Tokyo J. Math.
Vol. 11, No. 2, 1988

On Sullivan's Vanishing Cycles in Codimension-One Foliations

Shigeaki MIYOSHI

Waseda University

(Communicated by S. Suzuki)

Introduction.

A Sullivan's vanishing cycle is a cycle in a leaf of a foliation which is in a sense essential in the leaf and inessential if it is displaced to nearby leaves. This notion is first introduced by Novikov in his celebrated work [N] in order to prove his closed leaf theorem. In [Su] Sullivan defined a notion of higher dimensional vanishing cycles in foliations of arbitrary codimensions which includes Novikov's one as 1-dimensional case. Sullivan showed that a Sullivan's vanishing cycle yields a non-trivial foliation cycle and gave an alternative proof of Novikov's closed leaf theorem for S^3. In the previous paper [Miy] we gave a sufficient condition for the existence of Sullivan's vanishing cycles in codimension-one foliations and also showed a closed leaf theorem which in a sense generalizes Novikov's closed leaf theorem to the higher dimensional case.

In the present paper we will study topological aspects of codimension-one foliations which the existence of Sullivan's vanishing cycles yields. Our main theorem (Theorem B) gives some necessary and sufficient conditions for the existence of a Sullivan's vanishing cycle in case that there is no Novikov's vanishing cycle. Although Theorem A is the main theorem of [Miy], the proof in [Miy] is rather sketchy and it plays a key role in the proof of Theorem B. Therefore for the completeness we will give a complete proof of Theorem A which is also improved compared with the one given in [Miy]. Theorem C asserts that a Novikov's vanishing cycle yields a higher dimensional singular manifold chain bounded by a tangential cycle, which is the partial converse of Theorem A.

Contents are as follows: In Section 1 we state the results. We study a pull back of a foliation without Sullivan's vanishing cycles in Section 2 and by applying the results in part we study a foliation without

Received December 21, 1987
Revised May 25, 1988

Novikov's vanishing cycles in Section 3. In Section 4 we consider the case that a foliation has a Novikov's vanishing cycle. Theorems A, B and C are proved in Sections 2, 3 and 4 respectively.

The author wishes to thank Science and Engineering Research Laboratory of Waseda University which partially supports him as a post-doctoral fellow during the preparation of the paper.

§1. Statement of results.

Let M be an n-dimensional smooth manifold and $\mathscr{F}$ a codimension-one C^2 foliation on M. We will say that (C, f_C) (or simply f_C) is a *k-chain* in M if C is a finite oriented homogeneously k-dimensional simplicial complex and $f_C: C \to M$ is a continuous map from the *space* of C to M (see for example Spanier [Sp]). Here, by abuse of notation, we denote a simplicial complex and its space by a same letter. In addition, we call a k-chain (C, f_C) in M with $\partial C=0$ a *k-cycle* in M, where ∂C denotes the boundary of C in the homological sense. A k-chain in M is *tangential* to $\mathscr{F}$ if there exists a leaf of $\mathscr{F}$ which contains $f_C(C)$. For a tangential chain (C, f_C) the leaf which contains $f_C(C)$ is called the *support leaf*. A k-cycle (C, f_C) in M is said to be *$\mathscr{F}$-homologous to zero* if it is tangential and there exists a simply-connected tangential $(k+1)$-chain bounded by (C, f_C), i.e., if $f_C(C)$ is contained in a leaf L of $\mathscr{F}$ and there exists a $(k+1)$-chain (W, f_W) in L such that W is simply-connected with $\partial W=C$ and f_W is an extension of f_C. Note that if a k-cycle (C, f_C) in M is $\mathscr{F}$-homologous to zero by a $(k+1)$-chain (W, f_W), the induced germ of foliation around W is trivial. For a k-chain f_W we denote by ∂f_W a $(k-1)$-cycle $f_W|_{\partial W}$.

DEFINITION. A tangential cycle (C, f_C) to $\mathscr{F}$ is said to be a *Sullivan's vanishing cycle* if C is connected and there exists a homotopy of cycles $f: C\times[0, 1]\to M$ such that

1) for each $t\in[0, 1]$, $f_t=f|_{C\times\{t\}}$ is tangential, i.e. $f(C\times\{t\})$ is contained in a leaf L_t of $\mathscr{F}$,
2) for each $x\in C$, $f|_{\{x\}\times[0,1]}$ is a transverse arc,
3) $f_0=f_C$,
4) (C_0, f_0) is not $\mathscr{F}$-homologous to zero, and
5) (C_t, f_t) is $\mathscr{F}$-homologous to zero for $t>0$.

REMARK. Throughout this paper we suppose that a homotopy of tangential chains satisfies the condition 2) of this definition unless otherwise stated.

For a 1-cycle, to bound a simply-connected 2-chain and to be null-homotopic are equivalent. Therefore a Novikov's vanishing cycle [N] is just a 1-dimensional Sullivan's vanishing cycle. The following theorem is one of the conditions for the existence of a Novikov's vanishing cycle:

THEOREM 1 (Novikov [N]). *Let M be an n-dimensional smooth manifold and $\mathscr{F}$ a codimension-one C^0 foliation on M. Then the following conditions are equivalent*:

1) *There exists a leaf L of $\mathscr{F}$ such that the homomorphism $\pi_1(L)\rightarrow\pi_1(M)$ induced by the inclusion is not injective.*

2) *There exists a Novikov's vanishing cycle in $\mathscr{F}$.*

Novikov also gives some other (sufficient) conditions in [N] and proves the following closed leaf theorem in dimension three:

THEOREM 2 (Novikov [N]). *Assume the dimension of M is three and M is compact. If there exists a Novikov's vanishing cycle in $\mathscr{F}$, then $\mathscr{F}$ has a compact leaf.*

In fact, the compact leaf in Theorem 2 is the border leaf of a Reeb component in $\mathscr{F}$. For the details, the reader is referred to Novikov [N], also Haefliger [H1] and Camacho and Neto [C-N].

Now we state our key theorem which is given in [Miy].

THEOREM A. *Let M be an n-dimensional smooth manifold and $\mathscr{F}$ a codimension-one C^2 foliation on M. Suppose there exists a tangential m-cycle $f_Z\colon Z\rightarrow M$ $(1\leqq m\leqq n-2)$ which satisfies the following conditions;*

1) *Z is connected,*

2) *f_Z is not $\mathscr{F}$-homologous to zero, and*

3) *there exists an $(m+1)$-chain $f_X\colon X\rightarrow M$ such that X is a simply-connected manifold and f_X is bounded by f_Z, i.e. $\partial X=Z$ and $\partial f_X=f_Z$.*

Then there exists a Novikov's vanishing cycle or an m-dimensional Sullivan's vanishing cycle in $\mathscr{F}$.

As a corollary we have the following closed leaf theorem:

COROLLARY A. *Assume M is compact and there is no Novikov's vanishing cycle in $\mathscr{F}$. If there exists a leaf L of $\mathscr{F}$ such that the homomorphism $\pi_{n-2}(L)\rightarrow\pi_{n-2}(M)$ induced by the inclusion is not injective and the kernel has an element which is not $\mathscr{F}$-homologous to zero, then $\mathscr{F}$ has a compact leaf.*

REMARK. In Corollary A if we replace the assumption on a leaf L with the assumption of Theorem A with $m=n-2$ then we have also the same conclusion.

