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Some Skew Product Transformations Associated with Continued Fractions and Their Invariant Measures

Shunji ITO

Tsuda College

Introduction

In this paper we discuss the following number theoretical transformations defined on $[0,1)\times[0,1)$

$$T_1;\ (\alpha,\beta)\longrightarrow\left(\frac{1}{\alpha}-\left[\frac{1}{\alpha}\right],\ \frac{\beta}{\alpha}-\left[\frac{\beta}{\alpha}\right]\right)$$

and

$$T_2;\ (\alpha,\beta)\longrightarrow\left(\frac{1}{\alpha}-\left[\frac{1}{\alpha}\right],\ -\left[-\frac{\beta}{\alpha}\right]-\frac{\beta}{\alpha}\right).$$

These transformations T_1 and T_2 which can be found in [1] are examples of the so-called skew product transformations associated with the continued fraction transformation S; $\alpha\rightarrow(1/\alpha)-[1/\alpha]$. These transformations induce the following expansions, respectively (see §1 and §3 for details);

1) $\beta=\sum_{k=1}^{\infty}|\theta(k-1)|\cdot b(k)$

and

2) $\beta=\sum_{k=1}^{\infty}\theta(k-1)\cdot b'(k)$

where $\theta(n)=q_n\alpha-p_n$.

Therefore, the transformations T_1 and T_2 give the algorithms which will yield the approximations of the real number β by means of the set of all translates $\{n\alpha\}$ of an irrational number α.

In this paper we discuss the ergodic properties of the transformations T_1 and T_2. And we shall elaborate on number theoretical applica-

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tions of the algorithm T_1 in a subsequent paper [3]. Our main theorem is to state that the density functions of invariant measures μ_1 and μ_2 for T_1 and T_2, respectively, are given as follows:

$$\frac{d\mu_1}{d\lambda}=\begin{cases}\dfrac{1}{2\log 2}\dfrac{\alpha+2}{(1+\alpha)^2} & \text{if } \beta>\alpha\\[2ex] \dfrac{1}{2\log 2}\dfrac{\alpha+3}{(1+\alpha)^2} & \text{if } \beta\leqq\alpha\end{cases}$$

and

$$\frac{d\mu_2}{d\lambda}=\begin{cases}\dfrac{1}{2\log 2}\dfrac{\alpha+2}{(1+\alpha)^2} & \text{if } \alpha+\beta<1\\[2ex] \dfrac{1}{2\log 2}\dfrac{\alpha+3}{(1+\alpha)^2} & \text{if } \alpha+\beta>1\,.\end{cases}$$

But we feel that our procedure for obtaining the proof of this theorem, which should be called a realization of the natural extension, is more important than the specific form of the density functions obtained. Our proof proceeds as follows: first of all, we try to realize the natural extensions $\bar{T}_1$ and $\bar{T}_2$ of T_1 and T_2, respectively, on some subsets of $\boldsymbol{R}^4$. Secondly, we find the density function of invariant measures $\bar{\mu}_i$ for $\bar{T}_i$ $(i=1, 2)$. And, finally, we obtain the main theorem. In the course of this procedure we obtain a new and important transformation T_1^* (See §2) which will be called the dual algorithm of T_1, and we see that this algorithm T_1^* induces the so-called canonical form in the theory of discrepancy [5].

On the other hand, in the paper [2], we will discuss the ergodic properties of the following transformations defined on $[0, 1)\times[0, 1)$

$$T_3;\ (\alpha, \beta)\longrightarrow\left(-\left[-\frac{1}{\alpha}\right]-\frac{1}{\alpha},\ -\left[-\frac{\beta}{\alpha}\right]-\frac{\beta}{\alpha}\right)$$

and

$$T_4;\ (\alpha, \beta)\longrightarrow\left(-\left[-\frac{1}{\alpha}\right]-\frac{1}{\alpha},\ \frac{\beta}{\alpha}-\left[\frac{\beta}{\alpha}\right]\right).$$

And we will give the density function of invariant measures μ_i for T_i $(i=3, 4)$ explicitly as follows:

$$\frac{d\mu_3}{d\lambda}=\begin{cases}\dfrac{2-\alpha}{(1-\alpha)^2} & \text{if } \alpha<\beta\\[2ex] \dfrac{1}{(1-\alpha)} & \text{if } \alpha>\beta\end{cases}$$

and

$$\frac{d\mu_4}{d\lambda}=\begin{cases}\dfrac{2-\alpha}{(1-\alpha)^2} & \text{if} \quad \alpha+\beta<1 \\[2mm] \dfrac{1}{(1-\alpha)} & \text{if} \quad \alpha+\beta>1 .\end{cases}$$

In concluding this introduction, we would like to thank Professors Yuji ITO and Hitoshi NAKADA for their valuable discussions and advices.

§1. Definitions and fundamental properties of the map T_1.

In this section we define the first map T_1 and discuss fundamental properties of this map.

Let X be the set $\{(\alpha, \beta) \mid 0<\alpha, \beta<1\}$ and functions $a(\alpha)$ and $b(\alpha, \beta)$ on X be defined by

$$(1) \qquad a(\alpha)=\left[\frac{1}{\alpha}\right], \qquad b(\alpha, \beta)=\left[\frac{\beta}{\alpha}\right],$$

where $[x]$ for any real number x denotes its integer part. We define a map T_1 of X onto itself by

$$(2) \qquad T_1(\alpha, \beta)=\left(\frac{1}{\alpha}-\left[\frac{1}{\alpha}\right], \frac{\beta}{\alpha}-\left[\frac{\beta}{\alpha}\right]\right).$$

(see figure (1)). Note that $T_1^n(\alpha, \beta) \notin X$ may occur for some n. To avoid

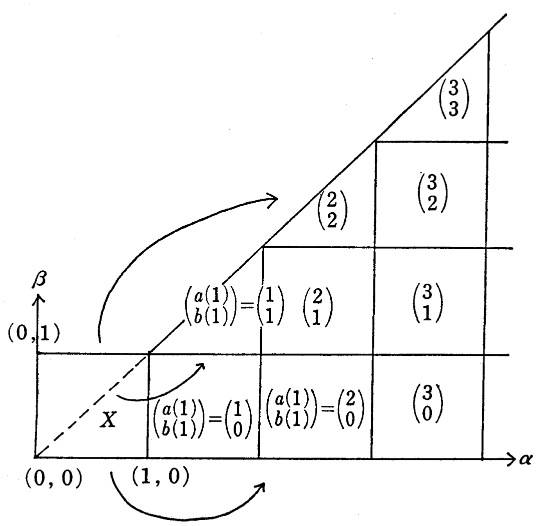


FIGURE 1. for $(\alpha, \beta) \to \left(\frac{1}{\alpha}, \frac{\beta}{\alpha}\right)$

