a}^0=\boldsymbol{F}_0 \oplus \mathfrak{a}^{-1}$. For every $\tilde{a} \in \mathfrak{a}^0/\mathfrak{a}^{-1}$, we denote by $\gamma(\tilde{a})$ the element of $\boldsymbol{F}_0$ such that $\gamma(\tilde{a})+\mathfrak{a}^{-1}=\tilde{a}$. Obviously, γ is a continuous linear mapping and hence C^∞. If $\tilde{a}$ is sufficiently close to 1, then $\gamma(\tilde{a}) \in \boldsymbol{F}_{0*}$. Remark that the multiplication in $\mathfrak{a}^0$ is a continuous bilinear mapping and hence C^∞. Hence setting

$$\gamma(\tilde{a})\gamma(\tilde{b})=\gamma(\tilde{a}\tilde{b})+\gamma_{-1}(\tilde{a}, \tilde{b}) , \qquad \gamma_{-1}(\tilde{a}, \tilde{b}) \in \mathfrak{a}^{-1} ,$$

we see that γ_{-1} is C^∞. Remark that $\gamma(\tilde{a})^{-1}=\gamma(\tilde{a}^{-1})(1+\gamma_{-1}(\tilde{a}, \tilde{a}^{-1}))^{-1}$. Thus, it is not hard to see that r_γ and α_γ are C^∞. Hence by the same reasoning as above, $G\mathfrak{a}^0$ is a regular Fréchet-Lie group. It is easy to see that $G\mathfrak{a}^0$ is an open subset of $\mathfrak{a}^0$, and the above topology for $G\mathfrak{a}^0$ is the relative topology. □

§3. Invertible elements in Fréchet algebras.

In this section, we shall consider a Fréchet algebra $\mathfrak{a}$ satisfying the following:

(A) $G\mathfrak{a}$ is an open subset of $1+\mathfrak{a}$.

(B) There is a system of semi-norms $\{|\ \ |_k; k\geqq 0\}$ where k's are integers, satisfying the following:

(i) $|a|_k \leqq |a|_{k+1}$, and $\{|\ \ |_k; k\geqq 0\}$ gives the topology of $\mathfrak{a}$.

(ii) There are constants C_k $(\geqq 1)$ and a function $\delta(k)$ $(\geqq k)$ such that $|ab|_k \leqq C_k |a|_k |b|_{\delta(k)}$.

The goal of this section is as follows:

THEOREM 3.1. *Assumptions being as above, $G\mathfrak{a}$ is a regular Fréchet-Lie group under the relative topology.*

We shall begin with the following:

LEMMA 3.2. *$G\mathfrak{a}$ is an FL-group under the relative topology.*

PROOF. Obviously, multiplication is C^∞. Hence, we have only to show the smoothness of the inversion. If $a\in\mathfrak{a}$ is sufficiently close to 0, then $1-a\in G\mathfrak{a}$ by the assumption. We shall show at first that $(1-a)^{-1}$ is continuous at $a=0$. For an arbitrarily fixed k, take a so small that $C'_k |a|_{\delta(k)} < 1$, where $C'_k = \max\{C_k |1|_k, C_k\}$. Then for any m $(<\infty)$,

$$\begin{aligned}|(1-a)^{-1}-1|_k &\leqq \sum_{i=1}^{m} (C'_k |a|_{\delta(k)})^i + |(1-a)^{-1}a^{m+1}|_k \\ &\leqq \sum_{i=1}^{m} (C'_k |a|_{\delta(k)})^i + |(1-a)^{-1}-1|_k (C'_k |a|_{\delta(k)})^{m+1} + (C'_k |a|_{\delta(k)})^{m+1} .\end{aligned}$$

Therefore,

$$\begin{aligned}|(1-a)^{-1}-1|_k &\leqq \frac{1}{1-(C'_k |a|_{\delta(k)})^{m+1}} \sum_{i=1}^{m+1} (C'_k |a|_{\delta(k)})^i \\ &\leqq \sum_{i=1}^{\infty} (C'_k |a|_{\delta(k)})^i = \frac{C'_k |a|_{\delta(k)}}{1-C'_k |a|_{\delta(k)}} .\end{aligned}$$

Thus, $(1-a)^{-1}$ is continuous at $a=0$, and hence $G\mathfrak{a}$ is a topological group.

Note that $1-(a+b)=(1-a)(1-(1-a)^{-1}b)$, hence if b is sufficiently close to 0, then for every m,

$$(1-(a+b))^{-1} = \sum_{i=0}^{m} ((1-a)^{-1}b)^i (1-a)^{-1} + ((1-a)^{-1}b)^{m+1}(1-(a+b))^{-1} .$$

By the continuity of the multiplication, we see that

$$(D^r i)(1-a)(b, \cdots, b) = r!((1-a)^{-1}b)^r (1-a)^{-1} , \qquad \text{where } i(c)=c^{-1} ,$$

defines a continuous mapping of $G\mathfrak{a}\times\mathfrak{a}\cdots\times\mathfrak{a}$ into $\mathfrak{a}$. It is now easy to prove that $i(1-a)=(1-a)^{-1}$ is C^∞ and the r-th derivative of i is given by $D^r i$. □

To prove that $G\mathfrak{a}$ is a regular Fréchet-Lie group, we have to consider product integrals on $G\mathfrak{a}$. So, let $\{(1+h_l(s, t), \Delta_l)\}$ be a sequence of step functions on $[0, \varepsilon]\times J$, $J=[a, b]$, such that $\lim_{l\to\infty}|\Delta_l|=0$ and $\{1+h_l\}$ converges uniformly on $[0, \varepsilon]\times J$ with their partial derivatives $\{\partial h_l/\partial s\}$ to a C^1-hair $1+h(t, s)$ defined on $[0, \varepsilon]\times J$. We set

$$g_l(t)=\prod_a^t (1+h_l, \Delta_l) \qquad (\text{cf. [7] } \S 3)\ . \tag{12}$$

We have only to show that $\lim_{l\to\infty} g_l(t)$ converges uniformly on J to a curve $g(t)$ in $G\mathfrak{a}$.

