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Classification of Finitely Determined Singularities of Formal Vector Fields on the Plane

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Introduction

The author gave a characterization of finite determinacy of formal vector fields in [6, 7]. Originally the problem of finite determinacy was proposed by R. Thom [8] for C^∞-map germs and J. Mather gave a complete answer to it [10]. Specially, for C^∞-functions this concept is very important in connection with the elementary catastroph theory [16]. Roughly speaking, for C^∞-functions k-determinacy has upper semi-continuity on k and the local structure of the orbit decomposition of function space by the action of the local diffeomorphisms is reduced to that of finite jet space.

On the other hand, for vector fields the situation is quite different. Upper semi-continuity of k-determinacy is lost and the local orbit decomposition is not reduced to that of finite jet space. Moreover, even local triviality of orbits does not hold. Thus we can not hope to construct an unfolding theory for vector fields except for some exceptional cases. However, in this paper we see that in 2-dimensional case the classification and the hierarchy can be simply described for finitely determined singularities of formal vector fields.

§1. Definitions and the results.

Let C be the field of complex numbers. Let $\mathscr{F}=C[[x, y]]$ be the formal power series algebra. We denote by $\mathfrak{X}^0$ the set of formal vector fields (i.e. derivations of $\mathscr{F}$) which have no constant terms. Naturally $\mathfrak{X}^0$ has Lie algebra structure and we denote by $[\ ,\ ]$ its Lie bracket. Let G be the group of algebra automorphisms of $\mathscr{F}$. The group G acts on $\mathfrak{X}^0$ as $\varphi_*X=\varphi^{-1}X\varphi$ where $\varphi\in G$ and $X\in\mathfrak{X}^0$. We say that two formal vector fields X and Y are *equivalent* if there is an element $\varphi\in G$ such

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that $\varphi_* X = Y$. By J^k we denote the k-jet space of formal vector fields. We identify J^k with the polynomial vector field of degree k which have no constant terms. Naturally J^k has finite dimensional Lie algebra structure and we denote by $[\ ,\ ]^k$ its Lie bracket. There is a canonical projection $j^k\colon \mathfrak{X}^0 \to J^k$ and we take on $\mathfrak{X}^0$ the topology induced by $\{j^k\}_{k\geq 1}$. We say that $X \in \mathfrak{X}^0$ is *k-determined* if for any $Y \in \mathfrak{X}^0$ such that $j^k X = j^k Y$, X and Y are equivalent. A formal vector field X is called *finitely determined* if X is k-determined for some positive integer k.

For $X \in \mathfrak{X}^0$ we denote by X_1 the 1-jet of X. By linear transformation, without loss of generality we can assume that X_1 is of the Jordan normal form. Let λ_1, λ_2 be the eigenvalues of X_1. There are the following cases:

(a) rank $X_1 = 2$ and

(i) Both λ_1/λ_2 and λ_2/λ_1 neither belong to Q^- (the negative rational numbers), nor to N^* (the positive integers larger than one). This case is classical (cf. [3]) and X is 1-determined.

(ii) $\lambda_1/\lambda_2 \in Q^-$ (Leads to Theorem 1).

(iii) λ_1/λ_2 or $\lambda_2/\lambda_1 \in N^*$ (Leads to Theorem 0).

(b) rank $X_1 = 1$ and X_1 is semi-simple (Leads to Theorem 2).

(c) rank $X_1 = 0$ or the case $X_1 = y\,\partial/\partial x$ (X is not finitely determined, see [6]).

THEOREM 0. *Let the 1-jet X_1 of $X \in \mathfrak{X}^0$ be of the form $\lambda_1 x\,\partial/\partial x + \lambda_2 y\,\partial/\partial y$ where $\lambda_1/\lambda_2 \neq 0$ and $\lambda_2 = m\lambda_1$ ($m \geq 2$). Then X is equivalent to one of the following*:

(0–1) $X_1 + x^m\,\partial/\partial y$,

(0–2) X_1.

REMARK. By the change of variables, the case $\lambda_2 = (1/m)\lambda_1$ is reduced to Theorem 0.