this difficulty we must sometimes consider the algorithm on the restricted set $X_{T_1}=\{(\alpha, \beta)\in X \mid T_1^n(\alpha, \beta)\in X \text{ for all } n\in N\}$. But it is easy to see that the Lebesgue measure of this set X_{T_1} is equal to 1. Therefore, for simplicity we write X for X_{T_1} throughout the paper. We then define the sequence of coordinates associated with $(\alpha, \beta)\in X$ as follows:

$$(3)\qquad \begin{aligned} &a(\alpha; n)=a(\alpha(n-1)) \\ &b(\alpha, \beta; n)=b(\alpha(n-1), \beta(n-1)) \quad (n\geqq 1) \end{aligned}$$

where $\alpha(n)$ and $\beta(n)$ are the coordinates of $T_1^n(\alpha, \beta)$, that is,

$$(4)\qquad (\alpha(n), \beta(n))=T_1^n(\alpha, \beta) \quad (n\geqq 0)\ .$$

From the definitions (2) and (3), it is easy to verify that for each $(\alpha, \beta)\in X(=X_{T_1})$

$$(5)\qquad \begin{cases} \alpha=\cfrac{1}{a(\alpha; 1)+\cfrac{1}{a(\alpha; 2)+\ddots+\cfrac{1}{a(\alpha; n)+\alpha(n)}}} \\ \beta=\sum_{k=1}^{n}\alpha(0)\alpha(1)\cdots\alpha(k-1)b(\alpha, \beta; k)+\alpha(0)\alpha(1)\cdots\alpha(n-1)\beta(n) \end{cases}$$

for $n\geqq 1$.

The first expresion in the expansion (5) is the usual continued fraction. So we denote

$$(6)\qquad \frac{p(\alpha; n)}{q(\alpha; n)}=\cfrac{1}{a(\alpha; 1)+\cfrac{1}{a(\alpha; 2)+\ddots+\cfrac{1}{a(\alpha; n)}}} \quad (n\geqq 1)$$

and

$$(7)\qquad \theta(\alpha; n)=q(\alpha; n)\alpha-p(\alpha; n) \quad (n\geqq 1)\ .$$

Then we have

$$(8)\qquad \begin{aligned} &q(\alpha; n)=a(\alpha; n)q(\alpha; n-1)+q(\alpha; n-2) \\ &p(\alpha; n)=a(\alpha; n)p(\alpha; n-1)+p(\alpha; n-2) \end{aligned} \quad (n\geqq 1)$$

where $(q(\alpha; 0), q(\alpha; -1))=(1, 0)$ and $(p(\alpha; 0), p(\alpha; -1))=(0, 1)$, and

$$\alpha(0)\alpha(1)\alpha(2)\cdots\alpha(n)=(-1)^n\theta(\alpha; n)=|\theta(\alpha; n)| . \tag{9}$$

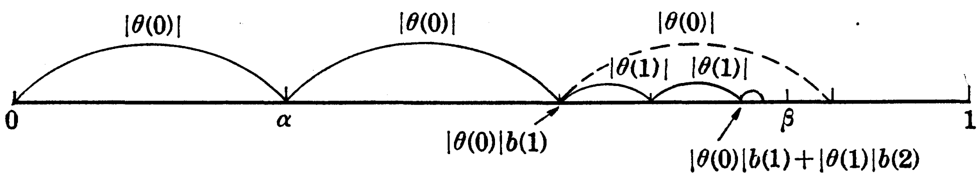
In the sequel for simplicity we sometimes write

$$a(n),\ b(n),\ p(n),\ q(n) \text{ and } \theta(n) \text{ for } a(\alpha; n),\ b(\alpha, \beta; n),\ p(\alpha; n) ,$$

$q(\alpha; n)$ and $\theta(\alpha; n)$, respectively. Now from the notation (6) and the relations (8) and (9) it follows that the expansion (5) of $(\alpha, \beta)\in X$ can be written in the form

$$\begin{cases} \alpha=\dfrac{p(n)+p(n-1)\alpha(n)}{q(n)+q(n-1)\alpha(n)} \\ \beta=\displaystyle\sum_{k=1}^{n} |\theta(k-1)|\, b(k)+\dfrac{\beta(n)}{q(n)+\alpha(n)q(n-1)} . \end{cases} \tag{10}$$

We call $(p(n)/q(n))$ and $\sum_{k=1}^{n} |\theta(k-1)|\, b(k)$ the n-th approximants of α and β with respect to the algorithm T_1. The second expression of (10) represents the approximation of the real number β corresponding to the following geometric picture:



We will discuss in detail geometric properties of this algorithm T_1 and its relation with the Weyl orbits $\{n\alpha\}$ in another paper [3].

The sequence of integer vectors $\left(\begin{pmatrix} a(1) \\ b(1) \end{pmatrix}, \begin{pmatrix} a(2) \\ b(2) \end{pmatrix}, \cdots\right)$, which will be called a T_1-admissible sequence, corresponds to each $(\alpha, \beta)\in X$, and it has the following "Markov" properties:

$$\text{(11)} \quad \begin{aligned} &(1)\quad a(i)\geqq b(i) \quad (i\geqq 1) \\ &(2)\quad \text{if } a(i)=b(i) \text{ then } b(i+1)=0 \quad (i\geqq 1) . \end{aligned}$$

Now we put

$$A(n)=\left\{\left(\begin{pmatrix} a(\alpha; 1) \\ b(\alpha, \beta; 1) \end{pmatrix}\begin{pmatrix} a(\alpha; 2) \\ b(\alpha, \beta; 2) \end{pmatrix}\cdots\begin{pmatrix} a(\alpha; n) \\ b(\alpha, \beta; n) \end{pmatrix}\right)\middle|\,(\alpha, \beta)\in X\right\} (n\geqq 1) \tag{12}$$

and for each $\left(\binom{a(1)}{b(1)}\binom{a(2)}{b(2)}\cdots\binom{a(n)}{b(n)}\right)\in A(n)$ we define

$$X\left(\binom{a(1)}{b(1)}\binom{a(2)}{b(2)}\cdots\binom{a(n)}{b(n)}\right)=\left\{(\alpha,\beta)\in X \,\middle|\, \binom{a(\alpha;i)}{b(\alpha,\beta;i)} =\binom{a(i)}{b(i)},\ 1\leqq i\leqq n\right\} \tag{13}$$

which will be called a fundamental cell of rank n. We sometimes denote $\left(\binom{a(1)}{b(1)}\binom{a(2)}{b(2)}\cdots\binom{a(n)}{b(n)}\right)$ and $X\left(\binom{a(1)}{b(1)}\cdots\binom{a(n)}{b(n)}\right)$ simply by $\boldsymbol{a}(n)$ and $X_{\boldsymbol{a}(n)}$. Then we have

$$X=\bigcup_{\boldsymbol{a}(n)\in A(n)} X_{\boldsymbol{a}(n)} \tag{14}$$

and

$$X_{\boldsymbol{a}(n)}\cap X_{\boldsymbol{a}'(n)}=\varnothing \quad \text{if} \quad \boldsymbol{a}(n)\neq\boldsymbol{a}'(n)\,,$$

that is, the set of all fundamental cells of rank n forms a partition of X. Moreover let sets U_0 and U_1 be defined by

$$\begin{aligned} U_0&=\{(\alpha,\beta);\,0<\alpha,\beta<1\} \quad (=X)\\ U_1&=\{(\alpha,\beta);\,0<\alpha<1,\,0<\beta<\alpha\}\,. \end{aligned} \tag{15}$$

Then from the definition of the map T_1 we can easily deduce the following lemma.