LEMMA 3.2. *Let* $M_{l,k}=\max_{s,t}|\partial h_l/\partial s|_k$, *and set* $N_{l,k}=C'_k M_{l,s(k)}$, *where* $C'_k=\max\{C_k|1|_k, C_k\}$. *Then,*

$$|g_l(t)-1|_k\leqq(1+(t-t_j)N_{l,k})(1+(t_j-t_{j-1})N_{l,k})\cdots(1+(t_1-t_0)N_{l,k})-1\ ,$$

where $\Delta_l=\{t_0, t_1, \cdots, t_m\}$ *and* j *is the number such that* $t\in[t_j, t_{j+1})$. *Thus, if we set* $N_k=\overline{\lim}_{l\to\infty} N_{l,k}$ *then*

$$\overline{\lim_{l\to\infty}}\,|g_l(t)-1|_k\leqq e^{(t-a)N_k}-1\ .$$

PROOF. Note that

$$g_l(t)=(1+h_l(t-t_j, t_j))(1+h_l(t_j-t_{j-1}, t_{j-1}))\cdots(1+h_l(t_1-t_0, t_0))\ .$$

Hence by using (B) (ii), we see

$$|g_l(t)-1|_k\leqq(1+C'_k|h_l(t-t_j, t_j)|_{s(k)})\cdots(1+C'_k|h_l(t_1-t_0, t_0)|_{s(k)})-1\ .$$

Remark that $|h_l(t_i-t_{i-1}, t_{i-1})|_{s(k)}\leqq\int_0^{t_i-t_{i-1}} M_{l,s(k)}dt$, hence the first inequality follows immediately. Since $|\Delta_l|=\max|t_{i+1}-t_i|$ tends to 0 as $l\to\infty$, we see easily the second inequality. □

LEMMA 3.3. *Notations and assumptions being as above, let* $\tilde{\Delta}_l$ *be a subdivision of* Δ_l. *Then* $(1+h_l, \tilde{\Delta}_l)$*'s are step functions on* $[0, \varepsilon]\times J$ *and* $g_l(t)-\prod_a^t(1+h_l, \tilde{\Delta}_l)$ *converges to 0 uniformly on* J *as* $\lim_{l\to\infty}|\Delta_l|=0$.

PROOF. Let $\Delta_l=\{t_0, t_1, \cdots, t_s\}$, $t_0=a$, $t_s=b$, and let j be the integer such that $t\in[t_j, t_{j+1})$. We set $1+w_l(j)=\prod_{t_j}^t(1+h_l, \tilde{\Delta}_l)$, and $1+w_l(i)=$

$\prod_{t_i}^{t_{i+1}}(1+h_l, \tilde{\Delta}_l)$ for $0 \leqq i \leqq j-1$. Then obviously

$$\prod_a^t (1+h_l, \tilde{\Delta}_l) = \prod_{i=0}^j (1+w_l(i)) .$$

Hence, using the telescope equality $a_s a_{s-1} \cdots a_2 a_1 - b_s b_{s-1} \cdots b_2 b_1 = \sum_{i=1}^s a_s \cdots a_{i+1}(a_i - b_i) b_{i-1} \cdots b_1$ and by the above lemma, we see that

$$\left| g_l(t) - \prod_{i=0}^j (1+w_l(i)) \right|_k \tag{13}$$

$$\leqq \text{Const.} \sum_{i=0}^j e^{Dk(t-t_{i+1})} |h_l(t_{i+1}-t_i, t_i) - w_l(i)|_{s(k)} \times e^{Dk(t_{i-1}-a)}$$

$$\leqq \text{Const.}\, e^{Dk(t-a)} \sum_{i=0}^j |h_l(t_{i+1}-t_i, t_i) - w_l(i)|_{s(k)} .$$

Let $t_i = \alpha_0 < \alpha_1 < \cdots < \alpha_\tau = t_{i+1}$ be the dividing points of $\tilde{\Delta}_l$ contained in $[t_i, t_{i+1}]$. Then

$$1+w_l(i) = \left(1+\int_0^{\alpha_\tau - \alpha_{\tau-1}} \frac{\partial h_l}{\partial s}(s, t_i) ds\right) \cdots \left(1+\int_0^{\alpha_1-\alpha_0} \frac{\partial h_l}{\partial s}(s, t_i) ds\right) .$$

Hence remarking $h_l(t_{i+1}-t_i, t_i) = \int_0^{t_{i+1}-t_i} (\partial h_l/\partial s)(s, t_i) ds$, we have

$$|h_l(t_{i+1}-t_i, t_i) - w_l(i)|_{s(k)} \leqq \int_0^{t_{i+1}-t_i} \left| \frac{\partial h_l}{\partial s}(s, t_i) - \frac{\partial h_l}{\partial s}(0, t_i) \right|_{s(k)} ds$$

$$+ \sum_{i=0}^{\tau-1} \int_0^{\alpha_{i+1}-\alpha_i} \left| \frac{\partial h_l}{\partial s}(s, t_i) - \frac{\partial h_l}{\partial s}(0, t_i) \right|_{s(k)} ds + O(|\tilde{\Delta}_l|^2) .$$

Note that $(\partial h_l/\partial s)(s, t)$ is continuous in s, and $\{(\partial h_l/\partial s)(s, t)\}$ converges uniformly to $(\partial h/\partial s)(s, t)$. Hence for any $\varepsilon > 0$, there is l_0 such that if $l \geqq l_0$, then $|(\partial h_l/\partial s)(s, t_i) - (\partial h_l/\partial s)(0, t_i)|_{s(k)} < \varepsilon$ for every s, $0 \leqq s \leqq |\Delta_l|$. Thus,

$$|h_l(t_{i+1}-t_i, t_i) - w_l(i)|_{s(k)} \leqq 2(t_{i+1}-t_i)\varepsilon + O(|\Delta_l|^2) .$$

Hence,

$$\left| g_l(t) - \prod_a^t (1+h_l, \tilde{\Delta}_l) \right|_k \leqq 2(t-a)\varepsilon' + O(|\Delta_l|) \longrightarrow 0 . \qquad \square$$

Proof of Theorem 3.1. First of all, we shall show that $\{g_l(t)\}$ is a Cauchy sequence in the uniform topology on J. Thus, consider $|g_{l'}(t) - g_l(t)|_k$. Assume $l' \geqq l$, and let $\tilde{\Delta}_l$ be a common subdivision of Δ_l and $\Delta_{l'}$. By the above lemma we have only to show

$$\lim_{l\to\infty} \left| \prod_a^t (1+h_l, \tilde{\Delta}_l) - \prod_a^t (1+h_{l'}, \tilde{\Delta}_l) \right|_k = 0 ,$$

uniformly on J. Let $\tilde{\Delta}_l=\{t_0, t_1, \cdots, t_s\}$. Then by the same manner as in (13), we have the following by setting $t\in[t_j, t_{j+1})$:

$$\left|\prod_a^t (1+h_l, \tilde{\Delta}_l)-\prod_a^t (1+h_{l'}, \tilde{\Delta}_l)\right|_k$$
$$\leqq \text{Const. } e^{D_k(t-a)} \sum_{i=0}^{j} |h_l(t_{i+1}-t_i, t_i)-h_{l'}(t_{i+1}-t_i, t_i)|_{s(k)} \,.$$

Now, remarking

$$(h_l-h_{l'})(t_{i+1}-t_i, t_i)=\int_0^{t_{i+1}-t_i} \frac{\partial(h_l-h_{l'})}{\partial s}(s, t_i)ds$$

and that $\{(\partial h_l/\partial s)(s, t)\}$ converges uniformly to $(\partial h/\partial s)(s, t)$, we can obtain easily the desired result.