A codimension-one foliation without Novikov's vanishing cycles is characterized by Theorem 1. Next we state results about foliations without Sullivan's vanishing cycles. For a pair of topological spaces (X, A) we denote by $\Omega_m(X, A)$ the m-th bordism homology group of (X, A) and by $\mu: \Omega_m(X, A) \to H_m(X, A: \boldsymbol{Z})$ the natural homomorphism defined by $\mu([V, f]) = f_*([V, \partial V])$, where $[V, f]$ is a bordism class of an m-dimensional singular manifold $f: (V, \partial V) \to (X, A)$ and $[V, \partial V]$ denotes the fundamental class of V (cf. Conner and Floyd [C-F]). The following theorem gives characterizations of a codimension-one foliation without Sullivan's vanishing cycles.

THEOREM B. *Suppose that M is an n-dimensional smooth manifold and $\mathscr{F}$ is a codimension-one C^2 foliation on M which has no Novikov's vanishing cycle. Let $\tilde{M}$ be the universal covering of M and $\tilde{\mathscr{F}}$ the lifted foliation on $\tilde{M}$. Suppose that m is an integer such that $2 \leqq m \leqq n-2$. Then the following conditions are equivalent:*

1) *there exists an m-dimensional Sullivan's vanishing cycle in $\mathscr{F}$,*

2) *there exist a leaf $\tilde{L}$ of $\tilde{\mathscr{F}}$ and a homology class of infinite order in the kernel of the homomorphism $H_m(\tilde{L}: \boldsymbol{Z}) \to H_m(\tilde{M}: \boldsymbol{Z})$ induced by the inclusion,*

3) *there exist a leaf $\tilde{L}$ of $\tilde{\mathscr{F}}$ and a homology class of infinite order in the image of the homomorphism $\mu \cdot \partial: \Omega_{m+1}(\tilde{M}, \tilde{L}) \to H_m(\tilde{L}: \boldsymbol{Z})$ where ∂ denotes the boundary homomorphism,*

4) *there exists a leaf $\tilde{L}$ of $\tilde{\mathscr{F}}$ such that* $\mathrm{Im}(\mu \cdot \partial: \Omega_{m+1}(\tilde{M}, \tilde{L}) \to H_m(\tilde{L}: \boldsymbol{Z})) \neq 0$.

We have some consequences of Theorem B as follows:

COROLLARY B. *Suppose that $H_m(\tilde{M}: \boldsymbol{Z})$ is a torsion group for any $m \in \boldsymbol{Z}$ such that $2 \leqq m \leqq n-2$, and that $\mathscr{F}$ has no Sullivan's vanishing cycle. Then for any leaf $\tilde{L} \in \tilde{\mathscr{F}}$, $H_m(\tilde{L}: \boldsymbol{Z})$ is also a torsion group* $(2 \leqq m \leqq n-2)$.

COROLLARY C. *Suppose that M is a closed 4-manifold with the contractible universal covering and that $\mathscr{F}$ has no Novikov's vanishing cycle. Then every leaf $L \in \mathscr{F}$ has the contractible universal covering or there exists a compact leaf in $\mathscr{F}$.*

COROLLARY D. *The following conditions are equivalent:*

1) *$\mathscr{F}$ has no m-dimensional Sullivan's vanishing cycle for m less than three,*

2) *for any leaf* $L \in \mathscr{F}$ *the homomorphisms* $\pi_m(L) \to \pi_m(M)$ *induced by the inclusion are injective for* m *less than three.*

In case there exists a Novikov's vanishing cycle we have the following result which is the partial converse of Theorem A.

THEOREM C. *Let* M *be an* n*-dimensional smooth manifold and* $\mathscr{F}$ *a codimension-one* C^0 *foliation on* M. *Suppose that there exists a Novikov's vanishing cycle in* $\mathscr{F}$. *Then for any* $m \in \boldsymbol{Z}$ *with* $3 \leqq m \leqq n-2$ *there exists an* $(m+1)$*-dimensional chain* $f_X: X \to M$ *such that* X *is a* 1*-connected* $(m+1)$*-manifold and* ∂f_X *is tangential and not* $\mathscr{F}$*-homologous to zero. In fact, we can choose* $D^2 \times S^{m-1}$ *as* X.

In case $m=2$ the following simple example shows that the assertion corresponding to Theorem C does not hold.

EXAMPLE. *There exists a codimension-one* C^∞ *foliation on* $S^1 \times S^2 \times S^1$ *which has a Novikov's vanishing cycle and admits no* 3*-chain* $f_X: X \to S^1 \times S^2 \times S^1$ *such that* X *is a* 1*-connected* 3*-manifold and* ∂f_X *is tangential but not* $\mathscr{F}$*-homologous to zero.*

§2. Proof of Theorem A.

First, we may assume $n>3$ and $m>1$ since in case $n=3$ or $m=1$ Theorem A holds by Theorem 1. Also, without loss of generality, we may assume M is orientable and $\mathscr{F}$ is transversely orientable by passing to a (at most) four-fold orientable covering if necessary (cf. Lemma 3.1).

We will give some lemmas and then apply them to prove Theorem A.

2.a. A pull back of a foliation without Novikov's vanishing cycles.

First by applying Thom's jet transversality theorem ([T]) repeatedly we have the following approximation lemma:

LEMMA 2.1. *Suppose* V *is a smooth manifold and* $g: V \to M$ *is a continuous map such that for each component* $\partial_i V$ *of* ∂V, *the image* $g(\partial_i V)$ *is contained in a leaf* $L_i \in \mathscr{F}$. *Then we can perturb* g *to obtain a smooth map* $f: V \to M$ *such that*

1) f *is an approximation of* g,

2) $f(\partial_i V) \subset L_i$ *for each* i,

3) f *is in general position with respect to* $\mathscr{F}$, *i.e.* f *is transverse to* $\mathscr{F}$ *except finitely many points where* f *is in contact with leaves of* $\mathscr{F}$ *generically.*

REMARK. We need the smoothness of class C^2 for the foliation only in this part of the proof.

Now suppose V is an $(m+1)$-dimensional compact oriented manifold $(m>1)$ and $f\colon V\to M$ is an approximated smooth map by Lemma 2.1. Then the pull back of $\mathscr{F}$ by f, denoted by $f^*\mathscr{F}$, is a codimension-one Haefliger structure on V whose singular points are of Morse type and lie in the interior of V. From now on we will analyze the structure of $(V, f^*\mathscr{F})$. We denote by $\Sigma(f^*\mathscr{F})$ the set of all singular points of $f^*\mathscr{F}$ and by $\mathscr{G}$ the restriction of $f^*\mathscr{F}$ to $V-\Sigma(f^*\mathscr{F})$. Then $\mathscr{G}$ is a codimension-one non-singular foliation on $V-\Sigma(f^*\mathscr{F})$. Recall that $\Sigma(f^*\mathscr{F})$ consists of finitely many points in the interior of V. We define a *leaf* of $f^*\mathscr{F}$ to be a path-connected component of the preimage of a leaf of $\mathscr{F}$ by f. Then a leaf of $f^*\mathscr{F}$ with singularity is a pinched codimension-one submanifold (i.e. a submanifold with Morse type singular points) of V or a singular point of index 0 or $m+1$, i.e. a local maximum or a local minimum. For a leaf F of $f^*\mathscr{F}$ we denote by L_F the leaf of $\mathscr{F}$ which contains $f(F)$.