LEMMA 1. *For each* $\boldsymbol{a}(n)=\left(\binom{a(1)}{b(1)}\cdots\binom{a(n)}{b(n)}\right)\in A(n)$

$$T_1^n X_{\boldsymbol{a}(n)}=\begin{cases} U_0 & \text{if } a(n)\neq b(n)\\ U_1 & \text{if } a(n)=b(n)\,. \end{cases} \tag{16}$$

Now for each $\boldsymbol{a}(n)=\left(\binom{a(1)}{b(1)}\binom{a(2)}{b(2)}\cdots\binom{a(n)}{b(n)}\right)\in A(n)$ we define a map $\psi_{\boldsymbol{a}(n)}$ of (α,β) by

$$\psi_{\boldsymbol{a}(n)}(\alpha,\beta)=\left(\frac{p(n)+p(n-1)\alpha}{q(n)+q(n-1)\alpha},\ \sum_{k=1}^{n}\alpha_0(\alpha)\alpha_1(\alpha)\cdots\alpha_{k-1}(\alpha)b(k) +\alpha_0(\alpha)\alpha_1(\alpha)\cdots\alpha_{n-1}(\alpha)\beta\right), \tag{17}$$

where

$$\alpha_{k-1}(\alpha)=\cfrac{1}{a(k)+\cfrac{1}{a(k+1)+\ddots+\cfrac{1}{a(n)+\alpha}}} \quad (1\leqq k\leqq n)\,.$$

Then, by the representation (5) and Lemma 1, the map $\psi_{a(n)}$ is seen to be the inverse map of the map T_1^n on $X_{a(n)}$, that is,

$$\psi_{a(n)}(T_1^n x)=x \quad \text{for each } x\in X_{a(n)}\,. \tag{18}$$

Moreover, the Jacobian $J(\psi_{a(n)})$ of the map $\psi_{a(n)}$ is given by $J(\psi_{a(n)})(\alpha, \beta)=1/(q(n)+q(n-1)\alpha)^3$ on each U_i, $i=0$ or 1. Therefore the function $\psi_{a(n)}$ satisfies "Renyi's condition" [2], [3], or [4], that is, there exists a constant C not depending on $a(n)\in A(n)$ such that

$$\sup_{(\alpha,\beta)} J(\psi_{a(n)}(\alpha, \beta))\leqq C\inf_{(\alpha,\beta)} J(\psi_{a(n)}(\alpha, \beta))\,. \tag{19}$$

Now, we can obtain the following Theorem 1.

THEOREM 1. *The map T_1 on X admits an invariant measure μ_1 equivalent to the Lebesgue measure λ, and T_1 is ergodic; moreover it is exact with respect to μ_1.*

PROOF. It is sufficient to see that the map T_1 satisfies all the conditions of Lemma 1 in [2] (or [3]). In fact, the assertion that the map T_1 has the Markov property and satisfies the Renyi's condition is derived from Lemma 1 and (19). The condition that $\operatorname{diam}(X_{a(n)})\to 0$ $(n\to\infty)$, which guarantees that the partition of X into fundamental cells of rank n tends, as $n\to\infty$, to the partition into individual points, is obtained from the expansion (10), that is, $\operatorname{diam}(X_{a(n)})\sim 0(1/q_n)$. The condition of ergodicity is easy to verify. Thus, we obtain this theorem. The form of $(d\mu_1/d\lambda)$ will be determined explicitly in §2.

§2. Dual algorithm and the natural extension of the map T_1.

In this section, we construct the natural extension $\bar{T}_1$ of the map T_1 by using the dual algorithm of T_1 and compute the density function for the invariant measure μ_1 explicitly.

Let Y be the set $\{(\gamma, \delta)\,|\,0<\gamma<1,\ -\gamma<\delta<1\}$ and integer valued functions $c(\gamma)$ and $d(\gamma, \delta)$ be defined by

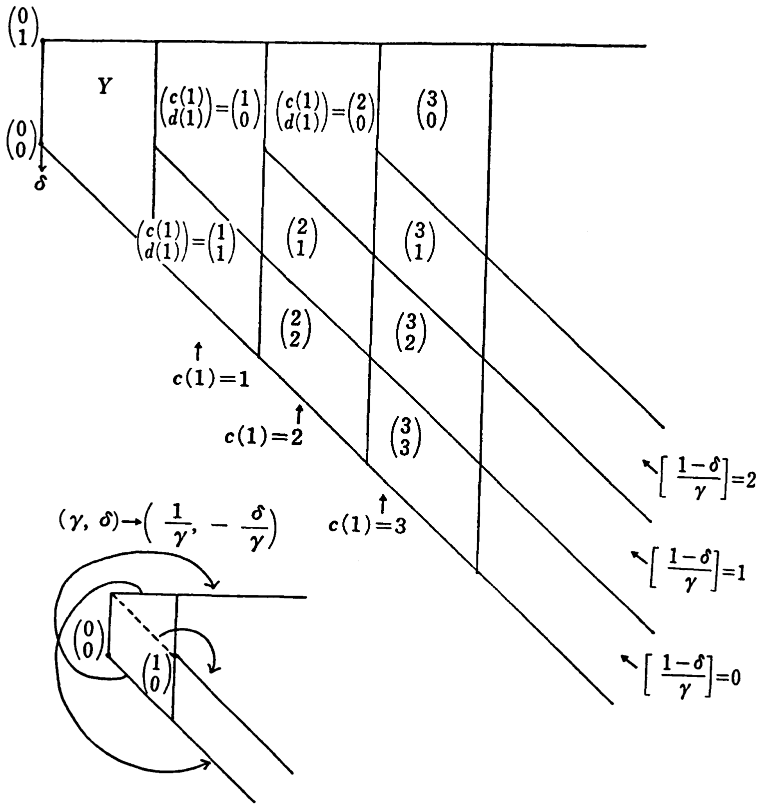


FIGURE 2

$$c(\gamma)=\left[\frac{1}{\gamma}\right], \qquad d(\gamma, \delta)=\max\left\{\left[\frac{1}{\gamma}\right]-\left[\frac{1-\delta}{\gamma}\right], 0\right\} . \tag{20}$$