THEOREM 1. *Let the 1-jet X_1 of X be of the form $\lambda_1 x\,\partial/\partial x + \lambda_2 y\,\partial/\partial y$ where $\lambda_1/\lambda_2 = -q/p$ and p, q are relatively prime positive integers. Then X is equivalent to one of the following*:

(1–1) $X_1 + \omega^k x\,\partial/\partial x + (b_k\omega^k + b_{2k}\omega^{2k})y\,\partial/\partial y$, ($b_k \neq -p/q$)

(1–2) $X_1 + q\omega^k x\,\partial/\partial x + (-p\omega^k + b_L\omega^L + \cdots + b_{2L-k}\omega^{2L-k} + b_{2L}\omega^{2L})y\,\partial/\partial y$, ($b_L \neq 0$ *and* $L > k$),

(1–3) $X_1 + q\omega^k x\,\partial/\partial x + (-p)\omega^k y\,\partial/\partial y$

(1–4) X_1,

where $\omega = x^p y^q$ *and* $1 \leq k < L$.

THEOREM 2. *Let the 1-jet X_1 of X be of the form $\lambda_1 x\,\partial/\partial x$ and $\lambda_1 \neq 0$.*

Then X is equivalent to one of the following:

(2–1) $X_1 + a_k xy^k\,\partial/\partial x + (y^k + b_{2k}y^{2k})y\,\partial/\partial y$,

(2–2) $X_1 + xy^k\,\partial/\partial x + (b_L y^L + \cdots + b_{2L-k}y^{2L-k} + b_{2L}y^{2L})y\,\partial/\partial y$, ($b_L \neq 0$ *and* $L > k$),

(2–3) $X_1 + xy^k\,\partial/\partial x$,

(2–4) X_1.

DEFINITION. For a subset $S \subset \mathfrak{X}^0$ we say that S is a *constructible set* (resp. *submanifold*) of $\mathfrak{X}^0$ if for any positive integer k, $j^k S$ is a constructible set (resp. submanifold) of J^k.

DEFINITION. For a submanifold M of $\mathfrak{X}^0$, we define a *codimension* $\tau(M)$ of M in $\mathfrak{X}^0$ as $\tau(M) = \lim \tau_k(j^k M)$ where $\tau_k(j^k M)$ is a codimension of $j^k M$ in J^k. Obviously GX is a submanifold of $\mathfrak{X}^0$. We use $\tau(X)$ instead of $\tau(GX)$.

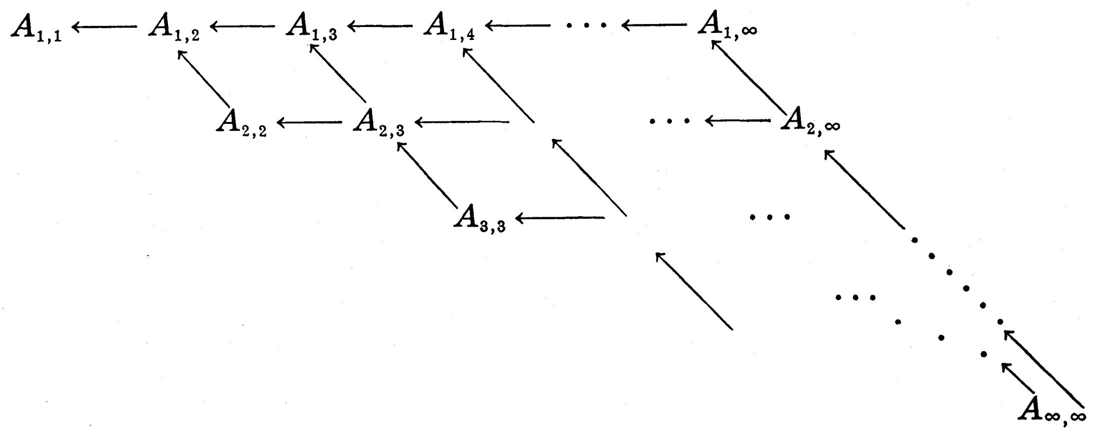
DEFINITION. For two submanifolds M and N of $\mathfrak{X}^0$, we say that M is *adjacent* to N if the closure of M contains N. We denote this adjacency by $M \leftarrow N$.

