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On Two Variable p-Adic L-Functions and a p-Adic Class Number Formula

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Introduction.

Let K be an imaginary quadratic field with class number 1 and discriminant $-d_K$ lying inside the complex number field $\boldsymbol{C}$, and denote by O the ring of integers of K. Let E be an elliptic curve defined over K with complex multiplication by O. We denote by ψ the Grössencharacter of E over K, and by $\mathfrak{f}$ the conductor of ψ. Fix a Weierstrass model for E

$$y^2=4x^3-g_2x-g_3 \tag{0.1}$$

such that g_2, $g_3\in O$ and the discriminant $\Delta=g_2^3-27g_3^2$ of (0.1) is divisible only by primes dividing $6\mathfrak{f}$. Let $P(z)$ be the Weierstrass pe-function associated with (0.1), and L the period lattice of $P(z)$. Fix an element $\Omega_\infty\in L$ such that $L=\Omega_\infty O$.

Let p be a rational prime number prime to $6d_K\mathfrak{f}$ and we assume that p splits in K, say $(p)=\mathfrak{p}\bar{\mathfrak{p}}$. We denote by $K_\mathfrak{p}$ the completion of K at $\mathfrak{p}$ and identify $K_\mathfrak{p}$ with the rational p-adic number field $\boldsymbol{Q}_p$. Let $\boldsymbol{C}_p$ be the completion of the algebraic closure of $K_\mathfrak{p}$, and denote by I the ring of integers of $\boldsymbol{C}_p$. Let $\bar{\boldsymbol{Q}}$ denote the algebraic closure of the rational number field $\boldsymbol{Q}$ in $\boldsymbol{C}$. Fixing an embedding of $\bar{\boldsymbol{Q}}$ in $\boldsymbol{C}_p$, we regard $\bar{\boldsymbol{Q}}$ also as a field contained in $\boldsymbol{C}_p$.

If Ψ is a Grössencharacter of K, we denote by $L(\Psi, s)$ the primitive complex Hecke L-function attached to Ψ. For each integral ideal $\mathfrak{a}$ of K, let $R_\mathfrak{a}$ denote the ray class field modulo $\mathfrak{a}$ of K. If $\mathfrak{a}$ is divisible by the conductor of Ψ, then, for each $\sigma\in\mathrm{Gal}(R_\mathfrak{a}/K)$, we denote by $L_\mathfrak{a}(\sigma, \Psi, s)$ the partial zeta function attached to Ψ and $\sigma\in\mathrm{Gal}(R_\mathfrak{a}/K)$.

If χ is a primitive class character of K, we put

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$$(0.2)\qquad L_\infty(\overline{\psi^{k+j}\chi}, k)=(1-\psi^{k+j}\chi(\mathfrak{p})/N\mathfrak{p}^{j+1})(1-\overline{\psi^{k+j}\chi}(\bar{\mathfrak{p}})/N\bar{\mathfrak{p}}^{k}) \times(2\pi/\sqrt{d_K})^j\Omega_\infty^{-(k+j)}L(\overline{\psi^{k+j}\chi}, k)$$

for integers $0\leqq j<k$. Damerell's theorem states that the right hand side of (0.2) lies in $\bar{\boldsymbol{Q}}$ ([6]).

In the present paper, we shall construct a two variable p-adic power series attached to χ which interpolates essentially the values (0.2), and show the p-adic class number formula for any finite abelian extension of K.

§1. p-adic properties of Eisenstein series.

For each positive integer k, we write $K_k(z, s)$ for the analytic continuation to the whole complex s-plane of

$$K_k(z, s)+\sum_{\substack{\omega\in L\\ \omega\neq -z}}(\bar{z}+\bar{\omega})^k|\bar{z}+\bar{\omega}|^{-2s}, \qquad \mathrm{Re}(s)>1+k/2,$$

and for integers $k>j\geqq 0$, we put

$$E_{j,k}(z)=(k-1)!(2\pi/\sqrt{d_K})^j|\Omega_\infty|^{-2j}K_{k+j}(z, k).$$

We put further $E_k(z)=E_{0,k}(z)$.