We define an algorithm T_1^* on Y which will be called the dual algorithm of T_1, by

$$T_1^*(\gamma, \delta)=\left(\frac{1}{\gamma}-c(\gamma),\ -\frac{\delta}{\gamma}+d(\gamma, \delta)\right) \tag{21}$$

(see Figure 2). We define similarly as in the preceeding section, sequences corresponding to "coordinates" associated with $(\gamma, \delta) \in Y$ as follows:

$$\begin{aligned} c(\gamma; n)&=c(\gamma(n-1)) \\ d(\gamma, \delta; n)&=d(\gamma(n-1), \delta(n-1)) \end{aligned} \qquad (n \geqq 1) \tag{22}$$

where

$$(\gamma(n), \delta(n))=T_1^{*n}(\gamma, \delta) \quad (n \geqq 0) . \tag{23}$$

In the sequel we sometimes denote, for the sake of simplicity, $c(n)$ and $d(n)$ for $c(\gamma; n)$ and $d(\gamma, \delta; n)$. From the definition (21) and (22), it is easy to verify that for each $(\gamma, \delta) \in Y$

$$
(24)\quad \begin{cases} \gamma = \cfrac{1}{c(1) + \cfrac{1}{c(2) + \ddots + \cfrac{1}{c(n)+\gamma(n)}}} \\ \delta = \sum_{k=1}^{n} (-1)^{k-1}\gamma(0)\gamma(1)\cdots\gamma(k-1)d(k) + (-1)^{n}\gamma(0)\gamma(1)\cdots\gamma(n-1)\delta(n) \end{cases} \quad (n \geqq 1) .
$$

So we denote

$$
(25)\quad \frac{p^*(n)}{q^*(n)} = \cfrac{1}{c(1) + \cfrac{1}{c(2) + \ddots + \cfrac{1}{c(n)}}} \quad (n \geqq 1) ,
$$

and

$$
(26)\quad \theta^*(n) = q^*(n)\gamma - p^*(n) \quad (n \geqq 1) .
$$

Then the expansion (24) of $(\gamma, \delta) \in Y$ can be written in the form

$$
(27)\quad \begin{cases} \gamma = \dfrac{p^*(n)+p^*(n-1)\gamma(n)}{q^*(n)+q^*(n-1)\gamma(n)} \\ \delta = \sum_{k=1}^{n} \theta^*(k-1)d(k) + \dfrac{(-1)^n\delta(n)}{q^*(n)+q^*(n-1)\gamma(n)} . \end{cases}
$$

The second identity in (27), which gives a representation of the real number δ, coincides with the so-called canonical form in the theory of discrepancy [5]. We will discuss more interesting facts about this representation in another paper [3]. Now, the sequence of integer vectors $\left(\binom{c(1)}{d(1)}, \binom{c(2)}{d(2)}, \cdots\right)$ which will be called a T_1^*-admissible sequence is associated with each $(\gamma, \delta) \in Y$, and it has the following "Markov" properties:

$$
(28)\quad \begin{aligned} &(1)\ \ c(i) \geqq d(i) \qquad (i \geqq 1) \\ &(2)\ \ \text{if}\ \ d(i) \neq 0\ \ \text{then}\ \ d(i+1) \neq c(i+1) . \end{aligned}
$$

These properties are equivalent to the assertion that, for any finite sequence of integer vectors $\left(\binom{c(1)}{d(1)} \cdots \binom{c(n)}{d(n)}\right)$ satisfying the property (28), the word $\left(\binom{c(n)}{d(n)} \cdots \binom{c(1)}{d(1)}\right)$ obtained by reading $\left(\binom{c(1)}{d(1)}\binom{c(2)}{d(2)} \cdots \binom{c(n)}{d(n)}\right)$

backwards is a word in $A(n)$.

For each pair c, d with $c\in N$, $d\in N\cup\{0\}$ and $c\geqq d$, we let

$$\varphi_{\binom{c}{d}}(\gamma, \delta)=\left(\frac{1}{\gamma+c}, \frac{d-\delta}{\gamma+c}\right) \tag{29}$$

and

$$Y_{\binom{c}{d}}=\begin{cases}\varphi_{\binom{c}{d}}(V_0) & \text{if } d=0\\ \varphi_{\binom{c}{d}}(V_1) & \text{if } c\geqq d\geqq 1\end{cases} \tag{30}$$

where

$$V_0=Y \quad\text{and}\quad V_1=\{(\gamma, \delta)\,|\,0<\gamma<1,\ -\gamma<\delta<1-\gamma\}\,.$$

Then from the definition of T_1^* we have the following lemma.

LEMMA 2. *The family of sets* $\{Y_{\binom{c}{d}};\ c\in N, d\in N\cup\{0\}$ *and* $c\geqq d\}$ *has the following properties*:

(1) *the family is a partition of* Y, *that is,*

$$Y_{\binom{c}{d}}\cap Y_{\binom{c'}{d'}}=\varnothing \qquad \textit{if} \quad \begin{pmatrix}c\\ d\end{pmatrix}\neq\begin{pmatrix}c'\\ d'\end{pmatrix},$$

(2) $T_1^*(Y_{\binom{c}{d}})=\begin{cases}V_0 & \textit{if } d=0\\ V_1 & \textit{if } d\neq 0\,.\end{cases}$

We put $V\begin{pmatrix}c\\ d\end{pmatrix}=T_1^*(Y_{\binom{c}{d}})$ and $U\begin{pmatrix}a\\ b\end{pmatrix}=T_1(X_{\binom{a}{b}})$, and let the set Z be defined by

$$Z=\left\{(\gamma, \delta, \alpha, \beta)\in Y\times X\,\middle|\,\left(\begin{pmatrix}c(\gamma; 1)\\ d(\gamma, \delta; 1)\end{pmatrix}\begin{pmatrix}a(\alpha; 1)\\ b(\alpha, \beta; 1)\end{pmatrix}\right)\in A(2)\right\}\,.$$

Then by (14), Lemma 1 and Lemma 2, the set Z is seen to have the following two partitions:

$$\begin{aligned} Z&=\bigcup_{\binom{a}{b}\in A(1)} V\begin{pmatrix}a\\ b\end{pmatrix}\times X_{\binom{a}{b}}\\ &=\bigcup_{\binom{a}{b}\in A(1)} Y_{\binom{a}{b}}\times U\begin{pmatrix}a\\ b\end{pmatrix}\,. \end{aligned} \tag{31}$$

Now, we define the map $\bar{T}_1$ of z as follows:

$$(32)\qquad \bar{T}_1(\gamma, \delta, \alpha, \beta)=(\varphi_{\binom{a(\alpha;1)}{b(\alpha,\beta;1)}}(\gamma, \delta), T_1(\alpha, \beta))$$
$$=\left(\frac{1}{\gamma+a(\alpha; 1)}, \frac{b(\alpha, \beta; 1)-\delta}{\gamma+a(\alpha; 1)}, \alpha_1, \beta_1\right).$$

Then for each $\binom{a}{b}\in A(1)$, the map $\bar{T}_1$ is a 1-1 map of $V\binom{a}{b}\times X_{\binom{a}{b}}$ onto $Y_{\binom{a}{b}}\times U\binom{a}{b}$. Thus, we see that the map $\bar{T}_1$ of Z onto itself gives the natural extension of the map T_1.