LEMMA 2.2. *If $\mathscr{F}$ has no Novikov's vanishing cycle and V is 1-connected then the pull back Haefliger structure $f^*\mathscr{F}$ on V is without holonomy. Precisely, for any map $h\colon S^1\times[0, 1[\to V$ with the property that* (1) *$h^{-1}(\Sigma(f^*\mathscr{F}))$ is contained in $S^1\times\{0\}$,* (2) *h restricted to the complement of $h^{-1}(\Sigma(f^*\mathscr{F}))$ is a smooth immersion transverse to $\mathscr{G}$, and* (3) *h restricted to $S^1\times\{0\}$ is a loop in a leaf of $f^*\mathscr{F}$, there exists a small collar N of $S^1\times\{0\}$ in $S^1\times[0, 1[$ such that $h^*(f^*\mathscr{F})|_N$ is a foliation by circles of class C^2 except at $h^{-1}(\Sigma(f^*\mathscr{F}))$.*

PROOF. Otherwise the map $f\cdot(h|_{S^1\times\{0\}})\colon S^1\to M$ is a loop in a leaf L of $\mathscr{F}$ with non-trivial (C^0) holonomy in $\mathscr{F}$. In particular $f\cdot(h|_{S^1\times\{0\}})$ is essential in L. On the other hand, since $f\cdot(h|_{S^1\times\{0\}})$ is a map via 1-connected manifold V, it is inessential in M. Therefore the loop represents a non-trivial element in the kernel of the homomorphism $\pi_1(L)\to\pi_1(M)$ induced by the inclusion. By Theorem 1 this contradicts the assumption that $\mathscr{F}$ has no Novikov's vanishing cycle.

LEMMA 2.3. *If $\mathscr{F}$ has no Novikov's vanishing cycle and V is 1-connected, then each leaf of $f^*\mathscr{F}$ is compact.*

PROOF. If there is a non-compact leaf F in $f^*\mathscr{F}$, then there is a loop γ in V such that $\gamma=\gamma_F*\gamma_{tr}$ where γ_F is a path in F and γ_{tr} is an arc transverse to $\mathscr{G}$. Passing to $(M, \mathscr{F})$ we have a loop with the same property. Precisely, we have $f\cdot\gamma=(f\cdot\gamma_F)*(f\cdot\gamma_{\mathrm{tr}})$ where $f\cdot\gamma_F$ is a path in a leaf L_F of $\mathscr{F}$ and $f\cdot\gamma_{\mathrm{tr}}$ is an arc transverse to $\mathscr{F}$. As is well known, we can perturb $f\cdot\gamma$ in order to obtain a closed transversal curve. Since it is homotopic to $f\cdot\gamma$ and $f\cdot\gamma$ is null homotopic in M, the closed transversal curve is also null homotopic in M. Therefore we can find a Novikov's vanishing cycle by a result of Novikov [N] and this is a contradiction.

In the sequel we assume that $\mathscr{F}$ has no Novikov's vanishing cycle and V is 1-connected. By Lemmas 2.2 and 2.3 we may consider that a leaf of $f^*\mathscr{F}$ is a (connected component of) level surface of a Morse function on V and a leaf with singularity is a (component of) critical level surface.

Now we describe the global structure of $f^*\mathscr{F}$. For that purpose we need some definitions. A leaf F of $f^*\mathscr{F}$ is said to be a *separatrix* if $F-\Sigma(f^*\mathscr{F})$ is non-compact or empty. A separatrix is a leaf with singularity (a pinched leaf) or a singular point of index 0 or $m+1$. For a separatrix F we call the closure of a component of $F-\Sigma(f^*\mathscr{F})$ a *component* of F. Note that for our purpose it is sufficient to consider the case that $\dim F=m$ is greater than one. We define a *subseparatrix* to be a union of components of a separatrix which is connected. We denote by Ω the complement of the union of all separatrices of $f^*\mathscr{F}$. Then Ω is connected if and only if f is transverse to $\mathscr{F}$ except at most two points. In this case Ω is diffeomorphic to $F\times I$ and $f^*\mathscr{F}|_\Omega$ is the product foliation $\{F\times\{t\}\}_{t\in I}$, where $I=[0, 1]$ or $]0, 1[$ and F is a component of ∂V or a spherical leaf. Hence we assume that Ω is not connected in the sequel. The following lemma is easily seen by Lemma 2.2 and Lemma 2.3, and therefore we omit the proof.

LEMMA 2.4. *Suppose Λ is any component of Ω. Then we have the following*:

1) *If a boundary compoent $\partial_i V$ is contained in Λ then $(\Lambda, \partial_i V)$ is diffeomorphic to $(\partial_i V\times[0, 1[, \partial_i V\times\{0\})$.*

2) *If $\partial V\cap\Lambda=\varnothing$ then there is a non-singular leaf F of $f^*\mathscr{F}$ in Λ such that Λ is diffeomorphic to $F\times]0, 1[$.*

3) *Λ is a $\mathscr{G}$-saturated set and $f^*\mathscr{F}|_\Lambda$ is the product foliation $\{F\times\{t\}\}_{t\in I}$ by identifying Λ with $F\times I$ by means of the above diffeomorphism, where F is the leaf in Λ and $I=[0, 1[$ or $]0, 1[$.*

4) *Let $\partial\Lambda$ denote $\mathrm{Cl}_V(\Lambda)-\Lambda$, where Cl_V denotes the closure in V. Then a component of $\partial\Lambda$ is a subseparatrix.*

We fix a transversal orientation of $\mathscr{F}$. We associate a linear graph

Γ (i.e. one-dimensional complex) with $f^*\mathscr{F}$ as follows: We assign an edge, a white vertex and a black vertex to a component of Ω, $\partial_i V$ and a separatrix, respectively. Also we assign to each edge the orientation determined by the transverse orientation of the corresponding component of Ω. The incident relation is defined by 4) of Lemma 2.4. Precisely, suppose that an edge e is assigned to a component Λ of Ω. Then each vertex of ∂e corresponds to the separatrix which contains a connected component of $\partial\Lambda$ in case of 2). In case of 1) one of the vertices ∂e corresponds to a boundary component $\partial_i V$ and the other to the separatrix containing $\partial\Lambda$. It is not hard to prove the following.

LEMMA 2.5. *The graph Γ satisfies the following*:

1) *Γ is a finite oriented tree,*

2) *Γ is homeomorphic to the leaf space $V/f^*\mathscr{F}$,*

3) *an initial (resp. terminal) black vertex corresponds to a singular point of index 0 (resp. $m+1$),*

4) *the valency (i.e. the number of edges which are incident with the vertex) of a black vertex is the number of components of the corresponding separatrix plus one.*

By abuse of language we will identify a point in Γ with a leaf of $f^*\mathscr{F}$.

