TOKYO J. MATH.
VOL. 11, No. 2, 1988

Knots in Certain Spatial Graphs

Miki SHIMABARA

Waseda University

(Communicated by S. Suzuki)

Abstract. In 1983, J. H. Conway and C. McA. Gordon showed in [1] that every embedding of the complete graph K_7 in the three-dimensional Euclidean space $\boldsymbol{R}^3$ contains a knotted cycle. In this paper we generalize their method and show that every embedding of the complete bipartite graph $K_{5,5}$ in $\boldsymbol{R}^3$ contains a knotted cycle.

§1. Introduction.

By a *spatial embedding* of a graph G we mean an embedding of G in the 3-space $\boldsymbol{R}^3$, which is tame, i.e., which has a polygonal representation and we call the image of a spatial embedding a *spatial graph.* In this paper, we consider knots in spatial embeddings of graphs.

A cycle of a spatial graph is said to be *knotted* if it bounds no 2-cell in $\boldsymbol{R}^3$. A graph G is *self-knotted* if every spatial embedding of G contains a knotted cycle. Conway and Gordon [1] proved that the complete graph K_7 is self-knotted and showed a spatial embedding of K_7 which contains exactly one knotted Hamiltonian cycle. Since the graph obtained from K_7 by removing one edge from the knotted cycle has no knotted cycles, any graph with $n \leqq 7$ vertices except K_7 is not self-knotted. The spatial embedding of the complete bipartite graph $K_{4,5}$ shown in Figure 1 has no knotted cycles. In this paper, we prove the following.

THEOREM 1. *The complete bipartite graph $K_{5,5}$ is self-knotted.*

Sharper statements of Theorem 1 will be given in Theorem 2 and its corollary. For the definitions and elementary terminology, we refer to Harary [2] in graph theory and Rolfsen [4] in knot theory.

§2. Lemmas.

For a spatial embedding $f: G \rightarrow \boldsymbol{R}^3$ of a graph G, we may suppose

Received January 9, 1988
Revised September 6, 1988

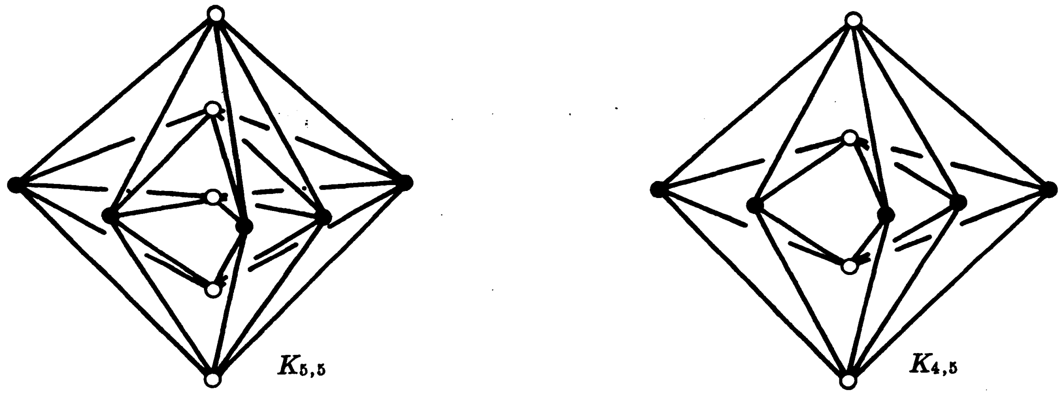


FIGURE 1

that, after a small ambient isotopy, the projection of $f(G)$ to the horizontal plane is regular, i.e., its multiple points are double points in the interiors of two edges of G. The projection of $f(G)$ indicating which edge is above and which edge is below at each double point is called the *diagram* of $f(G)$ and is denoted by G_f. We often consider a diagram of $f(G)$ as $f(G)$ itself. The following proposition is a standard fact in knot theory.