Now, we define $A_{k,k}$, $A_{k,L}$, $A_{k,\infty}$, $A_{\infty,\infty}$ as follows:

$$A_{k,k} := \{X \in \mathfrak{X}^0;\ X \text{ is equivalent to the form (1–1)}\}\,,$$
$$A_{k,L} := \{X \in \mathfrak{X}^0;\ X \text{ is equivalent to the form (1–2)}\}\,,$$
$$A_{k,\infty} := \{X \in \mathfrak{X}^0;\ X \text{ is equivalent to the form (1–3)}\}\,,$$
$$A_{\infty,\infty} := \{X \in \mathfrak{X}^0;\ X \text{ is equivalent to the form (1–4)}\}\,.$$

In the same way we define $B_{k,k}$, $B_{k,L}$, $B_{k,\infty}$, $B_{\infty,\infty}$ corresponding to Theorem 2 (2–1), (2–2), (2–3), (2–4).

THEOREM 3. *The subsets $A_{k,L}$, $B_{k,L}$ ($1 \leqq k \leqq L \leqq \infty$) are constructible submanifolds of $\mathfrak{X}^0$ and $\tau(A_{k,L}) = \tau(B_{k,L}) = k + L$. The adjacency of $\{A_{k,L}\}_{1 \leqq k \leqq L}$ is given by*



and for $\{B_{k,L}\}_{1\leq k\leq L}$ we have the same diagram of adjacency.

REMARK. After I had finished this work, I was informed that the classification problem of 2-dimensional singularities is very classical subject (cf. [3], [5] and J. Martinet's report at Bourbaki seminer n° 564 (1980) and its references), thus a part of the results of this paper might be already known.

§ 2. Preliminaries.

In this section we state several propositions without proofs. Their proofs are found in [6]. We denote by GL^k the k-jet space of automorphisms of $\mathscr{F}$. In a natural way GL^k has a Lie group structure. The group GL^k acts on J^k as follows; $\varphi_{k*}X_k=\varphi_k^{-1}X_k\varphi_k$ where $\varphi_k\in GL^k$ and $X_k\in J^k$. Obviously we have $j^kGX=GL^k(j^kX)$.

PROPOSITION 2.1. *The tangent space TGL^kX_k of the orbit GL^kX_k at X_k is given by*

$$TGL^kX_k=\{[X_k,\ Y_k];\ Y_k\in J^k\}\ .$$

In particular, the codimension $\tau_k(X_k)$ of GL^kX_k in J^k is given by

$$\tau_k(X_k)=\dim_C\{Y_k\in J^k;\ [X_k,\ Y_k]^k=0\}\ .$$

For $X\in\mathfrak{X}^0$ we decompose X as $X=X^s+X^n$ where X^s (resp. X^n) is the semi-simple (resp. nilpotent) part of the mapping $X:\mathscr{F}\to\mathscr{F}$. We see that X^s and X^n are also derivations of $\mathscr{F}$ and $[X^s, X^n]=0$. Moreover we see that for $\varphi\in G$, $(\varphi_*X)^s=\varphi_*X^s$ and $(\varphi_*X)^n=\varphi_*X^n$.

PROPOSITION 2.2. *If the 1-jet X_1 of X is of the form $X_1=\lambda_1x\,\partial/\partial x+\lambda_2y\,\partial/\partial y$, then there exists $\varphi\in G$ such that*

$$(*)\qquad \varphi_*X_1=X_1+\sum_{\mu_1\lambda_1+\mu_2\lambda_2=\lambda_1}a_{\mu_1\mu_2}x^{\mu_1}y^{\mu_2}\,\partial/\partial x+\sum_{\nu_1\lambda_1+\nu_2\lambda_2=\lambda_2}b_{\nu_1\nu_2}x^{\nu_1}y^{\nu_2}\,\partial/\partial y$$

where μ_1, μ_2, ν_1 and ν_2 are non-negative integers and $\mu_1+\mu_2\geqq 2$, $\nu_1+\nu_2\geqq 2$. Moreover the semi-simple part of (*) is X_1.

REMARK. We call (*) *the normal form* of X. A more general normal form theorem can be seen in [6, 14]. Note that the higher terms appeared in the normal form are the terms which commute X_1 with respect to Lie product.

PROPOSITION 2.3 (Takens [15]). *Let X, $Y\in\mathfrak{X}^0$. If $j^1Y=0$ and $j^k[X,$*

$Y]=0$, *then* $j^{k+1}(\exp Y)_* X = j^{k+1}(X+[X, Y])$.