For two points x and y of an edge e of Γ we denote by $[x, y]$ the closure of the connected component of $e-\{x, y\}$ which contains x and y. We choose an appropriate orientation and will identify it to the interval $[0, 1]$ if necessary. Intervals $[x, y[$, $]x, y]$ and $]x, y[$ in e can be defined naturally. Let $\pi: V\to\Gamma$ denote the natural projection. The following lemma describes the correspondence between Γ and $f^*\mathscr{F}$ more precisely:

LEMMA 2.6. *Let v be a black vertex of Γ and S the separatrix of $f^*\mathscr{F}$ which corresponds to v. Suppose that an edge e of Γ is incident with v and x is an interior point of e. Then the following hold*:

1) *the edge e determines a subseparatrix S_e of S in such a way that a non-singular leaf F corresponding to x approximates S_e, that is, there is an identification map $p: F\to S_e$ such that $p|_{F-p^{-1}(\Sigma(f^*\mathscr{F}))}$ is a smooth embedding onto $S_e-\Sigma(f^*\mathscr{F})$ and for any $y\in S_e\cap\Sigma(f^*\mathscr{F})$ the preimage $p^{-1}(y)$ is a sphere in F or a point,*

2) *the frontier of $\pi^{-1}([x, v[)$ in V is $F\amalg S_e$, where $\amalg$ denotes the disjoint union,*

3) *$(\mathrm{Cl}_V(\pi^{-1}([x, v[)); F, S_e)$ is homeomorphic to a mapping cylinder $(M_p; F, S_e)$ of $p: F\to S_e$, where $M_p=F\times[0, 1]\cup_{p\times 1} S_e$ and $p\times 1: F\times\{1\}\to S_e$ is the identification map,*

4) *$f|_{S_e}: S_e \to L_S$ is $\mathscr{F}$-homologous to zero if and only if $(f|_{S_e})\cdot p: F \to L_S$ is so.*

PROOF. By Lemma 2.4 $(\pi^{-1}([x, v[), \pi^{-1}(x))$ is diffeomorphic to $(F\times[0, 1[, F\times\{0\})$. Note that $F=\pi^{-1}(x)$. By definition the frontier $\mathrm{Fr}_V(\pi^{-1}([x, v[))=\mathrm{Cl}_V(\pi^{-1}([x, v[))-\mathrm{Int}(\pi^{-1}([x, v[))$, therefore $\mathrm{Fr}_V(\pi^{-1}([x, v[))-F\subset\pi^{-1}(v)=S$. As noted before, F and S are considered to be components of level surfaces of a Morse function on V. Since we may consider that F is sufficiently near S, there are a subseparatrix S_e of S and a map $p: F\to S_e$ as desired in 1), 2) and 3) (cf. [Mil]).

To prove 4) assume that $(f|_{S_e})\cdot p: F\to L_S$ is $\mathscr{F}$-homologous to zero. Suppose $g: W\to L_S$ is a chain such that $\partial g=(f|_{S_e})\cdot p$ and W is 1-connected. Let M_p be the (abstract) mapping cylinder of $p: F\to S_e$. Then we may define a map $h: M_p\to L_S$ by $h|_{F\times\{t\}}=(f|_{S_e})\cdot p$ and $h|_{S_e}=f|_{S_e}$. We set $W'=W\cup_F M_p$ and $g'=g\cup_F h: W'\to L_S$. Then we have $\partial W'=S_e$ and $\partial g'=f|_{S_e}$. It can be seen as in the proof of Lemma 2.2 that $(f|_{S_e})_*: \pi_1(S_e)\to\pi_1(L_S)$ is trivial. Since S_e is a deformation retract of M_p, $h_*: \pi_1(M_p)\to\pi_1(L_S)$ is also trivial. Therefore by applying Van Kampen's theorem the homomophism $g'_*: \pi_1(W')\to\pi_1(L_S)$ is seen to be trivial. Then by the following Sublemma we can modify g' to obtain a tangential $(m+1)$-chain $\hat{g}: \hat{W}\to L_S$ such that $\partial\hat{g}=\partial g'$ and $\hat{W}$ is 1-connected, which implies that $f|_{S_e}$ is $\mathscr{F}$-homologous to zero. Now the converse is clear by the same construction with turning upside down. In fact, the complex obtained by attaching the mapping cylinder is 1-connected in this case. This proves the lemma.

SUBLEMMA. *Let $g: W\to L$ be a connected $(m+1)$-chain. If the induced homomorphism $g_*: \pi_1(W)\to\pi_1(L)$ is trivial, then there is an $(m+1)$-chain $\hat{g}: \hat{W}\to L$ such that $\partial\hat{g}=\partial g$ and $\hat{W}$ is 1-connected.*

PROOF. Since W is compact we can choose a finite set of generators of $\pi_1(W)$. Represent each generator by a simplicial loop $l_i: S^1\to W$ $(i=1, 2, \cdots, k)$. By the assumption, each loop $g\cdot l_i$ in L bounds a disk $d_i: D^2\to L$. Let $S^{m+1}=S^1\times D^m\cup D^2\times S^{m-1}$ be the standard decomposition and C the core circle $S^1\times\{0\}\subset S^{m+1}$. By identifying C with S^1 we choose a (degree 1) map $\rho_i: C\to l_i(S^1)$ and attach S^{m+1} to W by ρ_i for each i. We denote the resulting complex by $\hat{W}$. Then $\hat{W}$ is a finite oriented 1-connected homogeneously $(m+1)$-dimensional complex and $\partial\hat{W}=\partial W$. Next we define a map $\hat{g}: \hat{W}\to L$. We only have to define a map on each $S^{m+1}=S^1\times D^m\cup D^2\times S^{m-1}$. Set $\hat{g}(x, y)=g(\rho_i(x))$ for $(x, y)\in S^1\times D^m\subset S^{m+1}$ and $\hat{g}(z, u)=d_i(z)$ for $(z, u)\in D^2\times S^{m-1}$. Then we have an $(m+1)$-chain $\hat{g}: \hat{W}\to L$ as desired.

2.b. The graph of a pull back of a foliation without Sullivan's vanishing cycles.

Suppose now that $\mathscr{F}$ has no Novikov's vanishing cycle nor m-dimensional Sullivan's vanishing cycle. Recall that V is an $(m+1)$-dimensional 1-connected compact oriented manifold and $f: V \to M$ is a smooth generic map as in Lemma 2.1. Also let Γ be the graph of $(V, f^*\mathscr{F})$ defined as in the part 2.a.

Suppose $c_1, \cdots, c_k$ are all singular points of index 0 or $m+1$, which may be identified with all black end vertices in Γ. Let D be the union of non-singular leaves or subseparatrices F such that $f|_F$ are $\mathscr{F}$-homologous to zero. Although the singular points $c_1, \cdots, c_k$ may not be $\mathscr{F}$-homologous to zero, we define that D contains $c_1, \cdots, c_k$.