PROPOSITION 1. *For any spatial embeddings f and g of G, there exist a diagram G_f of $f(G)$ and a diagram G_g of $g(G)$ such that G_g is obtained from G_f by crossing-changes at some double points of G_f.*

Let A and B be disjoint oriented arcs or circles in $\boldsymbol{R}^3$. We define the *writhe* $\varepsilon(c)$ at each crossing c in a regular diagram of $A \cup B$ as shown in Figure 2, and we define $\zeta(A, B)=\sum_c \varepsilon(c)$, the summation being taken over all crossings c where A crosses "under" B in the diagram. If A and B are circles, then $\zeta(A, B)$ is equal to the *linking number* $\mathrm{lk}(A, B)$ of A and B (see Rolfsen [4, p. 132]).

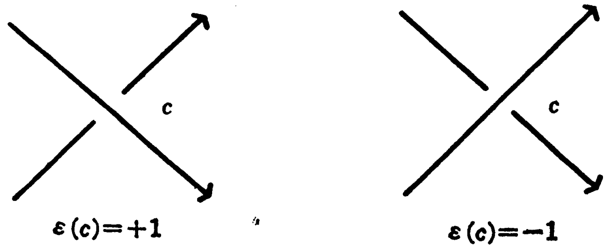
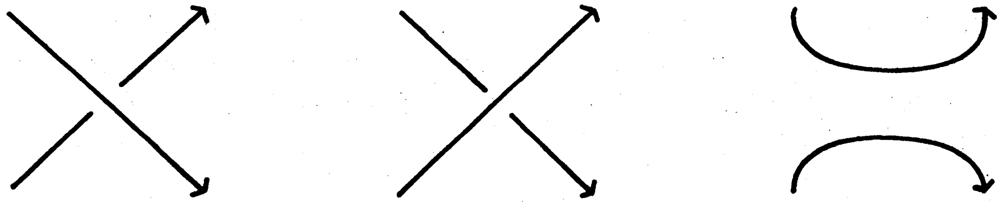


FIGURE 2

The *Conway polynomial* $\nabla_K(z)$ of an oriented knot or link K is the element of $\boldsymbol{Z}[z]$ defined recursively by

$$\nabla_{K_+}(z)-\nabla_{K_-}(z)=z\cdot\nabla_L(z)\ , \qquad \nabla_o(z)=1\ ,$$

where o is the trivial knot, and the oriented knots and links K_+, K_-, L have regular projections which are identical outside a small disk where they differ as indicated in Figure 3. Let $a_n(K)$ denote the coefficient of z^n in $\nabla_K(z)$. The following is shown by Kauffman [3].



K_+ K_- L

FIGURE 3

PROPOSITION 2 (Kauffman [3, Proposition 5.3 and p. 91]).

(1) *Let K^* be the knot obtained by reversing the orientation of an oriented knot K in $\boldsymbol{R}^3$, then*

$$\nabla_{K^*}(z)=\nabla_K(z), \text{ and in particular } a_2(K^*)=a_2(K) .$$

(2) *Let K_+ and K_- be the oriented knots and $L=L_1\cup L_2$ the oriented link in $\boldsymbol{R}^3$ which are identical except in a small ball where they differ as indicated in Figure 3. Then*

$$a_2(K_+)=a_2(K_-)+\mathrm{lk}(L_1, L_2) .$$

DEFINITION 1. Let Γ be a set of cycles in a graph G. For a spatial embedding f of G, define $\mu_f(G, \Gamma; n)\in \boldsymbol{Z}_n$ by

$$\mu_f(G, \Gamma; n)\equiv\sum_{\gamma\in\Gamma} a_2(f(\gamma)) \qquad (\mathrm{mod}\ n) ,$$

where $\sum_{\gamma\in\Gamma}$ is the summation over all cycles γ in Γ.

REMARK 1. By Proposition 2(1), $\mu_f(G, \Gamma; n)$ is well defined.

REMARK 2. Since the reduction of $a_2(K)$ modulo 2 gives the *Arf invariant* of K by Corollary 10.8 in Kauffman [3], $\mu_f(K_7, \Gamma; 2)$ is equal to Conway and Gordon's invariant σ in [1], where Γ is the set of all Hamiltonian cycles in K_7.