§ 3. Proof of Theorems 0, 1 and 2.

3.0. PROOF OF THEOREM 0. Since $\lambda_2 = m\lambda_1$ and $m \geqq 2$, from Proposition 2.2, the normal form of X is $X_1 + ax^m\, \partial/\partial y$. In the case $a \neq 0$, by linear transformation X is equivalent to $X_1 + x^m\, \partial/\partial y$. The case $a=0$ is (0–2). Thus in both cases X is m-determined. Note that in the case (0–1) $\tau(X)=2$ and in the case (0–2) $\tau(X_1)=3$.

3.1. PROOF OF THEOREM 1. From Proposition 2.2, the normal form of X is given by (1–4) or the following:

$$(**) \qquad X_1 + (\sum_{i=k} a_i \omega^i) x\, \partial/\partial x + (\sum_{i=k} b_i \omega^i) y\, \partial/\partial y$$

where $\omega = x^p y^q$ and $(a_k, b_k) \neq (0, 0)$. For simplicity we use $J^{(m)}$ (resp. $\tau_{(m)}$) instead of $J^{m(p+q)+1}$ (resp. $\tau_{m(p+q)+1}$).

LEMMA 3.1.1. *Let the notations be as above. Then k is uniquely determined by GX.*

PROOF. Let X be of the form (**). Then from Proposition 2.1,

$$\tau_{(k)}(X) = \dim\{\langle qx\, \partial/\partial x - py\, \partial/\partial y,\ \omega x\, \partial/\partial x,\ \cdots,\ \omega^k y\, \partial/\partial y \rangle_C\} = 2k+1 .$$

On the other hand

$$\tau_{(k)}(X_1) = \dim\{\langle x\, \partial/\partial x,\ y\, \partial/\partial y,\ \omega x\, \partial/\partial x,\ \cdots,\ \omega^k y\, \partial/\partial y \rangle_C\} = 2k+2 .$$

Now, we classify (**) into two cases (1)′ $pa_k + qb_k \neq 0$ and (2)′ $pa_k + qb_k = 0$. By linear change of coordinate we easily see that both (1)′ and (2)′ are equivalent respectively to the following (1), (2) in $J^{(k)}$.

(1) $X_1 + \omega^k x\, \partial/\partial x + b_k \omega^k y\, \partial/\partial y$, $(b_k \neq -p/q)$

(2) $X_1 + q\omega^k x\, \partial/\partial x - p\omega^k y\, \partial/\partial y$.

By $G_{(m)}$ (resp. $G_{(m),1}$) we denote the vector space spanned by $\{\omega^m x\, \partial/\partial x, \omega^m y\, \partial/\partial y\}$ (resp. $\{q\omega^m x\, \partial/\partial x - p\omega^m y\, \partial/\partial y\}$). The formal vector field X given by (**) can be expressed as $X_1 + X_{(k)} + X_{(k+1)} + \cdots$ where $X_{(j)} \in G_{(j)}$, $j = k, k+1, \cdots$.

LEMMA 3.1.2. *We fix the ordered basis $\{\omega^m x\, \partial/\partial x, \omega^m y\, \partial/\partial y\}$ of $G_{(m)}$ $(m=1, 2, \cdots)$. Then $X_{(j)} = a\omega^j x\, \partial/\partial x + b\omega^j y\, \partial/\partial y$ induces the linear mapping $[X_{(j)}, -]\colon G_{(m)} \to G_{(m+j)}$ and its representation matrix is given by*

$$\begin{bmatrix} (m-j)pa + mqb & -jqa \\ -jpb & (m-j)qb + mpa \end{bmatrix}$$

and the determinant of this matrix equals $m(m-j)(pa+qb)^2$.

PROOF. Direct computations.

