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On Automorphisms of Irrational Rotation Algebras

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Abstract. Let A_θ be an irrational rotation algebra and B_θ be the AF-algebra defined by Effros and Shen. Then we have monomorphisms of A_θ into B_θ. In the present paper we will show that there are automorphisms of A_θ which can not be extended to any automorphism of B_θ for any monomorphism of A_θ into B_θ.

§1. Preliminaries.

Let $C(\boldsymbol{T})$ be the C^*-algebra of all complex valued continuous functions on the one dimensional torus $\boldsymbol{T}$ and θ be an irrational number in $\boldsymbol{R}$. Let $C(\boldsymbol{T})\times_\sigma \boldsymbol{Z}$ be the crossed product for an action σ of the integer group $\boldsymbol{Z}$ on $C(\boldsymbol{T})$ by angle $2\pi\theta$. We denote $C(\boldsymbol{T})\times_\sigma \boldsymbol{Z}$ by A_θ and call it the *irrational rotation algebra by* θ. We identify $C(\boldsymbol{T})$ with the C^*-algebra $\{f\in C([0, 1])\mid f(0)=f(1)\}$. Let u and v be the unitary elements in A_θ defined by

$$u(n, t)=\begin{cases} e^{2\pi i t} & \text{if } n=0 \\ 0 & \text{if } n\neq 0 \end{cases}$$

and

$$v(n, t)=\begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n\neq 1 . \end{cases}$$

Then clearly A_θ is generated by u and v and $uv=e^{2\pi i\theta}vu$. Let p be a Rieffel projection in A_θ with $\tau(p)=\theta$. It is well known that $K_0(A_\theta)=\boldsymbol{Z}[1]\oplus\boldsymbol{Z}[p]$ and $K_1(A_\theta)=\boldsymbol{Z}[u]\oplus\boldsymbol{Z}[v]$.

Let B_θ be the AF-algebra defined in Effros and Shen [2]. Then Pimsner and Voiculescu [6] showed that there exist a monomorphism ρ of A_θ into B_θ and the unique tracial state τ_1 on B_θ such that $\tau=\tau_1\circ\rho$. Furthermore they showed that $K_0(B_\theta)\cong\boldsymbol{Z}^2$ and $\tau_{1*}(K_0(B_\theta))=\boldsymbol{Z}+\boldsymbol{Z}\theta$.

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LEMMA 1. *Let ϕ be an arbitrary monomorphism of A_θ into B_θ and let $p \in A_\theta$ be the Rieffel projection defined in the above. Then $\tau = \tau_1 \circ \phi$ and* [1], $[\phi(p)]$ *are generators of* $K_0(B_\theta)$.

PROOF. Let $\tau' = \tau_1 \circ \phi$. Then τ' is a tracial state on A_θ. Thus by the uniqueness of the tracial state on A_θ we obtain that $\tau = \tau' = \tau_1 \circ \phi$. Now let $[e_j] - [f_j]$, $j = 1, 2$, be the generators of $K_0(B_\theta)$ such that $\tau_{1*}([e_1] - [f_1]) = 1$ and $\tau_{1*}([e_2] - [f_2]) = \theta$ where e_j and f_j are projections in some matrix algebra $M_n(B_\theta)$ over B_θ. Then there are l, m in $\boldsymbol{Z}$ such that

$$[1] = l([e_1] - [f_1]) + m([e_2] - [f_2]) .$$

Hence

$$1 = \tau_{1*}([1]) = l + m\theta .$$

Thus $l = 1$ and $m = 0$. Therefore $[1] = [e_1] - [f_1]$. Similarly $[\phi(p)] = [e_2] - [f_2]$. Q.E.D.

§ 2. Automorphisms of B_θ.

Let ρ be the monomorphism of A_θ into B_θ defined in Pimsner and Voiculescu [6]. Then [1] and $[\rho(p)]$ are generators of $K_0(B_0)$ by Lemma 1. For any C^*-algebra A, $\mathrm{Aut}(A)$ denotes the group of all automorphisms of A.