Thus, we see that $\lim_{l\to\infty} g_l(t)$ converges uniformly in $1+\mathfrak{a}$. Note that $g_l(t)\in G\mathfrak{a}$ for every l. Hence $g_l(t)^{-1}g_{l'}(t)$ is sufficiently close to 1, if l, l' are sufficiently large. Since $G\mathfrak{a}$ is an open subset of $1+\mathfrak{a}$, we can conclude that $\lim_{l'\to\infty} g_l(t)^{-1}g_{l'}(t)\in G\mathfrak{a}$, if l is sufficiently large, and hence $\lim_{l\to\infty} g_l(t)\in G\mathfrak{a}$. Thus, by the above result and Lemma 3.2 we obtain that $G\mathfrak{a}$ is a regular Fréchet-Lie group. □

Proposition 1.2 ⟹ *Theorem*

In the remainder of this section, we shall show how Proposition 1.2 yields our main theorem.

By the above general results, we see easily that $G\mathscr{P}^{m}_{(m')}$ is a regular Fréchet-Lie group for every m, m' such that $m'\leqq m\leqq 0$, $-\infty<m'\leqq -n-1$. Remark $G\mathscr{P}^0/G\mathscr{P}^m=G(\mathscr{P}^0/\mathscr{P}^m)=G(\mathscr{P}^0_{(m)}/\mathscr{P}^m_{(m)})$ $m<0$. Hence for the proof that $G\mathscr{P}^0$ is a regular Fréchet-Lie group, we have only to show that $G\mathscr{P}^{-n-1}=\bigcap_{m'} G\mathscr{P}^{-n-1}_{(m')}$ is a regular Fréchet-Lie group. By Proposition 4.1 in [6], we see that $G\mathscr{P}^{-n-1}$ is an open subset of $1+\mathscr{P}^{-n-1}$. Hence $G\mathscr{P}^{-n-1}$ is an FL-group, because $G\mathscr{P}^{-n-1}$ is the inverse limit of $\{G\mathscr{P}^{-n-1}_{(m')}, m'\leqq -n-1\}$.

To prove the convergence of the product integrals, we remark at first that such ones converge in $G\mathscr{P}^{m}_{(m')}$ for every $m'\leqq -n-1$. Hence by definition of the inverse limit topology, we have the convergence of the product integrals in $G\mathscr{P}^m$. □

§4. Proof of Proposition 1.2.

In this section, we shall give the proof of Proposition 1.2, and hence of our main theorem because of the general results in §§2~3.

Before proving Proposition 1.2, we consider several lemmas for estimates which are used to get (c) of Proposition 1.2.

For $A(x;\xi)\in\sum_{C,(m)}^{m}$, we denote by $P(A,\nu)$ the operator in $\mathscr{P}_{(m)}^{m}$ defined by

$$(14)\qquad (P(A,\nu)f)(x)=\int_{T_x^*}\int_{T_x} A(x;\xi)e^{-i\langle\xi|X\rangle}(\nu f)^{\cdot}(x;X)dXd\xi\ ,$$

where ν is a cut off function of the breadth ε less than $3r_0/8$, where r_0 is the injectivity radius of N.

LEMMA 4.1. *Let $h(x,y)$ be a C^∞ function such that $h\equiv 0$ for $\rho(x,y)\leqq\delta_1$ and for $\rho(x,y)\geqq\delta_2$ $(\delta_1<\delta_2<r_0)$. Let A be an element of $\sum_{C,(m)}^{m}$. Set*

$$M(x,y)=h(x,y)\int_{T_x^*} A(x;\xi)e^{-i\langle\xi|X\rangle}d\xi\ ,\qquad y=\cdot_x X\ .$$

Then, if $m\leqq -n-1$, there is a constant $C_{m,k}(>0)$ such that $|M|_k\leqq C_{m,k}|A|_{m,k}$ for $A\in\sum_{C,(m)}^{m}$. Moreover, for a general m, $|M|_k\leqq C_{m,k}|A|_{m,k+m+n+1}$.

PROOF. For sufficiently small $Z, W\in T_x$, we have

$$M(\cdot_x Z,\cdot_y W)=h(\cdot_x Z,\cdot_y W)\int_{T_{\cdot_x Z}^*} A(\cdot_x Z;\xi)e^{-i\langle\xi|Y\rangle}d\xi\ ,$$

where $Y=Y(x;X,Z,W)$ is a smooth function such that $\cdot_{\cdot_x X}W=\cdot_{\cdot_x Z}Y$ and $Y(x;X,0,0)=X$. By using normal coordinate expression at x, that is, $\tilde{\xi}=(d\,\mathrm{Exp}_x)_Z^*\xi$, $\tilde{Y}=(d\,\mathrm{Exp}_x)_Z^{-1}Y$ and $\cdot_x(Z,\tilde{\xi})=(\cdot_x Z;(d\,\mathrm{Exp}_x)_Z^{*-1}\tilde{\xi})$, where $(d\,\mathrm{Exp}_x)_Z\colon T_x\to T_x$ is the differential of the exponential mapping Exp_x, the above integration can be rewritten as

$$\int_{T_x^*} A(\cdot_x(Z,\tilde{\xi}))e^{-i\langle\tilde{\xi}|\tilde{Y}\rangle}\frac{d\xi}{d\tilde{\xi}}d\tilde{\xi}\ .$$