LEMMA 2.7. *The union $D-\{c_1, \cdots, c_k\}$ is non-empty and if a non-singular leaf is contained in D then it is contained in* $\operatorname{Int} D$.

PROOF. Let U be a coordinate neighbourhood of a foliation chart of $f^*\mathscr{F}$ arround c_i and F a non-singular leaf of $f^*\mathscr{F}$ in U. Then F is diffeomorphic to S^m and bounds an $(m+1)$-ball B in U, that is, $\partial B = F$. If F is sufficiently near c_i then one can push B in M into the leaf of $\mathscr{F}$ which contains F. Precisely, it is clear to see that there is a homotopy $f_t: B \to M$ such that $f_0 = f|_B$, $f_t|_{\partial B} = f|_{\partial B}$ for any t, and $f_1(B) \subset L_F$. In particular $f|_F$ is $\mathscr{F}$-homologous to zero, therefore $F \subset D$.

Let F be a non-singular leaf such that $F \subset D$. Then there is an $(m+1)$-chain $f_W: W \to L_F$ such that W is simply-connected and $\partial f_W = f|_F$. By Lemma 2.4 there is a neighbourhood N of F in V which is diffeomorphic to $F \times]-\varepsilon, \varepsilon[$. Moreover $f^*\mathscr{F}|_N = \{F \times \{t\}\}$ and $F = F \times \{0\}$ by the diffeomorphism. On the other hand since W is compact there are finitely many foliation charts for $\mathscr{F}$ whose union covers $f_W(W)$ and each of which intersects $f_W(W)$ in a unique plaque. Now since W is simply-connected it is easy to see that there is a map $(f_W)_t: W \to L_{F\times\{t\}}$ such that $\partial (f_W)_t = f|_{F\times\{t\}}$ for $t \in]-\delta, \delta[$ and $(f_W)_0 = f_W$, where δ is a positive number such that $\delta \leqq \varepsilon$. This means $F \times \{t\} \subset D$ for $t \in]-\delta, \delta[$, which proves the lemma.

We denote by ∂D the frontier of D, i.e. $\partial D = \mathrm{Cl}_V(D) - \operatorname{Int} D$. It is easy to see that each connected component of ∂D must be a non-singular leaf or a subseparatrix. Let Δ and $\partial \Delta$ denote the subsets of Γ which correspond to D and ∂D respectively. The set Δ is identified with the subset $D/f^*\mathscr{F}$ of the leaf space $V/f^*\mathscr{F}$.

LEMMA 2.8. *Let e be an edge of Γ. If the interior of e intersects Δ then e is contained in Δ. In fact, the subseparatrices determined by* *e are contained in D.*

PROOF. Let $d=\Delta\cap\operatorname{Int} e$. Then d is non-empty and open in e by Lemma 2.7. We claim that $d=\operatorname{Int} e$. Otherwise choose a point x in the frontier of d in $\operatorname{Int} e$ and a point y in d which is sufficiently near x so that the interval $[x, y]$ in e satisfies $[x, y]\cap d=]x, y]$. We have a homotopy of tangential m-cycles $f|_{[x,y]}: F\times[x, y]\to M$, where F denotes the leaf corresponding to y (and to x). Since there is no Sullivan's vanishing cycle in $\mathscr{F}$ and $]x, y]\subset\Delta$, $f|_x$ must be $\mathscr{F}$-homologous to zero. Therefore x lies in d, which contradicts the assumption.

Next, let v be a black vertex of e and S the subseparatrix determined by e and v. Choose $x\in e$ sufficiently near v and suppose a non-singular leaf $F\in f^*\mathscr{F}$ corresponds to x. Then by Lemma 2.6 there is a homotopy of tangential m-cycles $g: F\times[x, v]\to M$ such that $g|_{F\times\{v\}}=(f|_S)\cdot p$ and $g|_{F\times[x,v[}=f|_{F\times[x,v[}$. Since there is no Sullivan's vanishing cycle in $\mathscr{F}$, $g|_{F\times\{v\}}$ must be $\mathscr{F}$-homologous to zero and therefore $f|_S$ is also $\mathscr{F}$-homologous to zero (cf. Lemma 2.6). This implies $v\in\Delta$. Now we have $e\subset\Delta$ and moreover $S\subset D$, which shows the lemma.

Now we analyze a global structure of Δ. By Lemma 2.8 it follows that Δ is closed and $\partial\Delta$ consists of vertices. Recall each black vertex of Γ corresponds to a whole separatrix. Therefore although a vertex in $\partial\Delta$ is contained in Δ the whole separatrix corresponding to the vertex may not be contained in D. However, we have the following:

LEMMA 2.9. *Suppose that there is only one white vertex in Γ and the white vertex does not lie in Δ. Let E denote the connected component of $\Gamma-\operatorname{Int}\Delta$ which contains the white vertex. Then there exists a vertex v in $\partial\Delta\cap E$ whose valency in the tree E is equal to one.*

PROOF. Assume for any vertex v of $\partial\Delta\cap E$ there exist at least two edges each of which is incident with v and does not intersect $\operatorname{Int}\Delta$. Then removing $\operatorname{Int}\Delta$ from Γ yields no new end vertex in E. All black end vertices in Γ, however, are contained in $\operatorname{Int}\Delta$. Therefore E has no end vertex except the white vertex. This contradicts the fact that Γ is a finite tree. This proves Lemma 2.9.

2.c. Proofs of Theorem A and Corollary A.

PROOF OF THEOREM A. Assuming that $\mathscr{F}$ has no Novikov's vanishing cycle nor m-dimensional Sullivan's vanishing cycle, we will prove it yields a contradiction. By Lemma 2.1 we may assume that f_X is a generic smooth map. Set $\mathscr{H}=(f_X)^*\mathscr{F}$ and define the graph Γ of $(X, \mathscr{H})$ as in the part 2.a. Also the subset D of X and the subgraph Δ of Γ can be defined as in the part 2.b. Now since $Z=\partial X$ is connected there is only one white vertex in Γ. Moreover $f_Z=\partial f_X$ is not $\mathscr{F}$-homologous to zero by the hypothesis of the theorem, which implies that the white vertex does not lie in Δ. Therefore we can apply Lemma 2.9 to the graph Γ so as to find a vertex v in $\partial\Delta\cap E$ whose valency in the tree E is equal to one. Recall that E is the connected component of $\Gamma-\operatorname{Int}\Delta$ which contains the white vertex. Let $S=\pi^{-1}(v)$ the separatrix which corresponds to v. Suppose $e_0, e_1, \cdots, e_l$ are all the edges which are incident with v in Γ where $e_0\subset E$ and $e_i\subset\Delta$ for $i=1, \cdots, l$. Then by Lemma 2.8 each subseparatrix S_i of S which is determined by each edge e_i is contained in D for $i>0$. Therefore there is a tangential $(m+1)$-chain $g_i: W_i\to L_S$ such that $\partial g_i=f|_{S_i}$. Then by attaching each W_i to $\partial W_i=S_i\subset S$ we define $g=\cup_{i=1}^l g_i: W=\cup_{i=1}^l W_i\to L_S$ and it follows $\partial g=f|_{S_0}$ since the vertex v is connected with the white vertex by the edge path starting from the edge e_0. Now we show the following claims.