THEOREM 2. *The natural extension $\bar{T}_1$ defined above has an invariant measure $\bar{\mu}$ satisfying $(d\bar{\mu}/d\bar{\lambda})=(1/\log 2)(1/(1+\gamma\alpha)^3)$ where $d\bar{\lambda}$ is the Lebesgue measure on Z.*

PROOF. The Jacobian $J(\bar{T}_1)$ of map $\bar{T}_1$ on $V\binom{a}{b}\times X_{\binom{a}{b}}$ is given by

$$(33)\qquad J(\bar{T}_1)=\frac{1}{(\gamma+a)^3\alpha^3}.$$

Now, we define the kernel function $K(\gamma, \delta, \alpha, \beta)$ on Z by

$$K(\gamma, \delta, \alpha, \beta)=\frac{1}{\log 2(1+\gamma\alpha)^3}.$$

Then the kernel function $K(\gamma, \delta, \alpha, \beta)$ satisfies the following relations:

$$(34)\qquad K(\bar{T}_1(\gamma, \delta, \alpha, \beta))J(\bar{T}_1(r, \delta, \alpha, \beta))$$
$$=\frac{1}{\log 2}\frac{1}{(1+((1/\alpha)-a)(1/(\gamma+a)))^3}\frac{1}{(\gamma+a)^3\alpha^3}$$
$$=K(\gamma, \delta, \alpha, \beta).$$

This relation (34) implies that $K(\gamma, \delta, \alpha, \beta)$ gives an invariant density function for $\bar{T}_1$.

Computing the marginal distribution, we obtain the following corollary.

COROLLARY.

(1) *The algorithm T_1 has a unique absolutely continuous invariant measure μ_1 satisfying*

$$\frac{d\mu_1}{d\lambda}=\begin{cases}\dfrac{1}{2\log 2}\dfrac{\alpha+3}{(1+\alpha)^2} & on\quad U_1\\[2ex] \dfrac{1}{2\log 2}\dfrac{\alpha+2}{(1+\alpha)^2} & on\quad X\backslash U_1\,,\end{cases}$$

(2) *The algorithm T_1^* has a unique absolutely continuous invariant measure μ_1^* satisfying*

$$\frac{d\mu_1^*}{d\lambda}=\begin{cases}\dfrac{1}{2\log 2}\dfrac{\gamma+2}{(1+\gamma)^2} & \text{on} \quad V_1\\[2ex] \dfrac{1}{2\log 2}\dfrac{1}{(1+\gamma)^2} & \text{on} \quad Y\backslash V_1\,.\end{cases}$$

§3. Definitions and fundamental properties of the map T_2.

In this section, we define another important map T_2 similar to the map T_1 given in §1.

Let X be the set $\{(\alpha, \beta)\,|\,0<\alpha, \beta<1\}$ and functions $a(\alpha)$ and $b(\alpha, \beta)$ on X be defined by

$$a(\alpha)=\left[\frac{1}{\alpha}\right], \qquad b(\alpha, \beta)=-\left[-\frac{\beta}{\alpha}\right]. \tag{35}$$

Then we define a map T_2 of X onto itself by

$$T_2(\alpha, \beta)=\left(\frac{1}{\alpha}-\left[\frac{1}{\alpha}\right], -\left[-\frac{\beta}{\alpha}\right]-\frac{\beta}{\alpha}\right) \tag{36}$$

(see Figure 3). We define the sequence of coordinates associated with

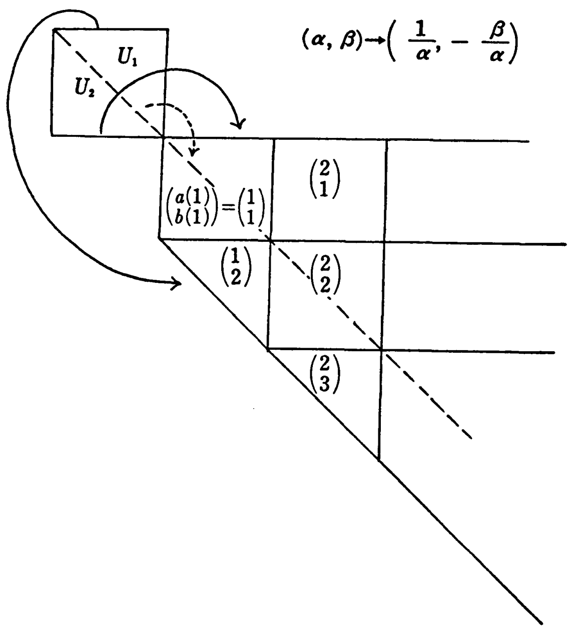


FIGURE 3

(α, β) as follows:

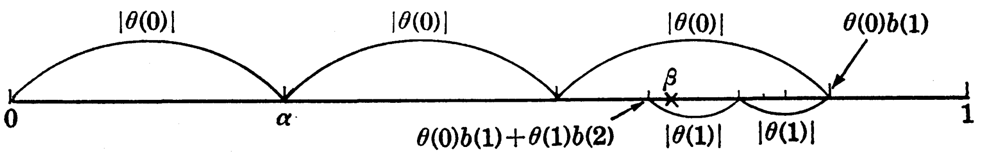
$$(37)\qquad \begin{aligned} &a(\alpha; n)=a(\alpha(n-1)) \\ &b(\alpha, \beta; n)=b(\alpha(n-1), \beta(n-1)) \quad (n\geqq 1) \end{aligned}$$

where $(\alpha(n), \beta(n))=T_2^n(\alpha, \beta)$ $(n\geqq 0)$.

Using the same notations as in §1, we have the following expansions

$$(38)\qquad \begin{cases} \alpha=\dfrac{p(n)+p(n-1)\alpha(n)}{q(n)+q(n-1)\alpha(n)} \\ \beta=\sum\limits_{k=1}^{n} \theta(k-1)b(k)+\dfrac{(-1)^n+\beta(n)}{q(n)+q(n-1)\alpha(n)} \end{cases}$$

The second identity of (38), which gives the approximation of the real number β by means of the algorithm T_2 corresponds to the following geometric picture:



The sequence of integer vectors $\left(\begin{pmatrix} a(1) \\ b(1) \end{pmatrix}, \begin{pmatrix} a(2) \\ b(2) \end{pmatrix}, \cdots \begin{pmatrix} a(n) \\ b(n) \end{pmatrix}, \cdots\right)$, which will be called a T_2-admissible sequence, is associated with each $(\alpha, \beta)\in X$, and it has the following "Multiple Markov" properties:

(1) $a(i)$, $b(i)\in N$ and $a(i)+1\geqq b(i)$,

(2) if there exist $k\geqq 1$ such that $b(k)=a(k)+1$ then

$$(39)\qquad b(k+1)=a(k+1) \quad \text{or} \quad a(k+1)+1\ ,$$

and if there exists j such that $a(k+i)=b(k+i)$ for $1\leqq i\leqq j$ and $a(k+j+1)\neq b(k+j+1)$ then $b(k+j+1)=a(k+j+1)+1$ in the case of even j's and $b(k+j+1)<a(k+j+1)$ in the case of odd j's.