Then, we get

$$\begin{aligned}\left(\frac{\partial}{\partial Z}\right)^{\alpha}\left(\frac{\partial}{\partial W}\right)^{\beta}M(\cdot_x Z,\cdot_y W)&=\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \beta_1+\beta_2=\beta}}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\beta!}{\beta_1!\beta_2!}\\ &\times\left(\frac{\partial}{\partial Z}\right)^{\alpha_1}\left(\frac{\partial}{\partial W}\right)^{\beta_1}h(\cdot_x Z,\cdot_y W)\int_{T_x^*}\left(\frac{\partial}{\partial Z}\right)^{\alpha_2}\left\{A(\cdot_x(Z,\tilde{\xi}))\frac{d\xi}{d\tilde{\xi}}(x;Z)\right\}\\ &\times\left(\frac{\partial}{\partial Z}\right)^{\alpha_3}\left(\frac{\partial}{\partial W}\right)^{\beta_2}e^{-i\langle\tilde{\xi}|\tilde{Y}\rangle}d\tilde{\xi}\ ,\end{aligned}$$

where $\alpha,\beta,\alpha_1,\cdots$ are multi-indices. Remark that

$$\left(\frac{\partial}{\partial Z}\right)^{\alpha}\left(\frac{\partial}{\partial W}\right)^{\beta}e^{-i\langle\tilde{\xi}|\tilde{Y}\rangle}\Big|_{\substack{Z=0\\ W=0}}=P_{(\alpha,\beta)}(x;X,\tilde{\xi})e^{-i\langle\tilde{\xi}|X\rangle}\ ,$$

where $P_{(\alpha,\beta)}(x; X, \tilde{\xi})$ is a polynomial in $\tilde{\xi}$ of degree $|\alpha+\beta|$ whose coefficients are C^∞-functions in X. Thus, we have

$$\begin{aligned}\partial_x^\alpha\partial_y^\beta M(x, y) &= \left(\frac{\partial}{\partial Z}\right)^\alpha\left(\frac{\partial}{\partial W}\right)^\beta M(\cdot_x Z, \cdot_y W)|_{Z,W=0} \\ &= \sum_{\substack{\alpha_1,\alpha_2,\beta_1 \\ |\alpha_1+\alpha_2|\leqq|\alpha| \\ |\beta_1|\leqq|\beta|}} F_{\alpha_1,\alpha_2,\beta_1}(x, y)\int_{T_x^*}\partial_x^{\alpha_1}A(x, \xi)P_{(\alpha_2,\beta_1)}(x; X, \xi)e^{-i\langle\xi|X\rangle}d\xi \\ &= \sum_{\alpha_1,\alpha_2,\beta_1} F_{\alpha_1,\alpha_2,\beta_1}(x, y)X^{-\gamma}\int_{T_x^*}\left(\frac{1}{i}\partial_\xi\right)^\gamma\{\partial_x^{\alpha_1}A(x; \xi)P_{(\alpha_2,\beta_1)}(x; X, \xi)\}e^{-i\langle\xi|X\rangle}d\xi ,\end{aligned}$$

where $F_{\alpha_1,\alpha_2,\beta_1}(x, y)$ is a smooth function vanishing on $\rho(x, y)\leqq\delta_1$, $\rho(x, y)\geqq\delta_2$, and the integrand of each term is of degree $m+|\alpha_2+\beta_1|-|\gamma|$ with respect to ξ. Take $|\gamma|\geqq m+|\alpha_2+\beta_1|+n+1$ or $|\gamma|\geqq|\alpha_2+\beta_1|$ in case $m\leqq-n-1$, then

$$\begin{aligned}|\partial_x^\alpha\partial_y^\beta M(x, y)| &\leqq \sum \text{Const. } |A|_{m,|\alpha_1|+|\gamma|}\int_{T_x^*}(1+|\xi|)^{m-|\gamma|+|\alpha_2|+|\beta_1|}d\xi \\ &\leqq \begin{cases}\text{Const. } |A|_{m,k+m+n+1} \cdot \\ \text{Const. } |A|_{m,k} \quad (\text{in case } m\leqq-n-1) .\end{cases}\end{aligned}$$ □

For $A\in\sum_{C,(m)}^m(m\leqq 0)$ and $M\in C^\infty(N\times N)$, $P(A, \nu)\circ M\circ$ is a smoothing operator whose kernel is given by

$$K(x, z)=\iint A(x; \xi)e^{-i\langle\xi|X\rangle}\nu(x, \cdot_x X)M(\cdot_x X, z)dXd\xi .$$

LEMMA 4.2. *Notations being as above, there is a constant* $C_{m,k}$ *such that* $|K|_k\leqq C_{m,k}|A|_{m,k}|M|_{k+\max\{0,m+n+1\}}$.

PROOF. For sufficiently small $W, Z\in T_x$, by using normal coordinate expressions, we have

$$\begin{aligned}K(\cdot_x W, \cdot_z Z) &= \int_{T^*_{\cdot_x W}}\int_{T_{\cdot_x W}} A(\cdot_x W; \xi)e^{-i\langle\xi|X\rangle}\nu(\cdot_x W, \cdot_{\cdot_x W}X)M(\cdot_{\cdot_x W}X, \cdot_z Z)dXd\xi \\ &= \int_{T_x^*}\int_{T_x} A(\cdot_x(W, \tilde{\xi}))e^{-i\langle\tilde{\xi}|\tilde{S}(x;Y,W)\rangle}\nu(\cdot_x W, \cdot_x Y)M(\cdot_x Y, \cdot_z Z) \\ &\quad\times J(x; Y, W)dYd\tilde{\xi} ,\end{aligned}$$

where $\cdot_{\cdot_x W}S(x; Y, W)=\cdot_x Y$ (cf. [5], (3)) and $J(x; Y, W)=(dX/dY)(d\xi/d\tilde{\xi})(x; Y, W)$. We see easily that

$$\partial_x^\alpha\partial_z^\beta K(x, z)=\left(\frac{\partial}{\partial W}\right)^\alpha\left(\frac{\partial}{\partial Z}\right)^\beta K(\cdot_x W, \cdot_z Z)\Big|_{\substack{W=0\\Z=0}}$$

$$=\sum_{|\alpha_1+\alpha_2|\leq|\alpha|}\int_{T_x^*}\int_{T_x} F_{\alpha_1,\alpha_2}(x;\,Y)\partial_x^{\alpha_1}A(x;\,\xi)Q_{(\alpha_2)}(x;Y,\,\xi)e^{-i\langle\xi|Y\rangle}\partial_z^{\beta}M(\cdot_x Y,\,z)dYd\xi\;,$$

where F_{α_1,α_2} is a smooth function which vanishes on $|Y|\geqq 2\varepsilon/3$ and $Q_{(\alpha_2)}$ is a polynomial in ξ of degree $|\alpha_2|$ such that

$$\left(\frac{\partial}{\partial W}\right)^{\alpha_2}e^{-i\langle\tilde{\xi}|\tilde{S}\rangle}|_{W=0}=Q_{(\alpha_2)}(x;\,Y,\,\xi)e^{-i\langle\xi|Y\rangle}\;.$$