CLAIM 1. The induced homomorphism $g_*: \pi_1(W)\to\pi_1(L_S)$ is trival.

PROOF OF CLAIM 1. Since $\partial g=f|_{S_0}$ induces the trivial homomorphism on π_1 (cf. the proof of Lemma 2.2) and $\operatorname{Int}W_i\cap\operatorname{Int}W_j=\emptyset$, it is easy to see that by repeated application of Van Kampen's theorem the homomorphism $g_*: \pi_1(W)\to\pi_1(L_S)$ is trivial.

CLAIM 2. The cycle $f|_{S_0}$ is $\mathscr{F}$-homologous to zero. In fact we may modify g to obtain a tangential $(m+1)$-chain $\hat{g}: \hat{W}\to L_S$ such that $\partial\hat{g}=\partial g$ and $\hat{W}$ is simply-connected.

PROOF OF CLAIM 2. By Claim 1 and Sublemma in the part 2.a the claim is clear.

Then we can displace the chain $\hat{g}$ along the remaining edge e_0. Precisely, choose $x\in\operatorname{Int}e_0$ sufficiently near v. Denote by $F=\pi^{-1}(x)$ a non-singular leaf corresponding to x and by $p: F\to S_0$ the natural projection. Then by Lemma 2.6 we have a leaf-preserving map $\phi: F\times[v, x]\to\pi^{-1}([v, x])$ such that $\phi|_{F\times]v,x]}$ is an embedding and $\phi|_{F\times\{v\}}=p$. By setting $h=f\cdot\phi$ we have a homotopy of tangential m-cycles $h: F\times[v, x]\to M$ such that $h_v=(f|_{S_0})\cdot p$, where $h_v=h|_{F\times\{v\}}$. By Claim 2 and Lemma 2.6 the cycle h_v is $\mathscr{F}$-homologous to zero. Now, as in the proof of Lemma 2.7, we can displace the chain bounded by h_v along the homotopy $h_t=h|_{F\times\{t\}}$. This contradicts the definition of D. The proof of Theorem A is now completed.

PROOF OF COROLLARY A. In case $n=3$ the assumption of Corollary A is empty by Theorem 1. Assume $n>3$. By Theorem A there is an $(n-2)$-dimensional Sullivan's vanishing cycle in $\mathscr{F}$. Then since M is compact it yields a non-trivial foliation cycle by a result of Sullivan (Theorem II.15 in [Su]). It is known by a work of Plante ([P1], [P2]) that a non-trivial foliation cycle in a codimension-one C^2 foliation on a compact manifold is supported on compact leaves or on the whole manifold. In the latter case $\mathscr{F}$ is without holonomy and then there is no leaf such that the homomorphism induced on $(n-2)$nd homotopy groups by the inclusion is not injective. Therefore the foliation cycle must be supported on compact leaves. This proves Corollary A.

§3. Foliations without Novikov's vanishing cycles.

In this section we prove Theorem B and give some consequences. Suppose M is an n-dimensional smooth manifold and $\mathscr{F}$ is a codimension-one C^2 foliation on M.

LEMMA 3.1. *Let $p\colon \tilde{M}\to M$ be a covering and set $\tilde{\mathscr{F}}=p^*\mathscr{F}$. Then we have the following*:

1) *If $\tilde{f}_Z\colon Z\to\tilde{M}$ is a Sullivan's vanishing cycle in $\tilde{\mathscr{F}}$ then $p\cdot\tilde{f}_Z\colon Z\to M$ is a Sullivan's vanishing cycle in $\mathscr{F}$.*

2) *If $f_Z\colon Z\to M$ is a Sullivan's vanishing cycle in $\mathscr{F}$ then there is a lift $\tilde{f}_Z\colon Z\to\tilde{M}$ of f_Z, which is also a Sullivan's vanishing cycle in $\tilde{\mathscr{F}}$.*

PROOF. Suppose $\tilde{f}_Z\colon Z\to\tilde{M}$ is a connected tangential cycle. Then it is clear by definition that $\tilde{f}_Z$ is $\tilde{\mathscr{F}}$-homologous to zero iff $p\cdot\tilde{f}_Z\colon Z\to M$ is $\mathscr{F}$-homologous to zero. Moreover if $f_Z\colon Z\to M$ is a Sullivan's vanishing cycle in $\mathscr{F}$ then, since the induced homomorphism $(f_Z)_*\colon \pi_1(Z)\to\pi_1(M)$ is trivial, there is a lift $\tilde{f}_Z\colon Z\to\tilde{M}$ of f_Z, i.e. $f_Z=p\cdot\tilde{f}_Z$. Since f_Z is tangential to $\mathscr{F}$, the lift $\tilde{f}_Z$ must be tangential to $\tilde{\mathscr{F}}$. This shows the lemma.

LEMMA 3.2. *Let (X, A) be a pair of CW-complexes and let $i\colon A\to X$ denote the inclusion. Suppose X and A are 1-connected. Then for any positive integer q and any element $\alpha\in\mathrm{Ker}(i_*\colon \Omega_q(A)\to\Omega_q(X))$ there exists a singular $(q+1)$-manifold $f\colon (V, \partial V)\to(X, A)$ such that $\alpha=\partial([V, f])$ and that V is 1-connected and ∂V is connected, where $\partial\colon \Omega_{q+1}(X, A)\to\Omega_q(A)$ is the boundary homomorphism.*

PROOF. In case $q=1$ the assertion is trivial. In case $q=2$ by the assumption and Hurewicz theorem we have Hurewicz isomorphism $h\colon \pi_2(A)\to H_2(A\colon \boldsymbol{Z})$ and $h\colon \pi_2(X)\to H_2(X\colon \boldsymbol{Z})$. It is known that for any CW-complex Y, $\mu\colon \Omega_k(Y)\to H_k(Y\colon \boldsymbol{Z})$ is an isomorphism if $0\leqq k\leqq 3$. Therefore we have isomorphisms $(\mu^{-1})\cdot h\colon \pi_2(A)\to\Omega_2(A)$ and $(\mu^{-1})\cdot h\colon \pi_2(X)\to\Omega_2(X)$, which commute $i_*\colon \pi_2(A)\to\pi_2(X)$ and $i_*\colon \Omega_2(A)\to\Omega_2(X)$. Then we can take a continuous map $f: (D^3, \partial D^3) \to (X, A)$ as desired.

Now we assume $q \geqq 3$. By exactness of the bordism homology sequence of a pair we may choose a singular $(q+1)$-manifold $g: (W, \partial W) \to (X, A)$ such that $\alpha = \partial([W, g])$. We can assume ∂W is connected by connecting boundary components with tubes if necessary. Since the dimension of W is greater than three we can perform surgeries on $g: W \to X$ in order to kill the generators of $\pi_1(W)$. This proves the lemma.

By Theorem 1 the following lemma is clear.