Now we put

$$(40)\qquad B(n)=\left\{\left(\begin{pmatrix} a(\alpha; 1) \\ b(\alpha, \beta; 1) \end{pmatrix}\cdots\begin{pmatrix} a(\alpha; n) \\ b(\alpha, \beta; n) \end{pmatrix}\right)\middle|(\alpha, \beta)\in X\right\} \quad (n\geqq 1)$$

and for each $\left(\begin{pmatrix} a(1) \\ b(1) \end{pmatrix}\cdots\begin{pmatrix} a(n) \\ b(n) \end{pmatrix}\right)\in B(n)$ we define

$$X_{\left(\binom{a(1)}{b(1)}\cdots\binom{a(n)}{b(n)}\right)}=\left\{(\alpha,\beta)\in X\,\middle|\,\binom{a(\alpha;i)}{b(\alpha,\beta;i)}=\binom{a(i)}{b(i)},\ 1\leqq i\leqq n\right\} \tag{41}$$

which will be called a fundamental cell of rank n. We sometimes denote $\left(\binom{a(1)}{b(1)}\cdots\binom{a(n)}{b(n)}\right)$ and $X_{\left(\binom{a(1)}{b(1)}\cdots\binom{a(n)}{b(n)}\right)}$ by $\boldsymbol{b}(n)$ and $X_{\boldsymbol{b}(n)}$. Then $\{X_{\boldsymbol{b}(n)};\boldsymbol{b}(n)\in B(n)\}$ forms a partition of X. Let $U_0=X$, $U_1=\{(\alpha,\beta)\in X\mid\alpha+\beta>1\}$ and $U_2=\{(\alpha,\beta)\in X\mid\alpha+\beta<1\}$. And we decompose $B(n)$ as follows:

$$B_0^{(n)}=\left\{\begin{array}{l}\boldsymbol{b}(n)\in B(n);\ a(n)>b(n)\text{ or there exists }k\text{ such}\\ \quad\text{that }a(k)>b(k)\text{ and }a(j)=b(j)\text{ for all}\\ \quad k+1\leqq j\leqq n.\end{array}\right\}$$

$$B_1^{(n)}=\left\{\begin{array}{l}\boldsymbol{b}(n)\in B(n);\ a(n)+1=b(n)\text{ or there exists }k\text{ such}\\ \quad\text{that }a(k)+1=b(k),\ a(j)=b(j)\text{ for all}\\ \quad k+1\leqq j\leqq n\text{ and }n-k\text{ is even.}\end{array}\right\} \tag{42}$$

and

$$B_2^{(n)}=\left\{\begin{array}{l}\boldsymbol{b}(n)\in B(n);\ \text{there exists }k\text{ such that }a(k)+1=b(k)\\ \quad a(j)=b(j)\text{ for all }k+1\leqq j\leqq n,\text{ and}\\ \quad n-k\text{ is odd.}\end{array}\right\}.$$

Using the above notations we can easily prove the following lemma.

LEMMA 3. *For each* $\boldsymbol{b}(n)\in B(n)$ $(n\geqq 1)$,

$$T_2^n X_{\boldsymbol{b}(n)}=U_i\ \textit{if}\ \boldsymbol{b}(n)\in B_i^{(n)}\quad(i=0,1,2)$$

Now, for each $\boldsymbol{b}(n)=\left(\binom{a(1)}{b(1)},\cdots\binom{a(n)}{b(n)}\right)\in B(n)$ we define a map $\psi_{\boldsymbol{b}(n)}$ of (α,β) by

$$\psi_{\boldsymbol{b}(n)}(\alpha,\beta)=\Big(\frac{p(n)+p(n-1)\alpha}{q(n)+q(n-1)\alpha},\ \sum_{k=1}^{n}(-1)^k\alpha_0(\alpha)\cdots\alpha_{k-1}(\alpha)b(k)$$
$$+(-1)^n\alpha_0(\alpha)\cdots\alpha_{n-1}(\alpha)\beta\Big), \tag{43}$$

where $\alpha_{k-1}(\alpha)$ has the same meaning as in (17). Then, by using the relation (38) and Lemma 3, we see that the map $\psi_{\boldsymbol{b}(n)}$ is the inverse of T_2^n on $X_{\boldsymbol{b}(n)}$, that is

$$\psi_{\boldsymbol{b}(n)}(T_2^n x)=x\quad\text{for each}\quad x\in X_{\boldsymbol{b}(n)}\,. \tag{44}$$

Moreover, the Jacobian $J(\psi_{\boldsymbol{b}(n)})$ of the map $\psi_{\boldsymbol{b}(n)}$ is given by

$$J(\psi_{b(n)})(\alpha, \beta)=\frac{1}{(q(n)+q(n-1)\alpha)^3} \tag{45}$$

on each U_i, $i=0, 1, 2$. Therefore, the map $\psi_{b(n)}$ satisfies "Renyi's condition". We can easily see that the map T_2 satisfies all the conditions of Lemma 1 in [2] just as the map T_1 does and therefore, we have the following theorem.

THEOREM 3. *The map T_2 on X admits an invariant measure μ_2 equivalent to the Lebesgue measure λ, and the map T_2 is ergodic; moreover, it is exact with respect to μ_2.*

§4. The natural extension of the map T_2.

In this section, we will construct the natural extension of the map T_2 and compute the density function of the invariant measure μ_2 explicitly. However, the sequences corresponding to T_2 are more complicated than those corresponding to T_1. Hence, we must prepare some notations in order to describe admissibility.