From now on, we consider directed graphs but any cycle below is an undirected one. Let E_1 and E_2 be two edges lying on a cycle γ. We say that E_1 and E_2 are *coherent* on γ if the directions of E_1 and E_2 induce the same orientation of γ.

For any distinct edges A, B and E, let n_1 denote the number of cycles in Γ containing $A \cup B \cup E$ on which A and E are coherent, and n_2 the number of cycles in Γ containing $A \cup B \cup E$ on which A and E is not coherent. Let $\nu_1(\Gamma; A, B, E)$ be $|n_1 - n_2|$.

For any pairs of non-adjacent edges $\{A, B\}$ and $\{E, F\}$, let Γ_1 denote the set of cycles in Γ along which the edges A, E, B, F lie in this order (see Figure 4). Let n_3 denote the number of cycles in Γ_1 on which even number of pairs of edges A, B, E, F are coherent, and n_4 the number of cycles in Γ_1 on which odd number of pairs of edges A, B, E, F are coherent. Let $\nu_2(\Gamma; A, B; E, F)$ be $|n_3 - n_4|$. Then we have:

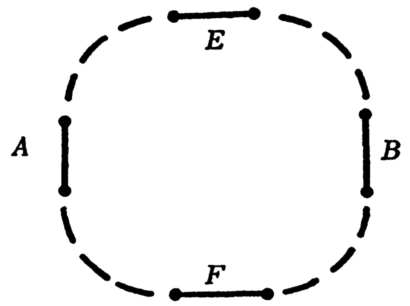


FIGURE 4

LEMMA 1. (1) *The number* $\nu_2(\Gamma; A, B; E, F)$ *is equal to the numbers* $\nu_2(\Gamma; A, B; F, E)$, $\nu_2(\Gamma; B, A; E, F)$ *and* $\nu_2(\Gamma; B, A; F, E)$.

(2) *The numbers* $\nu_1(\Gamma; A, B, E)$ *and* $\nu_2(\Gamma; A, B; E, F)$ *are independent of the direction of a graph G.*

PROOF. (1) It is clear by the definition of $\nu_2(\Gamma; A, B; E, F)$. (2) Any combination of reversing the direction of A, B, E, F fixes or interchanges the values of n_1 and n_2 and those of n_3 and n_4, respectively, and hence it does not change the values of $\nu_1(\Gamma; A, B, E) = |n_1 - n_2|$ and $\nu_2(\Gamma; A, B; E, F) = |n_3 - n_4|$. □

By (2) of Lemma 1, these two invariants $\nu_1(\Gamma; A, B, E)$ and $\nu_2(\Gamma; A, B; E, F)$ can be regarded as ones for undirected graphs. The following lemma for $n=2$ is essentially used by Conway and Gordon [1].

LEMMA 2. *Let* Γ *be a set of cycles in an undirected graph G. The invariant* $\mu_f(G, \Gamma; n)$ *does not depend on the spatial embedding f of G if the following two conditions hold:*

(1) *For any edges A, B, E such that A is adjacent to B, the reduction of* $\nu_1(\Gamma; A, B, E)$ *modulo n is equal to* 0.

(2) *For any pairs of non-adjacent edges* $\{A, B\}$ *and* $\{E, F\}$, *the reduction of* $\nu_2(\Gamma; A, B; E, F)$ *modulo n is equal to* 0.

PROOF. Suppose that G is a directed graph. We consider what happens to $\mu_f(G, \Gamma; n)$ under a crossing change on a diagram G_f of $f(G)$. The crossing change of an edge with itself can be always replaced by the crossing changes of distinct edges (see Figure 5). If we want to change a crossing of edges A and B, we may assume that G_f near the crossing point c is as shown in Figure 6 (a-1) or (b-1), possibly with the crossing reversed, according to whether A and B are adjacent or not. It suffices to show that μ_f is invariant under these two kinds of crossing changes by Proposition 1.