Thus, we have

$$\begin{aligned}\partial_x^{\alpha}\partial_z^{\beta}&K(x,\,z)\\ &=\sum_{|\alpha_1+\alpha_2|\leq|\alpha|}\iint \partial_x^{\alpha_1}A(x,\,\xi)Q_{(\alpha_2)}(x;\,Y,\,\xi)\|\xi\|^{-l}\|\xi\|^{l}e^{-i\langle\xi|Y\rangle}F_{\alpha_1,\alpha_2}(x;\,Y)\\ &\qquad\qquad\times\partial_z^{\beta}M(\cdot_x Y,\,z)dYd\xi\\ &=\sum_{|\alpha_1+\alpha_2|\leq|\alpha|}\iint \partial_x^{\alpha_1}A(x,\,\xi)\|\xi\|^{-l}e^{-i\langle\xi|Y\rangle}P_l(x;\,\xi,\,\partial_Y)\\ &\qquad\qquad\times\{Q_{(\alpha_2)}(x;\,Y,\,\xi)F_{\alpha_1,\alpha_2}(x;\,Y)\partial_z^{\beta}M(\cdot_x Y,\,z)\}dYd\xi\;,\end{aligned}$$

where $\|\xi\|=1+|\xi_1|+\cdots+|\xi_n|$, P_l is a differential operator of order l with respect to Y variable whose coefficients are uniformly bounded. Remark that the integrand is of degree $m+|\alpha_2|-l$ with respect to ξ. If we take $l\geqq m+|\alpha_2|+n+1$ in each term, we get desired result. □

For $M\in C^\infty(N\times N)$ and $A\in\sum_{C,(m)}^{m}$, we consider $M\circ P(A,\,\nu)$. We denote by ${}^{y\cdot}z$, the tangent vector $Y\in T_y$ such that $\cdot_y Y=z$ and $\rho(y,\,z)=|Y|$ (cf. [5] p. 359). Using these notations, the smooth kernel of $M\circ P(A,\,\nu)$ is given by

$$K(x,\,z)=\int_N\int_{T_y^*}M(x,\,y)A(y;\,\xi)e^{-i\langle\xi|{}^{y\cdot}z\rangle}\nu(y,\,z)\frac{dY}{dz}d\xi dy\;. \tag{15}$$

Since $\nu(y,\,z)=0$ whenever $\rho(y,\,z)\geqq 2\varepsilon/3$, one may assume $\rho(y,\,z)<2\varepsilon/3$, hence there exists uniquely $Z\in T_z$ such that $\cdot_z Z=y$, $\rho(y,\,z)=|Z|$. If $\cdot_y Y=z$, then Y and Z are related by $Y=-(d\,\mathrm{Exp}_z)_Z Z$. We shall denote $\eta=(d\,\mathrm{Exp}_z)_Z^*\xi$, and $\cdot_z(Z,\,\eta)=(\cdot_z Z;\,(d\,\mathrm{Exp}_z)_Z^{*-1}\eta)$ (cf. [5] p. 359). Thus, (15) can be rewritten as

$$K(x,\,z)=\int_{T_z^*}\int_{T_z}M(x,\,\cdot_z Z)A(\cdot_z(Z,\,\eta))e^{i\langle\eta|Z\rangle}\left(\nu\frac{dY}{dz}\,\frac{dy}{dZ}\,\frac{d\xi}{d\eta}\right)(z;\,Z)dZd\eta\;. \tag{16}$$

Note that $\nu(dY/dz)(dy/dZ)(d\xi/d\eta)$ is a C^∞ function in $(z;\,Z)$ which is identically zero, if $|Z|>2\varepsilon/3$.

LEMMA 4.3. *Notations being as above, if* $m\leqq -n-1$, *then*

$|K|_k \leqq C_{m,k}|M|_k|A|_{m,k}$. *Moreover, for a general m, we have* $|K|_k \leqq C_{m,k}|M|_{k+m+n+1}|A|_{m,k+m+n+1}$.

PROOF. We see easily that

$$\begin{aligned} K(\cdot_x Y, \cdot_z W) &= \int_{T^*_{\cdot_z W}} \int_{T_{\cdot_z W}} M(\cdot_x Y, \cdot_{\cdot_z W} Z) A(\cdot_{\cdot_z W}(Z, \eta)) e^{i\langle \eta | Z\rangle} \\ &\qquad \times \left(\nu \frac{dY}{dz} \frac{dy}{dZ} \frac{d\xi}{d\eta}\right)(\cdot_z W; Z) dZ d\eta \\ &= \int_{T^*_z} \int_{T_z} M(\cdot_x Y, \cdot_z X) A(\cdot_z(X, T(x; X, W)\tilde{\eta})) e^{i\langle \tilde{\eta} | \tilde{S}(z; X, W)\rangle} \\ &\qquad \times \nu(\cdot_z W, \cdot_z X) J(z; X, W) dX d\tilde{\eta}\ , \end{aligned}$$

where $\cdot_z X = \cdot_{\cdot_z W} Z$, $Z = S(z; X, W)$ and $T(z; X, W)$ is a linear mapping such that $T(x; X, W) = (d\,\mathrm{Exp}_z)^*_X (d\,\mathrm{Exp}_{\cdot_z W})^{*-1}_S (d\,\mathrm{Exp}_z)^{*-1}_W$, $T(z, X, 0) = I$ and $J = (dY/dz)(dy/dZ)(d\xi/d\eta)(dZ/dX)(d\eta/d\tilde{\eta})$. Differentiating K by Y and Z, we have

$$\begin{aligned} \partial_x^\alpha \partial_z^\beta K(x, z) &= \sum_{|\beta_1+\beta_2| \leq |\beta|} \iint N_{\beta_1,\beta_2}(z; X) \partial_x^\alpha M(x, \cdot_z X) \\ &\qquad \times \partial_\eta^{\beta_1} A(\cdot_z(X, \eta)) R_{(\beta_1)}(z; X, \eta) Q_{(\beta_2)}(z; X, \eta) e^{i\langle \eta | X\rangle} dX d\eta\ , \end{aligned}$$

where N_{β_1,β_2} is a smooth function vanishing on $|X| \geqq 2\varepsilon/3$ and $R_{(\beta_1)}(z; X, \eta)$ is a polynomial in η of degree $|\beta_1|$ such that

$$\left(\frac{\partial}{\partial W}\right)^\gamma A(\cdot_z(Z, T(z; X, W)\tilde{\eta}))|_{W=0} = \sum_{|\beta| \leq |\gamma|} \partial_\eta^\beta A(\cdot_z(Z, \eta)) R_{(\beta)}(z; X, \eta)\ .$$