LEMMA 2. *For any $\beta \in \mathrm{Aut}(B_\theta)$, $\beta_* = \mathrm{id}$ on $K_0(B_\theta)$. Then by Blackadar* [1, Theorem 3.1], *β is approximately inner.*

PROOF. Clearly $\beta_*([1]) = [1]$. Since $\beta_*([\rho(p)]) \in K_0(B_\theta)$, there are $l, m \in \boldsymbol{Z}$ such that

$$\beta_*([\rho(p)]) = l[1] + m[\rho(p)] .$$

Hence we obtain by Lemma 1 that

$$\tau_{1*}(\beta_*([\rho(p)])) = l + m\theta .$$

On the other hand

$$\tau_{1*}(\beta_*([\rho(p)])) = [(\tau_1 \circ \beta \circ \rho)(p)] .$$

By the uniqueness of τ_1 we obtain that

$$\tau_{1*}(\beta_*([\rho(p)])) = (\tau_1 \circ \rho)(p) = \tau(p) = \theta .$$

Thus $l = 1$ and $m = 0$. Hence $\beta_* = \mathrm{id}$ on $K_0(B_\theta)$. Q.E.D.

For any $\beta \in \mathrm{Aut}(B_\theta)$ let $\tilde{\tau}_1$ be a trace on $B_\theta \times_\beta \boldsymbol{Z}$ defined by $\tilde{\tau}_1(f) = \tau_1(f(0))$ for any $f \in l^1(\boldsymbol{Z}, B_\theta)$. Since τ_1 is the unique tracial state on B_θ, $\tilde{\tau}_1$ is well defined.

COROLLARY 3. *For any* $\beta \in \mathrm{Aut}(B_\theta)$, $\tilde{\tau}_{1*}(K_0(B_\theta \times_\beta \boldsymbol{Z})) = \boldsymbol{Z} + \boldsymbol{Z}\theta$.

PROOF. By Lemma 2, $\beta_* = \mathrm{id}$ on $K_0(B_\theta)$. Therefore by the Pimsner-Voiculescu six terms exact sequence we obtain that

$$0 \longrightarrow K_0(B_\theta) \xrightarrow{i_*} K_0(B_\theta \times_\beta \boldsymbol{Z}) \longrightarrow 0$$

where i_* is the homomorphism induced by the inclusion map i of B_θ into $B_\theta \times_\beta \boldsymbol{Z}$. Hence $i_*([1])$ and $i_*([p])$ are generators of $K_0(B_\theta \times_\beta \boldsymbol{Z}) \cong \boldsymbol{Z}^2$. Since $\tilde{\tau}_{1*}(i_*([1])) = 1$ and $\tilde{\tau}_{1*}(i_*([p])) = \theta$, we obtain that $\tilde{\tau}_{1*}(K_0(B_\theta \times_\beta \boldsymbol{Z})) = \boldsymbol{Z} + \boldsymbol{Z}\theta$.
Q.E.D.

§3. Automorphisms of A_θ.

For any $\alpha \in \mathrm{Aut}(A_\theta)$ let $\tilde{\tau}$ be a trace on $A_\theta \times_\alpha \boldsymbol{Z}$ defined by $\tilde{\tau}(f) = \tau(f(0))$ for any $f \in l^1(\boldsymbol{Z}, A_\theta)$.

DEFINITION. We say that an *automorphism* α *of* A_θ *can be extended to* $\beta \in \mathrm{Aut}(B_\theta)$ *for some monomorphism* ϕ *of* A_θ *into* B_θ if $\phi(\alpha(x)) = \beta(\phi(x))$ for any $x \in A_\theta$.