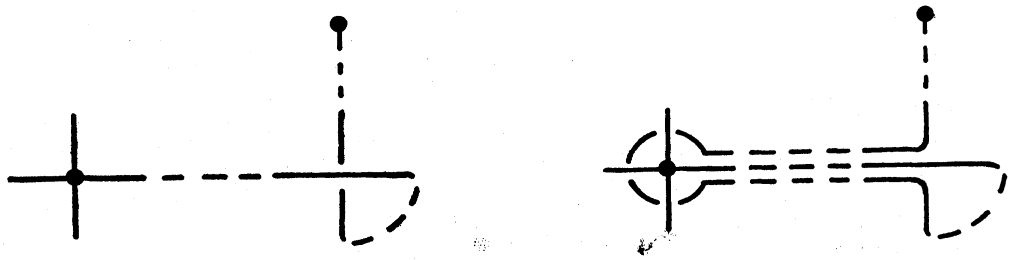
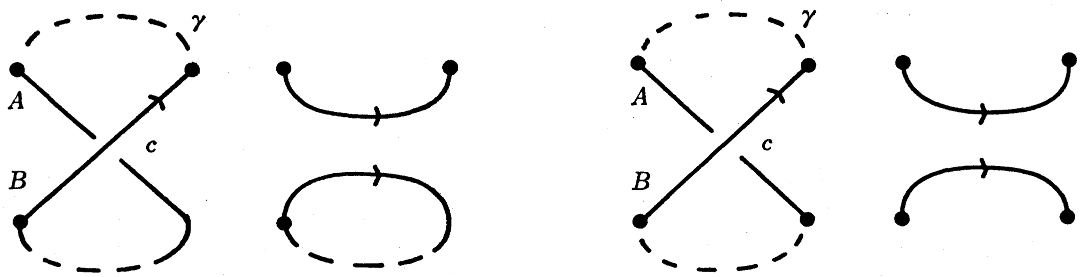


FIGURE 5



(a-1) (a-2) (b-1) (b-2)

FIGURE 6

Consider the spatial embedding g of G obtained from changing the crossing point in G_f. If a cycle γ in Γ does not contain both A and B, then the coefficient $a_2(\gamma)$ of z^2 in $\nabla_\gamma(z)$ is unchanged. We may assume that the orientation of $\gamma \supset A \cup B$ is induced from the direction of A. Let $\varepsilon(c)$ be the writhe of the crossing c, which depends on the orientation of γ but not on the direction of B, as shown in Figure 2, and $L = L_1 \cup L_2$ the oriented link determined by $f(\gamma)$ as shown in Figure 6. Let $\delta(\mu)$ be $\mu_f(G, \Gamma; n) - \mu_g(G, \Gamma; n)$, then we have by Proposition 2 (2)

$$\delta(\mu) \equiv \sum_{\gamma \in \Gamma, \gamma \supset A \cup B} \varepsilon(c) \cdot \mathrm{lk}(L_1, L_2) \pmod{n} .$$

To prove the invariance of $\mu_f(G, \Gamma; n)$, it suffices to show that $\delta(\mu) \equiv 0 \pmod{n}$ for the following two cases.

Case 1. The edge A is adjacent to B. Let $f_\gamma(E)$ be an edge $f(E)$ with direction induced by the orientation of γ, and $\zeta(f_\gamma(E), L_2)$ the total of the writhe of the crossings where $f_\gamma(E)$ crosses under L_2. Then

$$\begin{aligned}\delta(\mu) &\equiv \sum_{\gamma\in\Gamma,\gamma\supset A\cup B} \varepsilon(c)\cdot\Big(\sum_{E\subset\gamma-A\cup B} \zeta(f_\gamma(E), L_2)\Big)\\ &= \varepsilon(c)\cdot\sum_{E}\Big(\sum_{\gamma\in\Gamma,\gamma\supset A\cup B\cup E} \zeta(f_\gamma(E), L_2)\Big)\ ,\end{aligned}$$