Let $B(i, j)$, $i=1, 2$ and $j=1, 2, 3$ be the subsets of $\cup_{n=1}^{\infty} B(n)$ defined as follows:

$$B(1, 1)=\left\{\begin{pmatrix} a \\ b \end{pmatrix} \in B(1) \,\middle|\, a+1=b\right\},$$

$$B(1, 2)=\bigcup_{l=1}^{\infty}\left\{\left(\begin{pmatrix} a(1) \\ b(1) \end{pmatrix}, \cdots, \begin{pmatrix} a(2l) \\ b(2l) \end{pmatrix}\right) \in B(2l) \,\middle|\, a(j)=b(j) \right.$$
$$\left. \text{for all } 1\leqq j\leqq 2l-1 \text{ and } a(2l)>b(2l)\right\},$$

$$B(1, 3)=\bigcup_{l=1}^{\infty}\left\{\left(\begin{pmatrix} a(1) \\ b(1) \end{pmatrix}, \cdots, \begin{pmatrix} a(2l+1) \\ b(2l+1) \end{pmatrix}\right) \in B(2l+1) \,\middle|\, a(j)=b(j) \right.$$
$$\left. \text{for all } 1\leqq j\leqq 2l \text{ and } a(2l+1)+1=b(2l+1)\right\},$$

$$B(2, 1)=\left\{\begin{pmatrix} a \\ b \end{pmatrix} \in B(1) \,\middle|\, a>b\right\}, \tag{46}$$

$$B(2, 2)=\left\{\left(\begin{pmatrix} a(1) \\ b(1) \end{pmatrix}, \cdots, \begin{pmatrix} a(2l+1) \\ b(2l+1) \end{pmatrix}\right) \in B(2l+1) \,\middle|\, a(j)=b(j) \right.$$
$$\left. \text{for all } 1\leqq j\leqq 2l \text{ and } a(2l+1)>b(2l+1)\right\},$$

$$B(2,3)=\left\{\left(\begin{pmatrix}a(1)\\ b(1)\end{pmatrix},\cdots,\begin{pmatrix}a(2l)\\ b(2l)\end{pmatrix}\right)\in B(2l)\;\middle|\; a(j)=b(j)\right.$$

$$\left.\text{for all}\quad 1\leqq j\leqq 2l-1 \quad\text{and}\quad a(2l)+1=b(2l)\right\},$$

and put

$$B_1=\bigcup_{j=1}^{3} B(1,i) \quad\text{and}\quad B_2=\bigcup_{j=1}^{3} B(2,j)\,. \tag{47}$$

Then, we have

$$U_1=\bigcup_{b\in B_1} X_b \quad\text{and}\quad U_2=\bigcup_{b\in B_2} X_b\,.$$

For $\boldsymbol{b}=\left(\begin{pmatrix}a(1)\\ b(1)\end{pmatrix},\cdots,\begin{pmatrix}a(k)\\ b(k)\end{pmatrix}\right)\in B(k)$ and $\boldsymbol{b}'=\left(\begin{pmatrix}a'(1)\\ b'(1)\end{pmatrix},\cdots,\begin{pmatrix}a'(j)\\ b'(j)\end{pmatrix}\right)\in B(j)$, we denote $\left(\begin{pmatrix}a(1)\\ b(1)\end{pmatrix},\cdots,\begin{pmatrix}a(k)\\ b(k)\end{pmatrix},\begin{pmatrix}a'(1)\\ b'(1)\end{pmatrix},\cdots,\begin{pmatrix}a'(j)\\ b'(j)\end{pmatrix}\right)$ by $\boldsymbol{b}\cdot\boldsymbol{b}'$ and call $\boldsymbol{b}\cdot\boldsymbol{b}'$ admissible if $\boldsymbol{b}\cdot\boldsymbol{b}'\in B(k+j)$.

LEMMA 4. *Under the above notation and definition* (46), *we have the following*:

(1) *if* $\begin{pmatrix}a\\ b\end{pmatrix}\in B(1,1)$ *then* $\begin{pmatrix}a\\ b\end{pmatrix}\cdot\boldsymbol{b}$ *is admissible for all* $\boldsymbol{b}\in B_1$

(2) *if* $\begin{pmatrix}a\\ b\end{pmatrix}\in B(2,1)$ *then* $\begin{pmatrix}a\\ b\end{pmatrix}\cdot\boldsymbol{b}$ *is admissible for all* $\boldsymbol{b}\in B_1\cup B_2$

(3) *if* $\left(\begin{pmatrix}a(1)\\ b(1)\end{pmatrix},\cdots,\begin{pmatrix}a(2l)\\ b(2l)\end{pmatrix}\right)\in B(1,2)$
then $\left(\begin{pmatrix}a(2)\\ b(2)\end{pmatrix},\cdots,\begin{pmatrix}a(2l)\\ b(2l)\end{pmatrix}\right)\in\begin{cases}B(2,1) & \text{if } l=1\\ B(2,1) & \text{if } l\geqq 2\end{cases}$

(4) *if* $\left(\begin{pmatrix}a(1)\\ b(1)\end{pmatrix},\cdots,\begin{pmatrix}a(2l+1)\\ b(2l+1)\end{pmatrix}\right)\in B(1,3)$
then $\left(\begin{pmatrix}a(2)\\ b(2)\end{pmatrix},\cdots,\begin{pmatrix}a(2l+1)\\ b(2l+1)\end{pmatrix}\right)\in B(2,3)$

(5) *if* $\left(\begin{pmatrix}a(1)\\ b(1)\end{pmatrix},\cdots,\begin{pmatrix}a(2l+1)\\ b(2l+1)\end{pmatrix}\right)\in B(2,2)$
then $\left(\begin{pmatrix}a(2)\\ b(2)\end{pmatrix},\cdots,\begin{pmatrix}a(2l+1)\\ b(2l+1)\end{pmatrix}\right)\in B(1,2)$

(6) *if* $\left(\begin{pmatrix}a(1)\\ b(1)\end{pmatrix},\cdots,\begin{pmatrix}a(2l)\\ b(2l)\end{pmatrix}\right)\in B(2,3)$
then $\left(\begin{pmatrix}a(2)\\ b(2)\end{pmatrix},\cdots,\begin{pmatrix}a(2l)\\ b(2l)\end{pmatrix}\right)\in\begin{cases}B(1,1) & \text{if } l=1\\ B(1,3) & \text{if } l\geqq 2\end{cases}$

Using (1) and (2) in this lemma, we see that the following sets are well defined:

$$
\begin{aligned}
A_1 &= \cup \{X_{a\cdot b} \mid \boldsymbol{a} \in B(1, 1),\ \boldsymbol{b} \in B_1\} \\
A_2 &= \cup \{X_b \mid \boldsymbol{b} \in B(1, 2) \cup B(1, 3)\} \\
A_3 &= \cup \{X_{a\cdot b} \mid \boldsymbol{a} \in B(2, 1),\ \boldsymbol{b} \in B_1\} \\
A_4 &= \cup \{X_{a\cdot b} \mid \boldsymbol{a} \in B(2, 1),\ \boldsymbol{b} \in B_2\} \\
A_5 &= \cup \{X_b \mid \boldsymbol{b} \in B(2, 2) \cup B(2, 3)\},
\end{aligned}
\tag{48}
$$

and satisfy

$$U_1 = A_1 \cup A_2 \quad \text{and} \quad U_2 = A_3 \cup A_4 \cup A_5 . \tag{49}$$