By the above definition if $\alpha \in \mathrm{Aut}(A_\theta)$ can be extended to $\beta \in \mathrm{Aut}(B_\theta)$ there exists a monomorphism $\tilde{\phi}$ of $A_\theta \times_\alpha \boldsymbol{Z}$ into $B_\theta \times_\beta \boldsymbol{Z}$ such that $\tilde{\tau}(f) = \tilde{\tau}_1(\tilde{\phi}(f))$ for any $f \in A_\theta \times_\alpha \boldsymbol{Z}$. Thus we obtain that $\tilde{\tau}_*(K_0(A_\theta \times_\alpha \boldsymbol{Z})) \subset \tilde{\tau}_{1*}(K_0(B_\theta \times_\beta \boldsymbol{Z}))$. Since $\tilde{\tau}_{1*}(K_0(B_\theta \times_\beta \boldsymbol{Z})) = \boldsymbol{Z} + \boldsymbol{Z}\theta$, $\tilde{\tau}_*(K_0(A_\theta \times_\alpha \boldsymbol{Z})) \subset \boldsymbol{Z} + \boldsymbol{Z}\theta$.

LEMMA 4. *For any* $\alpha \in \mathrm{Aut}(A_\theta)$, $\tilde{\tau}_*(K_0(A_\theta \times_\alpha \boldsymbol{Z})) \supset \boldsymbol{Z} + \boldsymbol{Z}\theta$.

PROOF. Since τ is the unique tracial state on A_θ and $K_0(A_\theta) = \boldsymbol{Z}[1] \oplus \boldsymbol{Z}[p]$, $\alpha_* = \mathrm{id}$ on $K_0(A_\theta)$ for any $\alpha \in \mathrm{Aut}(A_\theta)$. Hence by the Pimsner-Voiculescu six terms exact sequence $\tilde{\tau}_*(K_0(A_\theta \times_\alpha \boldsymbol{Z})) \supset \tau_*(K_0(A_\theta)) = \boldsymbol{Z} + \boldsymbol{Z}\theta$.
Q.E.D.

COROLLARY 5. *If* $\alpha \in \mathrm{Aut}(A_\theta)$ *can be extended to* $\beta \in \mathrm{Aut}(B_\theta)$, $\tilde{\tau}_*(K_0(A_\theta \times_\alpha \boldsymbol{Z})) = \boldsymbol{Z} + \boldsymbol{Z}\theta$.

Next we will compute $\tilde{\tau}_*(K_0(A_\theta \times_\alpha \boldsymbol{Z}))$ for some automorphisms $\alpha \in \mathrm{Aut}(A_\theta)$. Let $SL(2, \boldsymbol{Z})$ be the group of all 2×2 matrices with integer entries and determinant 1. For each $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \boldsymbol{Z})$ we define an automorphism α_g by

$$\alpha_g(u^m v^n)=\exp(-\pi i\theta((am+bn)(cm+dn)-mn))u^{am+bn}v^{cm+dn}$$

for each m and $n\in \boldsymbol{Z}$. This definition is due to Watatani [9]. For each η_1 and $\eta_2\in \boldsymbol{R}$ let $\alpha_{(\eta_1,\eta_2)}$ be the automorphism of A_θ defined by $\alpha_{(\eta_1,\eta_2)}(u)=e^{2\pi i\eta_1}u$ and $\alpha_{(\eta_1,\eta_2)}(v)=e^{2\pi i\eta_2}v$. Let α be the automorphism of A_θ defined by

$$\alpha=\mathrm{Ad}(w)\circ\alpha_g\circ\alpha_{(\eta_1,\eta_2)}$$

where w is a unitary element in A_θ. We will consider the crossed products $A_\theta\times_\alpha \boldsymbol{Z}$. We can regard A_θ as a C^*-subalgebra of $A_\theta\times_\alpha \boldsymbol{Z}$. By the Pimsner-Voiculescu six terms exact sequence we can see that

$$K_0(A_\theta\times_\alpha \boldsymbol{Z})\cong\begin{cases}\boldsymbol{Z}^4 & \text{if } g=I\\ \boldsymbol{Z}^3 & \text{if } a+d=2 \text{ and } g\neq I\\ \boldsymbol{Z}^2 & \text{if } a+d\neq 2\,,\end{cases}$$

since $\alpha_*=\mathrm{id}$ on $K_0(A_\theta)$ where I is the unit element of the 2×2 matrix algebra M_2. If $a+d\neq 2$, then $[1]$ and $[p]$ are generators of $K_0(A_\theta\times_\alpha \boldsymbol{Z})$. Hence $\tilde{\tau}_*(K_0(A_\theta\times_\alpha \boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta$. Furthermore if $a+d=2$, we can see the following:

1) If $g=I$, $\mathrm{Ker}(\mathrm{id}-\alpha_*)=\boldsymbol{Z}[u]\oplus\boldsymbol{Z}[v]$.
2) If $a=1$, $b=0$ and $c\neq 0$, $\mathrm{Ker}(\mathrm{id}-\alpha_*)=\boldsymbol{Z}[v]$.
3) If $a=1$, $b\neq 0$ and $c=0$, $\mathrm{Ker}(\mathrm{id}-\alpha_*)=\boldsymbol{Z}[u]$.
4) If $a\neq 1$ and $b/(a-1)\in\boldsymbol{Z}$, $\mathrm{Ker}(\mathrm{id}-\alpha_*)=\boldsymbol{Z}((b/(a-1))[u]-[v])$.
5) If $a\neq 1$ and $(a-1)/b\in\boldsymbol{Z}$, $\mathrm{Ker}(\mathrm{id}-\alpha_*)=Z([u]-((a-1)/b)[v])$.
6) If $a\neq 1, b/(a-1)\notin\boldsymbol{Z}$ and $(a-1)/b\notin\boldsymbol{Z}$, $\mathrm{Ker}(\mathrm{id}-\alpha_*)=Z(b[u]-(a-1)[v])$.

PROPOSITION 6. *Let w be a unitary element in A_θ, $g=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in SL(2,\boldsymbol{Z})$ with $a+d=2$ and η_1, $\eta_2\in\boldsymbol{R}$. Let α be the automorphism of A_θ defined by $\alpha=\mathrm{Ad}(w)\circ\alpha_g\circ\alpha_{(\eta_1,\eta_2)}$. Then the following statements hold:*

1) *If $g=I$, then $\tilde{\tau}_*(K_0(A_\theta\times_\alpha\boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta+\boldsymbol{Z}\eta_1+\boldsymbol{Z}\eta_2$.*
2) *If $a=1$, $b=0$ and $c\neq 0$, then $\tilde{\tau}_*(K_0(A_\theta\times_\alpha\boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta+\boldsymbol{Z}\eta_2$.*
3) *If $a=1$, $b\neq 0$ and $c=0$, then $\tilde{\tau}_*(K_0(A_\theta\times_\alpha\boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta+\boldsymbol{Z}\eta_1$.*
4) *If $a\neq 1$ and $b/(a-1)\in\boldsymbol{Z}$,
then $\tilde{\tau}_*(K_0(A_\theta\times_\alpha\boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta+\boldsymbol{Z}((b/(a-1))\eta_1-\eta_2)$.*
5) *If $a\neq 1$ and $(a-1)/b\in\boldsymbol{Z}$,
then $\tilde{\tau}_*(K_0(A_\theta\times_\alpha\boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta+\boldsymbol{Z}(\eta_1-((a-1)/b)\eta_2)$.*
6) *If $a\neq 1$, $b/(a-1)\notin\boldsymbol{Z}$ and $(a-1)/b\notin\boldsymbol{Z}$,
then $\tilde{\tau}_*(K_0(A_\theta\times_\alpha\boldsymbol{Z}))=\boldsymbol{Z}+\boldsymbol{Z}\theta+\boldsymbol{Z}(b\eta_1-(a-1)\eta_2)$.*