where the summation $\sum_{E\subset\gamma-A\cup B}$ is taken over all edges $E\subset\gamma$, $E\neq A$, B in G, and $\sum_E$ is taken over all edges $E\neq A$, B in G. Let $f_\gamma^*(E)$ be the edge $f_\gamma(E)$ with direction reversed, then $\zeta(f_\gamma(E), L_2)=-\zeta(f_\gamma^*(E), L_2)$. Hence

$$\sum_{\gamma\in\Gamma,\gamma\supset A\cup B\cup E} \zeta(f_\gamma(E), L_2)=(n_1-n_2)\cdot\zeta(f(E), L_2) \pmod{n}\ .$$

If $\nu_1(\Gamma; A, B, E)=|n_1-n_2|\equiv 0 \pmod{n}$ for any three edges A, B and E, then $\delta(\mu)\equiv 0 \pmod{n}$.

Case 2. The edge A is not adjacent to B. In this case, the oriented link $L=L_1\cup L_2$ is as indicated in Figure 6 (b-2). Then we have;

$$\begin{aligned}\delta(\mu) &\equiv \sum_{\gamma\in\Gamma,\gamma\supset A\cup B}\ \sum_{E,F\subset\gamma} \varepsilon(c)\cdot\zeta(f_\gamma(E), f(F))\\ &=\sum_{E,F}\sum_{\gamma\in\Gamma_1} \varepsilon(c)\cdot\zeta(f_\gamma(E), f(F))\\ &=\sum_{E,F}(n_3-n_4)\cdot\zeta(f(E), f(F)) \pmod{n}\ .\end{aligned}$$

For each summation, E and F run over all distinct pairs of edges in G with $\{A, B\}\cap\{E, F\}=\emptyset$, but they are assumed to lie along γ in the order as shown in Figure 4 if γ contains them. Therefore for any pairs of disjoint edges $\{A, B\}$ and $\{E, F\}$, if $\nu_2(\Gamma; A, B; E, F)\equiv|n_3-n_4|\equiv 0 \pmod{n}$ then $\delta(\mu)\equiv 0 \pmod{n}$. □

§3. Proof of the theorem.

Let $G-\{e\}$ denote a graph obtained from a graph G by removing an edge and let $K_{l,m,n}$ denote a complete tripartite graph with part sizes l, m, n.

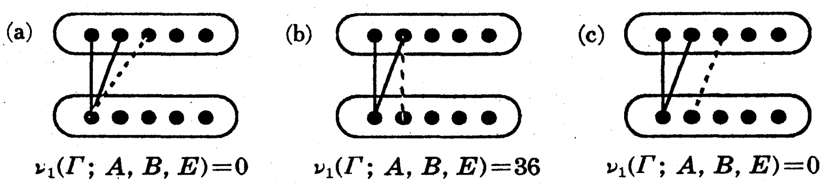
THEOREM 2. *Let G be one of the graphs $K_{5,5}-\{e\}$, $K_{4,4,1}$ and $K_{m,m}$ ($m\geqq 5$), and Γ the set of all Hamiltonian cycles in G. For any spatial embedding f of G, $\mu_f(G, \Gamma; 2)=0$ and $\mu_f(K_{5,5}, \Gamma; 4)=2$.*

PROOF. Let $V_1=\{1, 2, 3, 4, 5\}$ and $V_2=\{\hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5}\}$ be the canonical partite sets of $K_{5,5}$ and assume that each edge of $K_{5,5}$ is directed from V_1 to V_2 (see Figure 1). We shall evaluate $\nu_1(\Gamma; A, B, E)$ and $\nu_2(\Gamma; A, B; E, F)$