Thus, by the same manner as in the proof of Lemma 4.2, we have

$$\begin{aligned} \partial_x^\alpha \partial_z^\beta K(x, z) &= \sum_{|\beta_1+\beta_2| \leq |\beta|} \iint P_l(z; \eta, \partial_X)\{N_{\beta_1,\beta_2}(z; X) \cdot \\ &\qquad \times \partial_x^\alpha M(x, \cdot_z X) \partial_\eta^{\beta_1} A(\cdot_z(X, \eta)) R_{(\beta_1)}(z; X, \eta) Q_{(\beta_2)}(z; X, \eta) \|\eta\|^{-l}\} e^{i\langle \eta | X\rangle} dX d\eta\ . \end{aligned}$$

Take $l \geqq m + |\beta_2| + n + 1$, or $l \geqq |\beta_2|$ in case $m \leqq -n-1$, we get the desired result. □

For $A \in \sum^m_{C,(m)}$, and $B \in \sum^{m'}_{C,(m')}$, we consider an operator given by $P(A, \nu) \circ P(B, \nu)$.

By (85) in [6], $P(A, \nu) \circ P(B, \nu)$ is given as follows:

$$(17) \qquad (P(A, \nu) \circ P(B, \nu) f)(x) = \int_{T^*_x} \int_{T_x} A(x; \xi) B''(x; \xi, Z) e^{-i\langle \xi | Z\rangle} f(\cdot_x Z) dZ d\xi\ ,$$

where

$$B''(x;\,\xi,\,Z)=\iint B'(x,\,X'+Z,\,Z,\,\eta''+\xi)e^{i\langle\eta''|X'\rangle}dX'd\eta''\,,\tag{18}$$
$$B'(x,\,X,\,Z,\,\eta')=B(\cdot_x(X,\,E(x;\,Z,\,X)\eta'))F(x;\,Z,\,X)\,,$$
$$F(x;\,Z,\,X)=\nu(x,\,\cdot_x X)\nu(\cdot_x X,\,\cdot_x Z)J(x;\,Z,\,X)\,,$$

where $E(x; Z, X)$ is a linear transformation of T_x^* which is identity when X and Z are outside some compact set in T_x, and J is a C^∞ function of $(x; Z, X)$ involving various Jacobians. Let ν' be another cut off function of the breadth $4\varepsilon(<3r_0/2)$. Then, $B''(x;\xi, X)\nu'(x, \cdot_x X)\equiv B''(x; \xi, X)$ and by the result of [5] §4, (17) can be rewritten as

$$P(A,\,\nu)\circ P(B,\,\nu)=P(C,\,\nu')\,,\tag{19}$$

where

$$C(x;\,\xi)=\int_{T_x^*}\int_{T_x}A(x;\,\xi+\eta)B''(x;\,\xi+\eta,\,Y)e^{-i\langle\eta|Y\rangle}dYd\eta\,.\tag{20}$$

LEMMA 4.4. *Notations being as above, there is a constant $C_{m',k}$ such that*

$$|B''|_{m',k}\leqq C_{m',k}|B|_{m',k+n+1+\max\{m',|k-m'|\}}\,,$$

where

$$|B''|_{m',k}=\max_{l+s+t\leqq k}\,\max_{(x;\xi,Z)}\,(1+|\xi|)^{-m'+s}|\partial_x^l\partial_\xi^s\partial_Z^t B''|\,.\tag{21}$$

PROOF. For sufficiently small $W\in T_x$, using normal coordinate expression and the same manner as in the proof of Lemma 4.3, we can write

$$B''(\cdot_x(W,\,\xi,\,Z))=\int_{T_x^*}\int_{T_x}B(\cdot_x(H(x;\,Z+X',\,W),\,E(x;\,Z,\,Z+X',\,W)(\xi+\eta'')))$$
$$\times\tilde F(x;\,Z,\,X',\,W)e^{i\langle\eta''|\tilde S(x;X',W)\rangle}dX'd\eta''\,,$$

where H is a smooth function and $H(x; Z+X', 0)=Z+X'$, $E(x; Z, Z+X', W)$ is a linear mapping such that $E(x; Z, Z+X', 0)=(d\,\mathrm{Exp}_x)_H^* S_1^{-1*}(x; Z, Z+X')$ $S(x; X, Z)=S_1(x; X, Z)(X-Z)$ (cf. [5], (4)).

$$\partial_x^\alpha\partial_\xi^\beta\partial_Z^\gamma B''(x;\,\xi,\,Z)=\sum\iint G_{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\gamma_1,\gamma_2}(x;\,Z,\,X')\tag{22}$$
$$\times\partial_x^{\alpha_1+\gamma_1}\partial_\xi^{\alpha_2+\beta_1+\gamma_2}B(\cdot_x(Z+X'),\,S_1^{*-1}(x;\,Z,\,Z+X')(\xi+\eta''))$$
$$\times\partial_\xi^{\beta_2}\tilde R_{(\alpha_2,\gamma_2)}(x;\,Z,\,X',\,(\xi+\eta''))Q_{(\alpha_3)}(x;\,X',\,-\eta'')e^{i\langle\eta''|X'\rangle}dX'd\eta''\,,$$

where summation is taken over $|\alpha_1+\alpha_2+\alpha_3|\leqq|\alpha|$, $|\beta_1+\beta_2|\leqq|\beta|$, $|\gamma_1+\gamma_2|\leqq|\gamma|$, and $\tilde{R}_{(\alpha_2,\gamma_2)}$ is an polynomial in $(\xi+\eta'')$ of degree $|\alpha_2|+|\gamma_2|$. Deviding each term of integrands by $(1+|\eta''|)^l$ and integrating by parts, we get

$$\begin{aligned}&(1+|\xi|)^{-m'+|\beta|}|\partial_x^\alpha\partial_\xi^\beta\partial_Z^\gamma B''(x;\xi,Z)|\\&\quad\leqq \text{Const.}\ |B|_{m',|\alpha|+|\gamma|+|\beta|+l}\int_{T_x^*}\frac{1}{(1+|\eta''|)^l}\Big(\frac{1+|\xi|}{1+|\xi+\eta''|}\Big)^{-m'+|\beta|}d\eta''\\&\quad\leqq \text{Const.}\ |B|_{m',|\alpha|+|\beta|+|\gamma|+l}\int_{T_x^*}(1+|\eta''|)^{-l+|m'-|\beta||}d\eta''\,.\end{aligned}$$

Take $l\geqq n+1+|m'-|\beta||$, we get the desired result. □

Now, keeping Lemma 4.4 in mind, we see the following lemma.