LEMMA 3.3. *Let $p: \tilde{M} \to M$ be the universal covering and set $\tilde{\mathscr{F}} = p^*\mathscr{F}$. Then the following are equivalent*:

1) *$\mathscr{F}$ has no Novikov's vanishing cycle,*

2) *for any leaf $L \in \mathscr{F}$ and any component $\tilde{L}$ of $p^{-1}(L)$, the restriction map $p: \tilde{L} \to L$ is the universal covering.*

A foliation $\mathscr{F}$ on a manifold M is called *simple* if the leaf space $M/\mathscr{F}$ is a manifold (possibly non-Hausdorff).

LEMMA 3.4 (Haefliger [H2], Hector and Bouma [H-B]). *Let $\mathscr{F}$ be a codimension-one foliation on a simply-connected manifold M. Then the following conditions are equivalent*:

1) *$\mathscr{F}$ is simple.*

2) *$\mathscr{F}$ is without holonomy.*

3) *$\mathscr{F}$ does not admit a closed transversal.*

4) *All leaves of $\mathscr{F}$ are closed.*

We note that if $\mathscr{F}$ has no Novikov's vanishing cycle then each leaf $\tilde{L}$ of the universal covering $(\tilde{M}, \tilde{\mathscr{F}})$ is closed.

LEMMA 3.5. *Suppose M is 1-connected and $\mathscr{F}$ has no Novikov's vanishing cycle. Then for (non-trivial) m-dimensional Sullivan's vanishing cycle $f_Z: Z \to M$ in $\mathscr{F}$ the homology class $[f_Z] \in H_m(L: \boldsymbol{Z})$ is of infinite order, where L denotes the support leaf of f_Z.*

PROOF. Otherwise there is a positive integer $k \in \boldsymbol{Z}$ such that $k \cdot [f_Z] = 0$. That is, there is a $(m+1)$-chain $g: W \to L$ such that $\partial g = k \cdot f_Z$. Note that $k > 1$ since f_Z is not $\mathscr{F}$-homologous to zero and L is 1-connected (cf. Sublemma in Section 2, part 2.a). Since $g(W)$ is compact there is a codimension-zero compact connected submanifold V of L which contains $g(W)$. Then we may consider f_Z and g as chains in V. Set $\zeta = [f_Z] \in H_m(V: \boldsymbol{Z})$. Since $\zeta \neq 0$ and $k \cdot \zeta = 0$ in $H_m(V: \boldsymbol{Z})$ there is a homology class $\xi \in \mathrm{Tor}(H_{n-m-2}(V, \partial V: \boldsymbol{Z}))$ such that $\mathrm{lk}(\zeta, \xi) \neq 0$, where Tor denotes the torsion subgroup and lk denotes the linking form of $(V, \partial V)$. Suppose that a cycle $h:(C, \partial C)\to(V, \partial V)$ represents ξ. By definition $\mathrm{lk}(\xi, \zeta)=(1/k)\cdot I(g, h) \pmod{\boldsymbol{Z}}$, where I denotes the intersection of chains. Since $\mathscr{F}$ is without holonomy and V is compact, we can displace V to nearby leaves. Let $V_t\subset L_t\in\mathscr{F}$, $g_t: W\to V_t$ and $h_t:(C, \partial C)\to(V_t, \partial V_t)$ be the displacements of V, g and h respectively. Also we denote by $\zeta_t\in H_m(V_t: \boldsymbol{Z})$ the displacement of ζ. Then since ζ is a vanishing cycle we have $i_*\zeta_t=0$ in $H_m(L_t: \boldsymbol{Z})$ where $i: V_t\to L_t$ is the inclusion. Therefore it follows that there is a codimension-zero compact connected submanifold V_t' containing V_t such that $h_t:(C, \partial C)\to(V_t', \partial V_t')$ is a cycle and $j_*\zeta_t=0$ in $H_m(V_t': \boldsymbol{Z})$, where $j: V_t\to V_t'$ is the inclusion. Now we have $\mathrm{lk}(j_*\zeta_t, j_*\xi_t)=0$ in $(V_t', \partial V_t')$. However, since linking form is well defined under changing the bounded chain, it follows that $\mathrm{lk}(j_*\zeta_t, j_*\xi_t)=\mathrm{lk}(\zeta_t, \xi_t)$. This is a contradiction by the following formulae;

$$\begin{aligned} 0\neq \mathrm{lk}(\zeta, \xi) &= (1/k)\cdot I(g, h) \qquad \pmod{\boldsymbol{Z}} \\ &= (1/k)\cdot I(g_t, h_t) \\ &= \mathrm{lk}(\zeta_t, \xi_t) \qquad \pmod{\boldsymbol{Z}} \\ &= \mathrm{lk}(j_*\zeta_t, j_*\xi_t) \\ &= 0 . \end{aligned}$$

This proves the lemma.

Proof of Theorem B. We will proceed in numerical order of the conditions. First, we assume the condition 1). Let $f_Z: Z\to L$ be an m-dimensional Sullivan's vanishing cycle, where $L\in\mathscr{F}$. Then by Lemma 3.1 there is a lifted Sullivan's vanishing cycle $\tilde{f}_Z: Z\to\tilde{L}$. By Lemma 3.5 the homology class $[\tilde{f}_Z]$ is of infinite order in $H_m(\tilde{L}: \boldsymbol{Z})$ and since $\tilde{f}_Z$ is a Sullivan's vanishing cycle it lies in the kernel of the homomorphism $H_m(\tilde{L}: \boldsymbol{Z})\to H_m(\tilde{M}: \boldsymbol{Z})$. Now the condition 2) is satisfied.

Next, we assume the condition 2). We denote by α a homology class of infinite order in $\mathrm{Ker}(H_m(\tilde{L}: \boldsymbol{Z})\to H_m(\tilde{M}: \boldsymbol{Z}))$ for a leaf $\tilde{L}\in\tilde{\mathscr{F}}$. We consider the following commutative diagram;

$$\begin{array}{ccccc} \Omega_{m+1}(\tilde{M}, \tilde{L}) & \xrightarrow{\partial} & \Omega_m(\tilde{L}) & \xrightarrow{i_*} & \Omega_m(\tilde{M}) \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\ H_{m+1}(\tilde{M}, \tilde{L}: \boldsymbol{Z}) & \xrightarrow{\partial} & H_m(\tilde{L}: \boldsymbol{Z}) & \xrightarrow{i_*} & H_m(\tilde{M}, \boldsymbol{Z}) , \end{array}$$

where the rows are exact. Then there is $\beta\in H_{m+1}(\tilde{M}, \tilde{L}: \boldsymbol{Z})$ such that $\alpha=\partial(\beta)$. By a theorem of Conner and Floyd (Theorem 15.3 in [C-F]) there is an odd integer $q\in \boldsymbol{Z}$ such that $q\cdot\beta$ is Steenrod representable, i.e., there is $\gamma\in\Omega_{m+1}(\tilde{M}, \tilde{L})$ such that $q\cdot\beta=\mu(\gamma)$. By the commutativity $\mu\cdot\partial(\gamma)=\partial\cdot\mu(\gamma)=\partial(q\cdot\beta)=q\cdot\alpha$. Since α is of infinite order, $q\cdot\alpha$ is also of infinite order as desired in the condition 3).