Case (1). We use the same arguments as in [6, 15]. From Lemma 3.1.2, $[X_{(k)}, -]: G_{(m)} \to G_{(m+k)}$ is not surjective if and only if $m=k$. First we take $Y_{(1)} \in G_{(1)}$ such that $[X_{(k)}, Y_{(1)}] = -X_{(k+1)}$, then from Proposition 2.3, we have $j^{(k+1)}(\exp Y_{(1)})_* X = X_1 + X_{(k)}$. Moreover, from $[X_1, Y_{(1)}]=0$ we have $[X_1, (\exp Y_{(1)})_* X] = [(\exp Y_{(1)})_* X_1, (\exp Y_{(1)})_* X] = (\exp Y_{(1)})_* [X_1, X] = 0$. Thus $(\exp Y_{(1)})_* X$ is also of the normal form (**) with different coefficients. In the same way we can choose $Y_{(m)} \in G_{(m)}$ $(m=1, \cdots, k-1)$ such that $j^{(2k-1)}(\exp Y_{(k-1)})_* \cdots (\exp Y_{(1)})_* X = X_1 + X_{(k)}$. When $m=k$ we have $[X_{(k)}, G_{(k)}] = G_{(2k),1}$ and we decompose $G_{(2k)}$ as $G_{(2k),1} \oplus \langle \omega^{2k} y\, \partial/\partial y \rangle_C$. Then there is $Y_{(k)} \in G_{(k)}$ such that $j^{(2k)}(\exp Y_{(k)})_* \cdots (\exp Y_{(1)})_* X = X_1 + X_{(k)} + b_{2k}\omega^{2k} y\, \partial/\partial y$. Inductively, using Proposition 2.3, we can eliminate the higher terms and we obtain the normal form (1-1).

LEMMA 3.1.3. *For* $X_{(k)} = q\omega^k x\, \partial/\partial x - p\omega^k y\, \partial/\partial y$, *we have*

(i) $[X_{(k)}, G_{(m)}] = G_{(m+k),1}$,

(ii) $\ker\{[X_{(k)}, -]: G_{(m)} \to G_{(m+k)}\} = G_{(m),1}$,

(iii) $[G_{(j)}, G_{(m),1}] = G_{(m+j),1}$.

PROOF. This is an easy concequence of Lemma 3.1.2.

Now, we classify case (2) into two cases.

$(2)_1$ There is a positive integer L such that $pa_L + qb_L \neq 0$.

$(2)_2$ For any $j \geqq k$, $pa_j + qb_j = 0$.

In the case $(2)_1$ we denote by L the minimum L such that $pa_L + qb_L \neq 0$. We set $\mathscr{H} = \ker\{X^s: \mathscr{F} \to \mathscr{F}\}$ where X^s is the semi-simple part of X.

LEMMA 3.1.4. *In the case* $(2)_1$ *the above* L *is uniquely determined by* GX.

PROOF. Suppose that X is of the form (**). Then $\mathscr{H}$ is given by $\mathscr{H} = C[[\omega]]$ where $\omega = x^p y^q$. We denote by $\mathfrak{M}_{\mathscr{H}}$ the maximal ideal of $\mathscr{H}$. Then L is given by $\mathfrak{M}_{\mathscr{H}}^{L+1} = X(\mathfrak{M}_{\mathscr{H}})$. This completes the proof.

Case $(2)_1$. We decompose $G_{(j)}$ as $G_{(j),1} \oplus \langle \omega^j y\, \partial/\partial y \rangle_C$. From Lemma 3.1.3, using the same arguments as in the proof of Case (1), without loss of generality we can assume that X is of the following form;

$$X = X_1 + X_{(k)} + X_{(L)} + X_{(L+1)} + \cdots$$

where $X_{(j)} \in \langle \omega^j y\, \partial/\partial y \rangle_C$ $(j = L, L+1, \cdots)$ and $X_{(L)} \neq 0$. Now, we take $Y_{(m)} \in G_{(m),1}$, $Y_{(m)} \neq 0$. Then we have $[X, Y_{(m)}] = [X_{(L)}, Y_{(m)}] + [X_{(L+1)}, Y_{(m)}] + \cdots$.