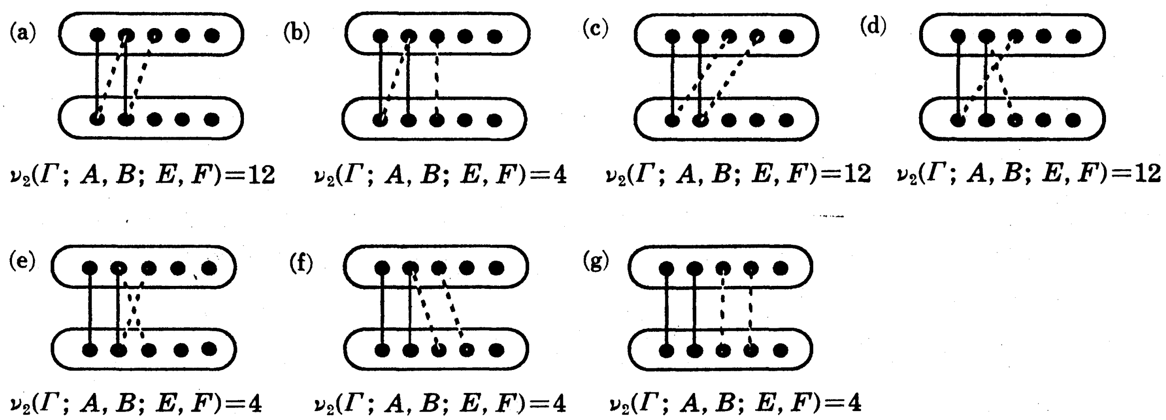
to show the invariance of $\mu_f(G, \Gamma; n)$.

Let A, B, E be edges as in (1) of Lemma 2. If A, B, E have a common vertex then the number of cycles in Γ containing $A \cup B \cup E$ is equal to 0. Hence $\nu_1(\Gamma; A, B, E)=0$. If E is adjacent to precisely one of A and B, say A (resp. B), then the number of cycles containing $A \cup B \cup E$ is equal to $3! \times 3! = 36$, $n_1 = 0$ and $n_2 = 36$ (resp. $n_1 = 36$ and $n_2 = 0$). Hence $\nu_1(\Gamma; A, B, E) = 36$. If E is adjacent to neither A nor B, then the number of cycles containing $A \cup B \cup E$ is equal to $72 = 3! \times 2! \times 6$ and $n_1 = n_2 = 36$. Hence $\nu_1(\Gamma; A, B, E) = 0$. Therefore, in each case, the reduction of $\nu_1(\Gamma; A, B, E)$ modulo 4 is equal to 0.

Let $\{A, B\}$ and $\{E, F\}$ be pairs of non-adjacent edges as in (2) of Lemma 2. We may assume that $A=(1\hat{1})$, $B=(2\hat{2})$. We consider the other pairs of edges $\{E, F\}$. By the condition as shown in Figure 4 and the fact described in Lemma 1 (1), it suffices to examine only the cases in which E and F are: (a) $(2\hat{1})$ $(3\hat{2})$, (b) $(2\hat{1})$ $(3\hat{3})$, (c) $(3\hat{2})$ $(4\hat{1})$, (d) $(2\hat{3})$ $(3\hat{1})$, (e) $(2\hat{3})$ $(3\hat{2})$, (f) $(2\hat{3})$ $(3\hat{4})$, (g) $(3\hat{3})$ $(4\hat{4})$. (See Figure 7.)



The case of Lemma 2 (1)



The case of Lemma 2 (2)

FIGURE 7

Let $n(A, B; E, F)$ be the number of Hamiltonian cycles in Γ containing $A \cup B \cup E \cup F$. It is a routine to determine the values of $n(A, B; E, F)$,

n_3 and n_4 for each case.

(a) $n(A, B; E, F)=3!\times 2!=12$, $n_3=12$ and $n_4=0$. Hence $\nu_2(\Gamma; A, B; E, F)=|n_3-n_4|=12$.

(b) $n(A, B; E, F)=20$, $n_3=8$ and $n_4=12$. Hence $\nu_2(\Gamma; A, B; E, F)=4$.

(c) $n(A, B; E, F)=12$, $n_3=12$ and $n_4=0$. Hence $\nu_2(\Gamma; A, B; E, F)=12$.