PROOF. 1) $\mathrm{Ker}(\mathrm{id}-\alpha_*)=\boldsymbol{Z}[u]\oplus\boldsymbol{Z}[v]$. $\alpha(u)u^*=e^{2\pi i\eta_1}wuw^*u^*$. Let ξ_1 be the continuously differentiable path on $[0,1]$ from $\alpha(u)u^*$ to wuw^*u^*

defined by $\xi_1(t)=e^{2\pi i(1-t)\eta_1}wuw^*u^*$ for $t\in[0, 1]$. Let ξ_2 be the continuously differentiable path on [1, 2] from $\begin{bmatrix} wuw^*u^* & 0 \\ 0 & 1\end{bmatrix}$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$ defined by

$$\xi_2(t)=\begin{bmatrix} w & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} \sin \pi t/2 & -\cos \pi t/2 \\ \cos \pi t/2 & \sin \pi t/2\end{bmatrix}\begin{bmatrix} u & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} \sin \pi t/2 & \cos \pi t/2 \\ -\cos \pi t/2 & \sin \pi t/2\end{bmatrix}$$
$$\times\begin{bmatrix} w^* & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} \sin \pi t/2 & -\cos \pi t/2 \\ \cos \pi t/2 & \sin \pi t/2\end{bmatrix}\begin{bmatrix} u^* & 1 \\ 0 & 1\end{bmatrix}\begin{bmatrix} \sin \pi t/2 & \cos \pi t/2 \\ -\cos \pi t/2 & \sin \pi t/2\end{bmatrix}.$$

Let

$$\xi(t)=\begin{cases}\begin{bmatrix} \xi_1(t) & 0 \\ 0 & 1\end{bmatrix} & \text{if} \quad t\in[0, 1) \\ \xi_2(t) & \text{if} \quad t\in[1, 2] .\end{cases}$$

Then ξ is a piecewise continuously differentiable path on [0, 2] from $\begin{bmatrix} \alpha(u)u^* & 0 \\ 0 & 1\end{bmatrix}$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$. Let Tr be the canonical trace on M_2. Then

$$\frac{1}{2\pi i}\int_0^2(\tau\otimes\mathrm{Tr})\Big(\xi(t)^*\frac{d}{dt}\xi(t)\Big)dt$$
$$=\frac{1}{2\pi i}\int_0^1\tau\Big(\xi_1(t)^*\frac{d}{dt}\xi_1(t)\Big)dt+\frac{1}{2\pi i}\int_1^2(\tau\otimes\mathrm{Tr})\Big(\xi_2(t)^*\frac{d}{dt}\xi_2(t)\Big)dt .$$

And

$$\frac{1}{2\pi i}\int_0^1\tau\Big(\xi_1(t)^*\frac{d}{dt}\xi_1(t)\Big)dt=\frac{1}{2\pi i}\int_0^1(-2\pi i\eta_1)dt=-\eta_1$$

and

$$(\tau\otimes\mathrm{Tr})\Big(\xi_2(t)^*\frac{d}{dt}\xi_2(t)\Big)=0 .$$

Hence we obtain that

$$\frac{1}{2\pi i}\int_0^2(\tau\otimes\mathrm{Tr})\Big(\xi(t)^*\frac{d}{dt}\xi(t)\Big)dt=-\eta_1 .$$

Similarly there is a piecewise continuously differentiable path ζ on [0, 2] from $\begin{bmatrix} \alpha(v)v^* & 0 \\ 0 & 1\end{bmatrix}$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$ such that

$$\frac{1}{2\pi i}\int_0^2(\tau\otimes\mathrm{Tr})\Big(\zeta(t)^*\frac{d}{dt}\zeta(t)\Big)dt=-\eta_2 .$$

Therefore we obtain by Pimsner [4, Theorem 3] that

$$\tilde{\tau}_*(K_0(A_\theta \times_\alpha \boldsymbol{Z})) = \boldsymbol{Z} + \boldsymbol{Z}\theta + \boldsymbol{Z}\eta_1 + \boldsymbol{Z}\eta_2 .$$