From Lemma 3.1.3 we can choose $Y_{(L+m+j-k)}\in G_{(L+m+j-k)}$ ($j=0, 1, \cdots, L-k-1$) such that $[X_{(L+j)}, Y_{(m)}]=-[X_{(k)}, Y_{(L+m+j-k)}]$. We set $\tilde{Y}_m=Y_{(m)}+Y_{(L+m-k)}+\cdots+Y_{(2L+m-2k-1)}$, then we have $[X, \tilde{Y}_m]=[X_{(2L-k)}, Y_{(m)}]+[X_{(L)}, Y_{(L+m-k)}]$+higher terms. From Lemma 3.1.2 we see that $[X_{(2L-k)}, Y_{(m)}]+[X_{(L)}, Y_{(L+m-k)}]\in G_{(2L+m-k)}$ and $\notin G_{(2L+m-k),1}$ if and only if $L+m-k\neq L$ i.e. $m\neq k$. Thus we have $[X, \langle\tilde{Y}_m\rangle_C\oplus G_{(2L+m-2k)}]=G_{(2L+m-k)}$+higher terms ($m\neq k$). From Proposition 2.3 we can eliminate the terms of $G_{(2L+m-k)}$ ($m=1, 2, \cdots$) except for the only term of $\langle\omega^{2L}y\,\partial/\partial y\rangle_C$. Thus we obtain the normal form (1–2).

Case $(2)_2$. In this case X is given by

$$X=X_1+X_{(k)}+X_{(k+1)}+\cdots$$

where $X_{(j)}\in G_{(j),1}$ ($j=k, k+1, \cdots$). Note that $X|_{\mathscr{X}}=0$ in this case. From Proposition 2.3 and Lemma 3.1.3, there is $Y_{(1)}\in G_{(1)}$ such that

$$(\exp Y_{(1)})_*X=X_1+X_{(k)}+X'_{(k+2)}+X'_{(k+3)}+\cdots .$$

Since the property $X|_{\mathscr{X}}=0$ is invariant under the action of G, so we have $X'_{(j)}\in G_{(j),1}$ ($j=k+2, k+3, \cdots$). Thus we can eliminate inductively the higher terms and we obtain the normal form (1–3).

REMARK. From the proof of Theorem 1, we easily see that we can choose the different normal forms (1–1)$\sim$(1–3) corresponding to the choice of the compliment linear subspace of $G_{(j),1}$ in $G_{(j)}$.

Theorem 2 can be proved in the same way, so we ommit the proof.

COROLLARY. *For any formal vector field X of $A_{k,L}$ (resp. $B_{k,L}$) X is $(2L(p+q)+1)$-determined (resp. $(2L+1)$-determined).*

§ 4. Proof of Theorem 3.

The following proposition was obtained by R. Thom as a corollary of Seidenberg-Tarski theorem.

PROPOSITION 4.1 ([8, 14]). *Let S' be a constructible set of J^k. Then $S=GL^kS'$ is also constructible set of J^k.*

From this proposition, we easily see that $A_{k,L}$ and $B_{k,L}$ ($1\leqq k\leqq L\leqq\infty$) are constructible sets of $\mathfrak{X}^0$.

LEMMA 4.2. *The subsets $A_{k,L}$ and $B_{k,L}$ ($1\leqq k\leqq L\leqq\infty$) are submanifolds of $\mathfrak{X}^0$.*

PROOF. The case $A_{k,\infty}$ ($k=1, 2, \cdots$) is trivial. For $A_{k,L}$ ($L<\infty$) it is enough to prove that $j^{(2L)}A_{k,L}$ is a submanifold of $J^{(2L)}$. We take $X\in A_{k,L}$ which is of the form (1-1) or (1-2). We prove that $j^{(2L)}A_{k,L}$ is a submanfold in a neighbourhood of X in $J^{(2L)}$. Note that

$$[X, x^{\alpha}y^{\beta}\,\partial/\partial x]=(\alpha\lambda_1+\beta\lambda_2-\lambda_1)x^{\alpha}y^{\beta}\,\partial/\partial x+\text{higher terms}\ ,$$
$$[X, x^{\alpha}y^{\beta}\,\partial/\partial y]=(\alpha\lambda_1+\beta\lambda_2-\lambda_2)x^{\alpha}y^{\beta}\,\partial/\partial y+\text{higher terms}\ .$$

From the above facts and the caluculations in the proof of Theorem 1 we easily see that

$$TGL^{(2L)}X\cap\langle\omega^k y\,\partial/\partial y,\ \omega^L y\,\partial/\partial y,\ \cdots,\ \omega^{(2L-k)}y\,\partial/\partial y,\ \omega^{2L}y\,\partial/\partial y\rangle=\{0\}\ .$$

Therefore the parameter directions of (1-1) and (1-2) are in the directions transversal to $TGL^{(2L)}X$ in $J^{(2L)}$. Thus $j^{(2L)}A_{k,L}$ is a submanifold of $J^{(2L)}$. We can prove in the same way for $B_{k,L}$.