(d) $n(A, B; E, F)=12$, and $n_3=12$. Hence $\nu_2(\Gamma; A, B; E, F)=12$.

(e) $n(A, B; E, F)=20$, and $n_3=8$. Hence $\nu_2(\Gamma; A, B; E, F)=4$.

(f) $n(A, B; E, F)=20$, and $n_3=8$. Hence $\nu_2(\Gamma; A, B; E, F)=4$.

(g) $n(A, B; E, F)=20$, and $n_3=12$. Hence $\nu_2(\Gamma; A, B; E, F)=4$.

Therefore the reduction of $\nu_2(\Gamma; A, B; E, F)$ modulo 4 is equal to 0.

We can divide the set Γ of 1440 Hamiltonian cycles of $K_{5,5}$ into ten disjoint subsets of 144 cycles so that cycles in each subset contains the following two edges, respectively: (1) $(1\hat{1})$ $(\hat{1}2)$, (2) $(1\hat{1})$ $(\hat{1}3)$, (3) $(1\hat{1})$ $(\hat{1}4)$, (4) $(1\hat{1})$ $(\hat{1}5)$, (5) $(2\hat{1})$ $(\hat{1}3)$, (6) $(2\hat{1})$ $(\hat{1}4)$, (7) $(2\hat{1})$ $(\hat{1}5)$, (8) $(3\hat{1})$ $(\hat{1}4)$, (9) $(3\hat{1})$ $(\hat{1}5)$, (10) $(4\hat{1})$ $(\hat{1}5)$. (See Figure 1.)

For the spatial embedding of $K_{5,5}$ in Figure 1, there is a homeomorphism h: $\boldsymbol{R}^3\to\boldsymbol{R}^3$ such that $h(K_{5,5})=K_{5,5}$, $h(\hat{i})=\hat{i}$ and $h(i)=i+1 \pmod 5$ for vertices $\hat{i}$ and i. So we consider the knottedness of cycles in the only two sets (1) and (2). We note that if the number of crossing of a cycle is less than 3, then the cycle can not be knotted. Then we find that every cycle in the set (1) is a trivial knot, and that the set (2) contains exactly two knotted cycles which are trefoil knots such that they are the mirror images of each other. Hence the embedding of $K_{5,5}$ shown in Figure 1 contains exactly ten Hamiltonian cycles which are trefoil knots. Since the Conway polynomial of the trefoil knot is z^2+1, $\mu_f(K_{5,5}, \Gamma; 4)=2$ and the proof is complete.

The cases for graphs $K_{5,5}-\{e\}$, $K_{4,4,1}$ and $K_{m,m}$ $(m\geqq 5)$ can be proved by the same method. □

We note that Theorem 2 contains Theorem 1, for if there were an embedding of $K_{5,5}$ such that every cycle of the embedding was a trivial knot, then $\mu_f(K_{5,5}, \Gamma; 4)$ would be 0.

By Remark 2, we have the following:

COROLLARY. *Every spatial embedding of the graphs* $K_{5,5}-\{e\}$, $K_{4,4,1}$ *and* $K_{m,m}$ $(m\geqq 5)$ *has even number of Hamiltonian cycles whose Arf invariants are one.*

ACKNOWLEDGEMENT. The author would like to express her hearty gratitude to Professor Shin'ichi Suzuki for valuable comments and suggestions.

References

[1] J. H. Conway and C. McA. Gordon, Knots and links in spatial graphs, J. Graph Theory, **7** (1983), 445–453.
[2] F. Harary, *Graph Theory*, Addison-Wesley, 1969.
[3] L. H. Kauffman, *Formal Knot Theory*, Math. Notes, **30**, Princeton Univ. Press, 1983.
[4] D. Rolfsen, *Knots and Links*, Math. Lecture Ser., **7**, Publish or Perish Inc., 1976.

Present Address:
Software Engineering Laboratory, NTT Software Laboratories
1-9-1 Kohnan, Minato-ku, Tokyo 108, Japan