LEMMA 4.5. *Notations and assumptions being as in* (17-20), *there is a constant* C_k *such that* $|C|_{m+m',k}\leqq C_k|A|_{m,k}|B''|_{m',\delta_0(k)}$, *where* $\delta_0(k)=k+n+1+\max\{m+m',|k-m-m'|\}$ *and* $|B''|_{m',k}$ *is defined by* (21).

PROOF. For sufficiently small $W\in T_x$,

$$\begin{aligned}&C(\cdot_x(W,\tilde{\xi}))\\&\quad=\int_{T_x^*}\int_{T_x}A(\cdot_x(W;\tilde{\xi}+\tilde{\eta}))B''(\cdot_x(W,\tilde{\xi}+\tilde{\eta},\tilde{S}(x;Z,W)))\\&\qquad\times e^{-i\langle\tilde{\eta}|\tilde{S}(x;Z,W)\rangle}J(x;Z,W)dZd\tilde{\eta}\,.\end{aligned}$$

Then,

$$\begin{aligned}\partial_x^\alpha\partial_\xi^\beta C(x;\xi)&=\sum\iint F_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2}(x;Z)\partial_x^{\alpha_1}\partial_\xi^{\beta_1}A(x;\xi+\eta)\\&\qquad\times\partial_x^{\alpha_2}\partial_\xi^{\beta_2}\partial_Z^{\alpha_3}B''(x;\xi+\eta,Z)Q_{(\alpha_4)}(x;Z,\eta)e^{-i\langle\eta|Z\rangle}dZd\eta\,.\\&=\sum\iint\partial_x^{\alpha_1}\partial_\xi^{\beta_1}A(x;\xi+\eta)P_l(x;\eta,\partial_Z)\{F_{\alpha_1,\dots,\beta_2}\\&\qquad\times\partial_x^{\alpha_2}\partial_\xi^{\beta_2}\partial_Z^{\alpha_3}B''(x;\xi+\eta,Z)Q_{(\alpha_4)}(x;Z,\eta)\}\|\eta\|^{-l}e^{-i\langle\eta|Z\rangle}dZd\eta\,.\end{aligned}$$

Thus, we have

$$\begin{aligned}&(1+|\xi|)^{-m-m'-|\beta|}|\partial_x^\alpha\partial_\xi^\beta C(x;\xi)|\\&\quad\leqq\text{Const.}\ |A|_{m,|\alpha|+|\beta|}|B''|_{m',|\alpha|+|\beta|+l}\cdot\int_{T_x^*}\frac{1}{(1+|\eta|)^l}\Big(\frac{1+|\xi|}{1+|\xi+\eta|}\Big)^{-m-m'+|\beta|}d\eta\\&\quad\leqq\text{Const.}\ |A|_{m,|\alpha|+|\beta|}|B''|_{m',|\alpha|+|\beta|+l}\cdot\int_{T_x^*}(1+|\eta|)^{-l+|m+m'-|\beta||}d\eta\,.\end{aligned}$$

If we take $l\geqq|m+m'-|\beta||+n+1$, we get the desired result. □

Now, we shall prove (c) of Proposition 1.2. Let $K\circ$, $L\circ\in\mathscr{P}_{(m)}^m$ $(m\leqq$

$-n-1)$. For any $\delta>0$, there exist K_1, K_2, L_1 and L_2 such that $K=K_1+K_2$, $L=L_1+L_2$, $\operatorname{supp} K_1$, $\operatorname{supp} L_1 \subset \Delta(r_0/12)$, $K_2, L_2 \in C^\infty(N\times N)$ and

$$\begin{aligned} \|K\circ\|_{m,k}+\delta &\geqq |K_1|_{m,k}+|K_2|_k\ , \\ \|L\circ\|_{m,k}+\delta &\geqq |L_1|_{m,k}+|L_2|_k\ . \end{aligned} \tag{23}$$

Moreover, there exist $A, B\in\sum^m_{C,(m)}$ and $K_1\circ$, $L_1\circ$ can be written as $(K_1\circ f)(x)=(P(A, \nu_0)f)(x)$, $(L_1\circ f)(x)=(P(B, \nu_0)f)(x)$ where ν_0 is a cut off function of breadth $r_0/4$. Compute $K\circ L\circ$ by using above decomposition and we get

$$\begin{aligned} \|K\circ L\circ\|_{m,k} &= \|(K_1\circ+K_2\circ)(L_1\circ+L_2\circ)\|_{m,k} \\ &\leqq \|K_1\circ L_1\circ\|_{m,k}+\|K_1\circ L_2\circ\|_{m,k}+\|K_2\circ L_1\circ\|_{m,k}+\|K_2\circ L_2\circ\|_{m,k} \\ &\leqq \|K_1\circ L_1\circ\|_{m,k}+|K_1\circ L_2\circ|_k+|K_2\circ L_1\circ|_k+|K_2\circ L_2\circ|_k\ . \end{aligned} \tag{24}$$

The last inequality results from the fact that $K_1\circ L_2\circ=P(A, \nu_0)\circ L_2\circ$ and $K_2\circ L_1\circ=K_2\circ P(B, \nu_0)$ are smoothing operators (cf. Lemmas 4.2, 4.3) and the definition of operator norms (cf. (11)).

By (23) and (24), we have only to prove the following:

(i) $$\|K_1\circ L_1\circ\|_{m,k}\leqq C_{m,k}|K_1|_{m,k}|L_1|_{m,\delta(k)}\ ,$$

(ii) $$|K_1\circ L_2|_k\leqq C_{m,k}|K_1|_{m,k}|L_2|_{\delta(k)}\ ,$$

(iii) $$|K_2\circ L_1|_k\leqq C_{m,k}|K_2|_k|L_1|_{m,\delta(k)}\ ,$$

(iv) $$|K_2\circ L_2|_k\leqq C_k|K_2|_k|L_2|_k\ ,$$

where $\delta(k)=3k+2(n+1)-3m$.