2)-6) Let $(l, m) = (1, 0)$, $(0, 1)$, $(b/(a-1), -1)$, $(1, (a-1)/b)$ or $(b, 1-a)$. Then $\text{Ker}(\text{id} - \alpha_*) = \boldsymbol{Z}[u^l v^m]$. By trivial computation we can see that $\alpha_g(u^l v^m) = u^l v^m$. Since $\alpha = \text{Ad}(w) \circ \alpha_g \circ \alpha_{(\eta_1, \eta_2)}$, we obtain that

$$\begin{aligned} \alpha(u^l v^m)(u^l v^m)^* &= (\text{Ad}(w) \circ \alpha_g(e^{2\pi i (l\eta_1 + m\eta_2)} u^l v^m)) v^{-m} u^{-l} \\ &= e^{2\pi i (l\eta_1 + m\eta_2)} w \alpha_g(u^l v^m) w^* v^{-m} u^{-l} \\ &= e^{2\pi i (l\eta_1 + m\eta_2)} w u^l v^m w^* v^{-m} u^{-l} . \end{aligned}$$

Hence if we repeat the same discussion as 1), we can easily obtain the conclusions. Q.E.D.

§4. The main theorem.

Now we will state the main theorem.

THEOREM 7. *Let* $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \boldsymbol{Z})$ *with* $a + d = 2$ *and* $\eta_1, \eta_2 \in \boldsymbol{R}$. *Let* α *be an automorphism of* A_θ *with* $\alpha = \text{Ad}(w) \circ \alpha_g \circ \alpha_{(\eta_1, \eta_2)}$ *where* w *is a unitary element in* A_θ. *Moreover* α *satisfies one of the following conditions*:

1) *If* $g = I$, *then* $\eta_1 \notin \boldsymbol{Z} + \boldsymbol{Z}\theta$ *or* $\eta_2 \notin \boldsymbol{Z} + \boldsymbol{Z}\theta$.
2) *If* $a = 1$, $b \neq 0$ *and* $c = 0$, *then* $\eta_2 \notin \boldsymbol{Z} + \boldsymbol{Z}\theta$.
3) *If* $a = 1$, $b = 0$ *and* $c \neq 0$, *then* $\eta_1 \notin \boldsymbol{Z} + \boldsymbol{Z}\theta$.
4) *If* $a \neq 1$ *and* $b/(a-1) \in \boldsymbol{Z}$, *then*, $(b/(a-1))\eta_1 - \eta_2 \notin \boldsymbol{Z} + \boldsymbol{Z}\theta$.
5) *If* $a \neq 1$ *and* $(a-1)/b \in \boldsymbol{Z}$, *then* $\eta_1 - ((a-1)/b)\eta_2 \notin \boldsymbol{Z} + \boldsymbol{Z}\theta$.
6) *If* $a \neq 1$, $b/(a-1) \notin \boldsymbol{Z}$ *and* $(a-1)/b \notin \boldsymbol{Z}$, *then* $b\eta_1 - (a-1)\eta_2 \notin \boldsymbol{Z} + \boldsymbol{Z}\theta$.

Then α *can not be extended to any automorphism of* B_θ *for any monomorphism of* A_θ *into* B_θ.

PROOF. Let α be an automorphism of A_θ satisfying the assumptions. We suppose that α can be extended to some automorphism of B_θ for some monomorphism of A_θ into B_θ. Then, by Corollary 5, $\tilde{\tau}_*(K_0(A_\theta \times_\alpha \boldsymbol{Z})) = \boldsymbol{Z} + \boldsymbol{Z}\theta$. However this fact contradicts Proposition 6. Q.E.D.

COROLLARY 8. *Let* α *be an automorphism of* A_θ *with* $\alpha = \text{Ad}(w) \circ \alpha_{(\eta_1, \eta_2)}$ *where* $\eta_1, \eta_2 \in \boldsymbol{R}$ *and* w *is a unitary element in* A_θ. *Then* α *can be extended to some automorphism of* B_θ *if and only if* α *is inner*.

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