On the other hand, we put D and E as follows:

$$
\begin{aligned}
D &= \{(\gamma, \delta) \mid 0 < \gamma, \delta < 1\} \\
E &= \{(\gamma, \delta) \mid 0 < \gamma < 1,\ 0 < \delta < \gamma + 1\}
\end{aligned}
$$

and we define the partitions of D and E

$$D = D_1 \cup D_2 \quad \text{and} \quad E = E_1 \cup E_2 \cup E_3 \tag{50}$$

where

$$
\begin{aligned}
D_1 &= \bigcup_{k=1}^{\infty} \left\{(\gamma, \delta) \,\middle|\, \frac{1}{k+1} < \gamma < \frac{1}{k},\ (k-1)\gamma < \delta < 1\right\} \\
D_2 &= \bigcup_{k=2}^{\infty} \bigcup_{j=1}^{k-1} \left\{(\gamma, \delta) \,\middle|\, \frac{1}{k+1} < \gamma < \frac{1}{k},\ (j-1)\gamma < \delta < j\gamma\right\} \\
E_1 &= \bigcup_{k=1}^{\infty} \left\{(\gamma, \delta) \,\middle|\, \frac{1}{k+1} < \gamma < \frac{1}{k},\ k\gamma < \delta < 1+\gamma\right\} \\
E_2 &= \bigcup_{k=1}^{\infty} \left\{(\gamma, \delta) \,\middle|\, \frac{1}{k+1} < \gamma < \frac{1}{k},\ (k-1)\gamma < \delta < k\gamma\right\} \\
E_3 &= D_2 \qquad \text{(see Figure 4)} .
\end{aligned}
\tag{51}
$$

Now, we define the map T_2 as follows:

$$\bar{T}_2(\gamma, \delta, \alpha, \beta) = \left(\frac{1}{\gamma + a(1)}, \frac{(b(1)-1)+\delta}{\gamma + a(1)}, \frac{1}{\alpha} - a(1), b(1) - \frac{\beta}{\alpha}\right) . \tag{52}$$

LEMMA 5. *Let M be*

$$M = E \times U_1 \cup D \times U_2 , \tag{53}$$

then the map T_2 is a one to one, onto map on M.

PROOF. The set M is divided into five sets $E \times A_i$ $i=1, 2$ and $D \times A_i$ $i=3, 4, 5$. From the definition (51) and (52) and lemma 4, we see that

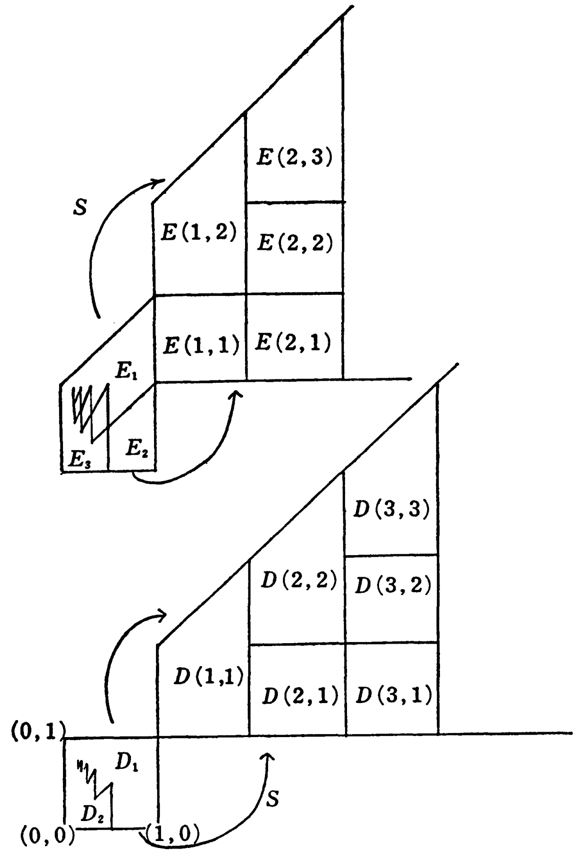


FIGURE 4.
$$S: (\gamma, \delta) \to \left(\frac{1}{\gamma}, \frac{\gamma+\delta}{\gamma}\right)$$

the map T_2 is a one to one map on each set, and the range sets are given by

$$\begin{aligned}
\bar{T}_2(E\times A_1)&=E_1\times U_1\\
\bar{T}_2(E\times A_2)&=D_1\times U_2\\
\bar{T}_2(D\times A_3)&=E_3\times U_1\\
\bar{T}_2(D\times A_4)&=D_2\times U_2\\
\bar{T}_2(D\times A_5)&=E_2\times U_1 .
\end{aligned}$$

This completes the proof.

THEOREM 4. *The map $\bar{T}_2$ on M has an invariant probability measure $\bar{\mu}_2$ satisfying*

$$\frac{d\bar{\mu}_2}{d\bar{\lambda}}=\frac{1}{\log 2(1+\gamma\alpha)^3} \tag{54}$$

where $\bar{\lambda}$ is the Lebesgue measure. Therefore, the dynamical system

$(M, \bar{T}_2, \bar{\mu}_2)$ is a realization of a natural extension of (X, T_2, μ_2).

PROOF. The Jacobian $J(\bar{T}_2)$ of the map $\bar{T}_2$ is given by

$$J(\bar{T}_2)=\frac{1}{(\gamma+a(1))^3\alpha^3} .$$

The kernel function defined by

$$K(\gamma, \delta, \alpha, \beta)=\frac{1}{\log 2(1+\alpha\gamma)^3} ,$$

satisfies the relation:

$$K(\bar{T}_2(\gamma, \delta, \alpha, \beta))J(\bar{T}_2(\gamma, \delta, \alpha, \beta))=K(\gamma, \delta, \alpha, \beta) .$$

The relation above implies that the kernel $K(\gamma, \delta, \alpha, \beta)$ gives an invariant density function for $\bar{T}_2$. That is, the system $(M, \bar{T}_2, \bar{\mu}_2)$ is a dynamical system. From the definition of $\bar{T}_2$ as a skew product transformation with respect to the map T_2, we conclude that the dynamical system $(M, \bar{T}_2, \bar{\mu}_2)$ is a natural extension of (X, T_2, μ_2).

Computing the marginal distribution, we obtain the following corollary.

COROLLARY. *The map T_2 on X has the following invariant density function:*

$$\frac{d\mu_2}{d\lambda}=\begin{cases}\dfrac{\alpha+3}{2\log 2(1+\alpha)^2} & \text{if } (\alpha, \beta)\in U_1 \\ \dfrac{\alpha+2}{2\log 2(1+\alpha)^2} & \text{if } (\alpha, \beta)\in U_2 .\end{cases}$$

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Present Address:
DEPARTMENT OF MATHEMATICS
TSUDA COLLEGE,
TSUDA-MACHI, KODAIRA, TOKYO 187