Since the condition 3) is included in the condition 4), now we only have to show that the condition 4) implies the condition 1) in order to complete the proof. Let $\alpha\in\Omega_{m+1}(\tilde{M}, \tilde{L})$ such that $\mu\cdot\partial(\alpha)\neq 0$ in $H_m(\tilde{L}: \boldsymbol{Z})$. By Lemma 3.3 and Lemma 3.4 each leaf of $\tilde{\mathscr{F}}$ is 1-connected and closed, hence we can apply Lemma 3.2 to $(\tilde{M}, \tilde{L})$ to obtain a singular $(m+1)$-manifold chain $f_X: (X, \partial X)\to(\tilde{M}, \tilde{L})$ which represents α such that X is 1-connected and ∂X is connected. Now we may apply Theorem A to f_X to obtain an m-dimensional Sullivan's vanishing cycle in $\tilde{\mathscr{F}}$, and by Lemma 3.1 the m-cycle in M induced by the covering projection is also a Sullivan's vanishing cycle in $\mathscr{F}$. This completes the proof.

Proof of Corollary B. Otherwise the condition 2) of Theorem B is satisfied and it yields a Sullivan's vanishing cycle by Theorem B.

Proof of Corollary C. Let $p: (\tilde{M}, \tilde{\mathscr{F}})\to(M, \mathscr{F})$ be the universal covering. We assume that there is a leaf $L\in\mathscr{F}$ such that $H_2(\tilde{L}: \boldsymbol{Z})\neq 0$ or $H_3(\tilde{L}: \boldsymbol{Z})\neq 0$, where $\tilde{L}$ is a component of $p^{-1}(L)$. In case that $H_3(\tilde{L}: \boldsymbol{Z})\neq 0$, $\tilde{L}$ must be compact. Hence $L=p(\tilde{L})$ is compact. Next we consider the case that $H_2(\tilde{L}: \boldsymbol{Z})\neq 0$. Since $\tilde{M}$ is contractible there is a non-zero homology class $\zeta\in H_2(\tilde{L}: \boldsymbol{Z})$ which vanishes in $H_2(\tilde{M}: \boldsymbol{Z})$. Since $\tilde{M}$ (resp. $\tilde{L}$) is the universal covering of M (resp. L), $\pi_2(M)$ (resp. $\pi_2(L)$) is isomorphic to $H_2(\tilde{M}: \boldsymbol{Z})$ (resp. $H_2(\tilde{L}: \boldsymbol{Z})$). Therefore ζ is mapped to a non-zero class $z\in\pi_2(L)$ which vanishes in $\pi_2(M)$. We claim that z is not $\mathscr{F}$-homologous to zero. For otherwise the simply-connected bounded chain can be lifted to $\tilde{L}$, which contradicts the assumption that ζ is non-zero in $H_2(\tilde{L}: \boldsymbol{Z})$. Now we can apply Corollary A to $z\in\pi_2(L)$ to obtain a compact leaf in $\mathscr{F}$. This completes the proof.

Proof of Corollary D. By Theorem 1 we only have to show that $\mathscr{F}$ has no 2-dimensional Sullivan's vanishing cycle iff for any leaf $L\in\mathscr{F}$ the homomorphism $\pi_2(L)\to\pi_2(M)$ induced by the inclusion is injective. Let $p: (\tilde{M}, \tilde{\mathscr{F}})\to(M, \mathscr{F})$ be the universal covering. For any leaf $\tilde{L}\in\tilde{\mathscr{F}}$ we have the following commutative diagram;

$$\begin{array}{ccccccc}
\Omega_2(\tilde{L}) & \xrightarrow{\mu} & H_2(\tilde{L}: \boldsymbol{Z}) & \longleftarrow & \pi_2(\tilde{L}) & \xrightarrow{p_*} & \pi_2(L) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega_2(\tilde{M}) & \xrightarrow{\mu} & H_2(\tilde{M}: \boldsymbol{Z}) & \longleftarrow & \pi_2(\tilde{M}) & \xrightarrow{p_*} & \pi_2(M)
\end{array}$$

where the rows are all isomorphisms since $\tilde{L}$ and $\tilde{M}$ are 1-connected. Therefore the assertion holds by Theorem A and Lemma 3.2. This proves the corollary.

§4. Proof of Theorem C.

In this section we consider the case that there is a Novikov's vanishing cycle. First we prove Theorem C.

PROOF OF THEOREM C. Let $f: S^1 \to L$ be a Novikov's vanishing cycle, where $L \in \mathscr{F}$. We may assume f is a smooth immersion. Moreover, in view of Lemma 3.1, we may assume M is orientable and $\mathscr{F}$ is transversely orientable without loss of generality. Therefore we may choose a tubular neighbourhood $F: S^1 \times D^{n-2} \to L$ of f. Note that since f is essential in L the cycle $F|_{\partial(S^1 \times D^{n-2})}$ is not $\mathscr{F}$-homologous to zero. Since f is a Novikov's vanishing cycle, there is a continuous map $g: D^2 \to M$ such that $g|_{\partial D^2} = f$. Choose a tubular neighbourhood $G: D^2 \times D^{n-2} \to M$ of g by approximating g by a smooth immersion. Here we can choose G such that $G|_{\partial D^2 \times D^{n-2}} = F$. Now we consider the standard embedding $D^2 \times S^{m-1} \subset D^2 \times \partial D^{n-2} \subset D^2 \times D^{n-2}$. Then $G|_{\partial D^2 \times S^{m-1}} = F|_{\partial D^2 \times S^{m-1}}$ induces a non-trivial homomorphism $\pi_1(\partial D^2 \times S^{m-1}) \to \pi_1(L)$ since $G|_{\partial D^2 \times \{*\}} \simeq f$ where $* \in S^{m-1}$. Therefore $G|_{\partial D^2 \times S^{m-1}}$ is not $\mathscr{F}$-homologous to zero. Set $X = D^2 \times S^{m-1}$ and $f_X = G|_{D^2 \times S^{m-1}}$. Then we have the desired chain. This proves the theorem.

Now we construct an example. Let $(S^1 \times D^2, \mathscr{F}_R)$ be a Reeb component. We set $(M, \mathscr{F}) = D(S^1 \times D^2, \mathscr{F}_R) \times S^1$ where D denotes the double. Then $\mathscr{F}$ has a Novikov's vanishing cycle. We claim that $\mathscr{F}$ admits no singular 3-manifold chain such as in the assertion of Theorem C. It is enough to show the claim for the universal covering $(\tilde{M}, \tilde{\mathscr{F}})$. Since the boundary of a compact orientable 1-connected 3-manifold is a union of 2-spheres, we only have to show that there is no tangential 2-cycle which is spherical and not $\mathscr{F}$-homologous to zero. It is easy to see that all leaves of $\tilde{\mathscr{F}}$ are diffeomorphic to $\boldsymbol{R}^3$ except only one leaf which is diffeomorphic to $S^1 \times \boldsymbol{R}^2$. Therefore any spherical tangential 2-cycle must be inessential in the support leaf. This shows the claim.

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Present Address:
Department of Mathematics
School of Science and Engineering, Waseda University
Okubo, Shinjuku-ku, Tokyo 160, Japan