Let $\sigma(z)$ be the Weierstrass σ-function of L and put $\theta(z)=\Delta\cdot\exp(-6s_2z^2)\sigma(z)^{12}$, where $s_2=\lim_{s\to 0+}\sum_{\omega\in L,\omega\neq 0}\omega^{-2}|\omega|^{-2s}$.

For any integral ideal $\mathfrak{a}$ of K prime to $\mathfrak{f}$, we put

$$\Theta(z, \mathfrak{a})=\theta(z)^{N\mathfrak{a}}/\theta(\psi(\mathfrak{a})z),$$

which is a rational function of $P(z)$ with coefficients in K ([7] §4).

For any integral ideal $\mathfrak{a}$ of K, we write $E_\mathfrak{a}$ for the group of points of $E(\bar{\boldsymbol{Q}})$ of order dividing $\mathfrak{a}$. If $a\in O$, we put $E_a=E_{(a)}$. Further, we put $E_{\mathfrak{a}\mathfrak{p}^\infty}=\bigcup_{n=0}^\infty E_{\mathfrak{a}\mathfrak{p}^n}$, $E_{\mathfrak{a}\bar{\mathfrak{p}}^\infty}=\bigcup_{m=0}^\infty E_{\mathfrak{a}\bar{\mathfrak{p}}^m}$. We write $e_\mathfrak{a}$ for the number of units in O which are congruent to 1 modulo $\mathfrak{a}$.

Concerning $E_{j,k}(z)$, we have the following properties ([8] §2):

For any integer $k>0$, any $\alpha\in \boldsymbol{C}\backslash L$ and any integral ideal $\mathfrak{a}$ of K prime to $\mathfrak{f}$,

$$(1.1)\qquad (d/dz)^k\log\Theta(z+\alpha, \mathfrak{a})|_{z=0}=12(-1)^{k-1}\{N\mathfrak{a}E_k(\alpha)-\psi(\mathfrak{a})^kE_k(\psi(\mathfrak{a})\alpha)\}.$$

If $k>j\geqq 0$, there is a polynomial $P_{j,k}(X_1, \cdots, X_{k+j})\in\boldsymbol{Z}[X_1, \cdots, X_{k+j}]$ such that $\deg P_{j,k}\leqq j+1$, $\deg_{X_1}P_{j,k}(X_1, \cdots, X_{k+j})\leqq j-1$ and

$$(1.2)\qquad E_{j,k}(z)=(-E_1(z))^jE_k(z)+2^{-j}P_{j,k}(E_1(z), \cdots, E_{k+j}(z)).$$

For any $g \in O$ and $\alpha \in g^{-1}L \setminus L$,

$$E_{j,k}(\alpha) \in K(E_g) . \tag{1.3}$$

If $\mathfrak{b}$ is an integral ideal of K prime to $g\mathfrak{f}$, then

$$E_{j,k}(\Omega_\infty/g)^{(\mathfrak{b}, K(E_g)/K)} = E_{j,k}(\psi(\mathfrak{b})\Omega_\infty/g) . \tag{1.4}$$

If the conductor of ψ^{k+j} divides $g \in O$, then for any integral ideal $\mathfrak{b}$ of K prime to $g\mathfrak{f}$,

$$E_{j,k}(\psi(\mathfrak{b})\Omega_\infty/g) = (k-1)!(2\pi/\sqrt{d_K})^j \Omega_\infty^{-(k+j)}(g^{k+j}/N\mathfrak{g}^j)e_\mathfrak{g} L_\mathfrak{g}(\sigma_\mathfrak{b}, \overline{\psi^{k+j}}, k) \tag{1.5}$$

where $\mathfrak{g}=(g)$ and $\sigma_\mathfrak{b}=(\mathfrak{b}, R_\mathfrak{g}/K)$.