Now, the adjacency is obvious from the normal forms (1-1)$\sim$(1-4) and (2-1)$\sim$(2-4).

LEMMA 4.3. $\tau(A_{k,k})=\tau(B_{k,k})=2k$.

PROOF. We assume that $X\in A_{k,k}$ is of the form (1-1). Then from Proposition 2.1, we have

$$\begin{aligned}\tau_{(2k)}(X)&=\dim\{\langle qx\,\partial/\partial x-py\,\partial/\partial y,\ \omega^k x\,\partial/\partial x+b_k\omega^k y\,\partial/\partial y,\\ &\qquad \omega^{k+1}x\,\partial/\partial x,\ \cdots,\ \omega^{2k}y\,\partial/\partial y\rangle_C\}\\ &=2k+2\ .\end{aligned}$$

Since the dimension of parmeters of (1-1) is two, so we have $\tau(A_{k,k})=\tau_{(2k)}(j^{(2k)}A_{k,k})=2k$. For the case $B_{k,k}$ we can prove in the same way.

PROPOSITION 4.4. $\tau(A_{k,L})=\tau(B_{k,L})=k+L$.

PROOF. We prove this proposition by the induction on $L-k$. The case $L-k=0$ is Lemma 4.3. We assume that $\tau(W_{k,k+s})=2k+s$ for $k=1, 2, 3, \cdots$, where W stands for A and B. Then by the adjacency $W_{k,k+s}\leftarrow W_{k,k+s+1}\leftarrow W_{k+1,k+s+1}$ we have $2k+s<\tau(W_{k,k+s+1})<2k+s+2$. Thus we have $\tau(W_{k,k+s+1})=2k+s+1$. This completes the proof.

§ 5. Real case.

Let X be a germ of C^∞-vector field at $(R^2, 0)$ with $X(0)=0$. We denote by λ_1, λ_2 the eigenvalues of 1-jet X_1 of X. From Sternberg's linearization theorem, if (i) $\mathcal{R}e\,\lambda_1=\mathcal{R}e\,\lambda_2\neq0$ or (ii) λ_1, λ_2 are non-zero real numbers and $\lambda_1/\lambda_2\notin Q^-$, then X is 1-determined as C^∞-germ. For other case (Section 1 case (a) (ii) (iii), case (b) and case (c)), we have the similar theorems with Theorems 0, 1 and 2 in the formal category. However, from Sternberg's work, we see that a C^∞-vector field germ X which has a hyperbolic singularity at the origin is k-determined as C^∞-germ if and only if ∞-jet of X at the origin is formally k-determined. Thus Theorem 0 and Theorem 1 (1–1), (1–2) hold. And Theorem 3 holds replacing "constructible submanifolds" by "semi-algebraic submanifolds". Finally we state the pure imaginary eigenvalue case.

THEOREM 4. *Let the 1-jet X_1 of real formal vector field X be of the form $\theta x\,\partial/\partial y-\theta y\,\partial/\partial x$ where $\theta\in R$ and $\theta\neq 0$. Then X is equivalent to one of the following*:

(4–1) $X_1+(\delta\gamma^k+a_{2k}\gamma^{2k})(x\,\partial/\partial x+y\,\partial/\partial y)+b_k\gamma^k(x\,\partial/\partial y-y\,\partial/\partial x)$,

(4–2) $X_1+(a_L\gamma^L+\cdots+a_{2L-k}\gamma^{2L-k}+a_{2L}\gamma^{2L})(x\,\partial/\partial x+y\,\partial/\partial y)+\delta\gamma^k(x\,\partial/\partial y-y\,\partial/\partial x)$, $(a_L\neq 0$ *and* $L>k)$,

(4–3) $X_1+\delta\gamma^k(x\,\partial/\partial y-y\,\partial/\partial x)$,

(4–4) X_1,

where $\delta=\pm 1$ *and* $\gamma=x^2+y^2$.

REMARK. See Takens [14] for the normal form of X.

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