Inequality (iv) is trivial and (ii), (iii) are proved by Lemma 4.2 and Lemma 4.3.

By (19),

$$K_1\circ L_1\circ=P(A, \nu_0)\circ P(B, \nu_0)=P(C, \nu')\ ,$$

so we can write as

$$K_1\circ L_1\circ=P(C, \nu_1)+P(C, (1-\nu_1)\nu')\ ,$$

where ν_1 is a cut off function of breadth less than $r_0/12$. The symbol function of $P(C, \nu_1)$ is given by

$$C*\hat{\nu}_1(x, \xi)=\int_{T^*_x} C(x; \xi+\eta)\hat{\nu}_1(x; \eta)d\eta\ ,$$

where $\hat{\nu}_1$ is a Fourier transform of ν_1 (cf. (3)) and $P(C, (1-\nu_1)\nu')$ is a smoothing operator, denote its kernel function by M, then

$$\|K_1\circ L_1\circ\|_{m,k}\leqq \text{Const. } \{|C*\hat{\nu}_1|_{m,k}+|M|_k\} .$$

It is easy to see $|C*\hat{\nu}_1|_{m,k}\leqq \text{Const. } |C|_{m,k}$. By (8) and Lemma 4.5, $|C|_{m,k}\leqq |C|_{2m,k}\leqq \text{Const. } |A|_{m,k}|B|_{m,s(k)}$. On the other hand, by Lemma 4.1, $|M|_k\leqq \text{Const. } |C|_{m,k}\leqq \text{Const. } |A|_{m,k}|B|_{m,s(k)}$. Thus, we have

$$\|K_1\circ L_1\circ\|_{m,k}\leqq C_{m,k}|A|_{m,k}|B|_{m,s(k)}=C_{m,k}|K_1|_{m,k}|L_1|_{m,s(k)} .$$

This proves (i), hence we get (c) of Proposition 1.2.

Now, we shall compute $*$-operation to prove (d) of Proposition 1.2. For $K\circ\in\mathscr{P}^{m}_{(m')}$, consider a decomposition K_1, K_2 given by (2), (3) such that $K=K_1+K_2$, and

$$K_1(x, \cdot_x X)\frac{dy}{dX}(x; X)=\int_{T_x^*} A(x; \xi)e^{-i\langle\xi|X\rangle}d\xi$$

for some $A\in\sum^{m}_{C,(m')}$. The adjoint operators of K_1, K_2 can be written as

$$(K_1^*\circ u)(y)=(P(C, \nu_2)u)(y) , \tag{25}$$

$$(K_2^*\circ u)(y)=\int_N \bar{K}_2(x; y)u(x)dx ,$$

where

$$C(y; \zeta)=\iint B(y; \zeta+\eta, Y)e^{-i\langle\eta|Y\rangle}dYd\eta ,$$

$$B(y; \zeta, Y)=\bar{A}(\cdot_y(Y, \zeta))\frac{dX}{dy}\frac{d\xi}{d\zeta}\frac{dx}{dY}(y; Y)\nu_0(\cdot_y Y, y) ,$$

and ν_2 is a cut off function of breadth $3r_0/4$.

By the above argements, we see that K^* is an element of $\mathscr{P}^{m}_{(m')}$ and has the following decomposition satisfying the conditions in (2), (3),

$$K^*=L_1+L_2 , \tag{26}$$

where

$$L_1(y, z)=\int_{T_y^*}\int_{T_y^*} C(y, \xi+\zeta)\hat{\nu}_1(y; \zeta)e^{-i\langle\xi|Z\rangle}d\zeta d\xi\frac{dZ}{dz}(y, z) ,$$

$$L_2(y, z)=\int_{T_y^*} C(y; \zeta)e^{-i\langle\zeta|Z\rangle}((1-\nu_1)\nu_2)(y, z)d\zeta\frac{dZ}{dz}(y, z)$$
$$+K_2^*(y, z) , \quad z=\cdot_y Z .$$

Compute $|L_1|_{m',k}(m'\leqq m)$, $|L_2|_k$ in (25), (26), then one gets,

(27) $$|L_1|_{m',k}\leqq C_{m',k}|A|_{m',\delta_1(m',k)}\ ,$$
$$|L_2|_k\leqq C_{m',k}|A|_{m',\delta_2(m',k)}+|K_2|_k\ ,$$

where $\delta_1(m', k)=k+3(m-m')+n+1+\max\{m', |k+m-2m'|\}$,
$$\delta_2(m', k)=2k+3(n+1)-m'\ .$$

Thus, by (27) and the definition of operator norms, we obtain
$$\|K^*\circ\|_{m',k}\leqq C_{m',k}\|K\circ\|_{m',\max\{\delta_1(m',k),\delta_2(m',k)\}}\ .$$

This proves (d) of Proposition 1.2.

Now, it remains to show that $(\mathscr{P}^m_{(m')}, T_{m'})$ is a Fréchet algebra for every m', $m'\leqq m\leqq 0$. Since $\mathscr{P}^i_{(m')}\cdot\mathscr{P}^j_{(m')}\subset\mathscr{P}^{\max\{i+j,m'\}}_{(m')}$ $(m'\leqq i, j\leqq m\leqq 0)$ is well known, and $\mathscr{P}^i_{(m')}$ is a closed subspace of $\mathscr{P}^m_{(m')}$ by Lemma 1.1, we have only to consider the case $m=0$. Namely, what we have to prove is that $\mathscr{P}^0_{(m')}$ is a Fréchet algebra for every $m'\leqq 0$.

Suppose $K\circ$, $L\circ\in\mathscr{P}^0$. By the same manner as in the proof (i)-(iv) and the same calculation in the proof of Lemmas 4.1-4.5 for the estimates of kernel function of $K\circ L\circ$, we have
$$\|K\circ L\circ\|_{m',k}\leqq C_{m',k}\|K\circ\|_{m',\delta_3(m',k)}\|L\circ\|_{m',\delta_4(m',k)}$$
where
$$\delta_3(m', k)=\max\{k-m', k+n+1\}-m'\ ,$$
$$\delta_4(m', k)=2\max\{k-m', k+n+1\}-7m'+3(n+1)\ .$$

Therefore, we can conclude $\mathscr{P}^0_{(m')}$ is a Fréchet algebra for every $m'\leqq 0$. This completes the proof of Proposition 1.2.

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