Let $\hat{E}$ be the formal group giving the kernel of reduction modulo $\mathfrak{p}$ on E. The parameter of $\hat{E}$ is $t=-2x/y$. We write $\lambda: \hat{E} \xrightarrow{\sim} G_a$ for the logarithm of $\hat{E}$.

For any $g \in O$ and any integers n, $m \geqq 0$, we put $g_m = g\psi(\bar{\mathfrak{p}}^m)$ and $g_{n,m} = g\psi(\mathfrak{p}^n\bar{\mathfrak{p}}^m)$.

PROPOSITION 1.1. *Let $g \in O$ be such that $(g, \bar{\mathfrak{p}})=(1)$. Then, for any positive integer m,*

$$p^m E_1(\Omega_\infty/g_m) \in I .$$

If $k \geqq 2$, there is an integer $\delta(g, k)$ such that for all $m>0$,

$$p^{\delta(g,k)} E_k(\Omega_\infty/g_m) \in I .$$

PROOF. In a similar way as in the proof of Theorem 10 of [7], we see that, for any integral ideal $\mathfrak{a}$ of K prime to $p\mathfrak{f}$, the Taylor series expansion of $\Theta(\lambda(t)+\Omega_\infty/g_m, \mathfrak{a})$ at $t=0$ has coefficients in $K(E_{g_m}) \cap I$ and that the constant term $\Theta(\Omega_\infty/g_m, \mathfrak{a})$ is a unit of I. Since $\lambda'(t)$ is a unit power series of $\boldsymbol{Z}_p[[t]]$,

$$\begin{aligned}&(d/dz)^k \log \Theta(z+\Omega_\infty/g_m, \mathfrak{a})|_{z=0}\\ &\quad = (\lambda'(t)^{-1}d/dt)^k \log \Theta(\lambda(t)+\Omega_\infty/g_m, \mathfrak{a})|_{t=0} \in I .\end{aligned}$$

By equation (1.1), we obtain

$$N\mathfrak{a}E_k(\Omega_\infty/g_m) - \psi(\mathfrak{a})^k E_k(\psi(\mathfrak{a})\Omega_\infty/g_m) \in I . \tag{1.6}$$

Choose a generator f of $\mathfrak{f}$. Putting $\mathfrak{a}=(1+fg_{1,m})$, we see $(N\mathfrak{a}-\psi(\mathfrak{a})^k)\times E_k(\Omega_\infty/g_m) \in I$. If $k=1$, then $(N\mathfrak{a}-\psi(\mathfrak{a}))/p^m \in I^\times$, and so, $p^m E_1(\Omega_\infty/g_m) \in I$. If $k \geqq 2$, it is easy to see that there is an integer $\delta(g, k)$ independent of m such that $p^{\delta(g,k)} \in (N\mathfrak{a}-\psi(\mathfrak{a})^k)I$. Hence, $p^{\delta(g,k)}E_k(\Omega_\infty/g_m)$ lies in I for all $m>0$.

From Proposition 1.1 and equation (1.2), we deduce the following

PROPOSITION 1.2. *Let $g \in O$ be such that $(g, \bar{\mathfrak{p}})=(1)$. Then for integers $k>j\geqq 0$, there is an integer $\delta(g, k, j)$ satisfying*

$$\overline{g_m}^j E_{j,k}(\Omega_\infty/g_m)-(-\overline{g_m}E_1(\Omega_\infty/g_m))^j E_k(\Omega_\infty/g_m) \in p^{m-\delta(g,k,j)}I \,.$$

PROPOSITION 1.3. *Let $g \in O$ be such that $(g, \bar{\mathfrak{p}})=(1)$. Then, $\lim_{m\to\infty}(-\overline{g_m}E_1(\Omega_\infty/g_m))$ exists in I and is independent of g.*

PROOF. For each $m>0$, choose a set B_m of integral ideals of K such that $\{\psi(\mathfrak{b})\colon \mathfrak{b}\in B_m\}$ is a set of representatives of $1+g\bar{\mathfrak{p}}^m \bmod g\bar{\mathfrak{p}}^{m+1}$. Then

$$\overline{g_m}E_1(\Omega_\infty/g_m)=\overline{g_m}/\psi(\bar{\mathfrak{p}})\sum_{\mathfrak{b}\in B_m} E_1(\psi(\mathfrak{b})\Omega_\infty/g_{m+1}) \,.$$

We can choose a set B_m with an additional condition that $(\mathfrak{b}, p\mathfrak{f})=(1)$ for all $\mathfrak{b}\in B_m$. Then, by (1.6), $\bar{\psi}(\mathfrak{b})E_1(\Omega_\infty/g_{m+1})-E_1(\psi(\mathfrak{b})\Omega_\infty/g_{m+1})\in I$. Further, we have $\sum_{\mathfrak{b}\in B_m}\bar{\psi}(\mathfrak{b})\equiv p \pmod{\mathfrak{p}^{m+1}}$. Hence, by Proposition 1.1,

$$\overline{g_m}E_1(\Omega_\infty/g_m)\equiv \overline{g_m}/\psi(\bar{\mathfrak{p}})\cdot pE_1(\Omega_\infty/g_{m+1}) \pmod{\mathfrak{p}^m} \,,$$
$$\overline{g_m}/\psi(\bar{\mathfrak{p}})\cdot pE_1(\Omega_\infty/g_{m+1})=\overline{g_{m+1}}E_1(\Omega_\infty/g_m)$$

and we see that $\lim_{m\to\infty}(-\overline{g_m}E_1(\Omega_\infty/g_m))$ exists in I.

Next, let us show that it is independent of g. Taking account of Theorem 4.3 of [8], it is sufficient to show that

$$\overline{g'_{l,m}}E_1(\Omega_\infty/g'_{l,m})\equiv\overline{g'_{l+1,m}}E_1(\Omega_\infty/g'_{l+1,m}) \pmod{\mathfrak{p}^{m-1}}$$

holds for all $g'\in O$ with $(g', p)=(1)$ and integers $l\geqq 0$ and $m>0$. Choose a set $B'_{l,m}$ of integral ideals of K prime to $p\mathfrak{f}$ such that $\{\psi(\mathfrak{b})\colon \mathfrak{b}\in B'_{l,m}\}$ is a set of representatives $1+(g'_{l,m}) \bmod (g'_{l+1,m})$. Then,

$$\begin{aligned}\overline{g'_{l,m}}E_1(\Omega_\infty/g'_{l,m})&=\overline{g'_{l,m}}/\psi(\mathfrak{p})\cdot\sum_{\mathfrak{b}\in B'_{l,m}}E_1(\psi(\mathfrak{b})\Omega_\infty/g'_{l+1,m})\\ &\equiv\overline{g'_{l,m}}/\psi(\mathfrak{p})\cdot\sum_{\mathfrak{b}\in B'_{l,m}}\psi(\mathfrak{b})E_1(\Omega_\infty/g'_{l+1,m}) \pmod{\mathfrak{p}^{m-1}}\\ &\equiv\overline{g'_{l,m}}/\psi(\mathfrak{p})\cdot pE_1(\Omega_\infty/g'_{l+1,m}) \pmod{\mathfrak{p}^{m-1}}\,.\end{aligned}$$

Hence

$$\overline{g'_{l,m}}E_1(\Omega_\infty/g'_{l,m})\equiv\overline{g'_{l+1,m}}E_1(\Omega_\infty/g'_{l+1,m}) \pmod{\mathfrak{p}^{m-1}}\,.$$

Thus, we obtain the required result.

As shown in [8] § 4, there exists an integer $r_\mathfrak{p}\geqq 0$ such that

$p^{-r_\mathfrak{p}} \lim_{m\to\infty}(-\overline{g_m}E_1(\Omega_\infty/g_m)) \in I^\times$ and we have $r_\mathfrak{p}=0$ for almost all $\mathfrak{p}$. Further, we can choose an isomorphism η from $\hat{E}$ to the formal multiplicative group G_m over I such that $\eta(T)=\Omega_\mathfrak{p}T+\cdots$, where

$$\Omega_\mathfrak{p}=p^{-r_\mathfrak{p}}\lim_{m\to\infty}(-\overline{g_m}E_1(\Omega_\infty/g_m)) . \tag{1.7}$$

If F is a Galois extension of K containing $K(E_{\mathfrak{p}^\infty})$ (resp. $K(E_{\bar{\mathfrak{p}}^\infty})$), we denote by κ_1: $\mathrm{Gal}(F/K)\to \boldsymbol{Z}_p^\times$ (resp. κ_2: $\mathrm{Gal}(F/K)\to\boldsymbol{Z}_p^\times$) the homomorphism giving the action of $\mathrm{Gal}(F/K)$ on $E_{\mathfrak{p}^\infty}$ (resp. $E_{\bar{\mathfrak{p}}^\infty}$).

We see from (1.6) that for any integral ideal $\mathfrak{a}$ of K prime to $p\mathfrak{f}g$,

$$\bar{\psi}(\mathfrak{a})\Omega_\mathfrak{p}=p^{-r_\mathfrak{p}}\lim_{m\to\infty}(-\overline{g_m}E_1(\psi(\mathfrak{a})\Omega_\infty/g_m)) . \tag{1.8}$$

By (1.4), we have, for any $\sigma\in\mathrm{Gal}(K(E_{g\bar{\mathfrak{p}}^\infty})/K)$,

$$\kappa_2(\sigma)\Omega_\mathfrak{p}=p^{-r_\mathfrak{p}}\lim_{m\to\infty}(-\overline{g_m}E_1(\Omega_\infty/g_m)^\sigma) . \tag{1.9}$$

§ 2. Construction of two variable p-adic L-functions.

Let χ be a class character of K and $\mathfrak{f}_\chi$ the conductor of χ. We denote by R_χ the subfield of $R_{\mathfrak{f}_\chi}$ corresponding to $\mathrm{Ker}\,\chi$. If F is an abelian extension of K containing R_χ, we can regard χ also as a character of $\mathrm{Gal}(F/K)$.

If ν is a finite character of $\boldsymbol{Z}_p^\times$, $\nu\circ\kappa_1$ (resp. $\nu\circ\kappa_2$) is a character of $\mathrm{Gal}(K(E_{\mathfrak{p}^\infty})/K)$ (resp. $\mathrm{Gal}(K(E_{\bar{\mathfrak{p}}^\infty})/K)$). We denote by $\nu_\mathfrak{p}$ (resp. $\nu_{\bar{\mathfrak{p}}}$) the class character of K induced by $\nu\circ\kappa_1$ (resp. $\nu\circ\kappa_2$).

We can express χ as $\chi=\chi_1(\varphi\omega^{i_1})_\mathfrak{p}(\varphi'\omega^{i_2})_{\bar{\mathfrak{p}}}$, where χ_1 is a class character of K with $(\mathfrak{f}_{\chi_1}, p)=(1)$, φ and φ' are characters of the second kind of $\boldsymbol{Z}_p^\times$, ω is the Teichmüller character and $i_1, i_2\in\boldsymbol{Z}/(p-1)\boldsymbol{Z}$. We put $\nu_\chi=\varphi\omega^{i_1}$ and $\nu'_\chi=\varphi'\omega^{i_2}$.

Let $\mathfrak{f}_\chi=\mathfrak{g}_\chi\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^{n'_\chi}$, where $(\mathfrak{g}_\chi, p)=(1)$, and write simply $\mathfrak{g}$ for $\mathfrak{g}_\chi$. Note that $\mathfrak{g}$ depends only on χ_1 and (i_1, i_2), and that $R_{\chi_1}\subset K(E_\mathfrak{g})$. We fix a generator g of $\mathfrak{g}$. It is easy to see that p^{n_χ} (resp. $p^{n'_\chi}$) is a conductor of ν_χ (resp. ν'_χ).

For any integer $m>0$, and any integral ideal $\mathfrak{a}$ of K, $\Theta(z+\Omega_\infty/g_m, \mathfrak{a})$ is a rational function of $P(z)$ and $P'(z)$ with coefficients in $K(E_{g_m})$, and so, for each $\sigma\in\mathrm{Gal}(K(E_{g_m})/K)$, we can define a function $\Theta(z+\Omega_\infty/g_m, \mathfrak{a})^\sigma$ by applying σ on the coefficients. We put

$$\Lambda_m(z, \mathfrak{a})=\prod_{\sigma\in\mathrm{Gal}(K(E_{g_m})/R_{\chi_1}(E_{\bar{\mathfrak{p}}^m}))}\Theta(z+\Omega_\infty/g_m, \mathfrak{a})^\sigma ,$$

which is a rational function of $P(z)$ and $P'(z)$ with coefficients in $R_{\chi_1}(E_{\bar{\mathfrak{p}}^m})$.

Let I_{χ_1} denote the set of integral ideals of K which are prime to $6p\mathfrak{f}\mathfrak{f}_{\chi_1}$, and let

$$\delta_{\chi_1}=\{\mu : I_{\chi_1}\to \boldsymbol{Z} \mid \mu(\mathfrak{a})=0 \text{ for almost all } \mathfrak{a}\in I_{\chi_1} \text{ and } \sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})(N\mathfrak{a}-1)=0\} .$$

For each $\mu\in\delta_{\chi_1}$, put

$$\Theta(z;\mu)=\prod_{\mathfrak{a}\in I_{\chi_1}}\Theta(z,\mathfrak{a})^{\mu(\mathfrak{a})} \quad\text{and}\quad \Lambda_m(z;\mu)=\prod_{\mathfrak{a}\in I_{\chi_1}}\Lambda_m(z,\mathfrak{a})^{\mu(\mathfrak{a})} .$$

We put further

$$c_{m,\mu}(T)=\Lambda_m(\psi(\bar{\mathfrak{p}})^{-m}\lambda(T);\mu) \quad\text{and}\quad g_{m,\mu}(T)=\lambda'(T)^{-1}d/dT\log_p c_{m,\mu}(T) ,$$

which are in $R_{\chi_1}(E_{\bar{\mathfrak{p}}^m})[[T]]$ and also in $I[[T]]$.

For each $\tau\in\mathrm{Gal}(R_{\chi_1}(E_{\bar{\mathfrak{p}}^m})/K)$, $\kappa_2(\tau)$ is well defined modulo p^m, and so, $(1+T)^{\kappa_2(\tau)}$ is well defined modulo $((1+T)^{p^m}-1)$.

Proposition 2.1. *For each $\sigma\in\mathrm{Gal}(R_{\chi_1}/K)$, there exists a unique power series $g_{\sigma,\mu}(T_1, T_2)\in I[[T_1, T_2]]$ such that*

$$g_{\sigma,\mu}(T_1,T_2)\equiv \sum_{\substack{\tau\in\mathrm{Gal}(R_{\chi_1}(E_{\bar{\mathfrak{p}}^m})/K)\\ \tau|R_{\chi_1}=\sigma}} g_{m,\mu}{}^{\tau}(T_1)(1+T_2)^{\kappa_2(\tau)}$$
$$\mod((1+T_2)^{p^m}-1)I[[T_1, T_2]] .$$

Proof. From Lemma 6 of [1], we see that, for each $m>0$,

$$\prod_{\gamma\in\mathrm{Gal}(K(E_{\mathfrak{g}^{m+1}})/K(E_{\mathfrak{g}^m}))}\Theta(z+\Omega_\infty/g_{m+1};\mu)^{\gamma}=\Theta(z+\Omega_\infty/g_m;\mu) .$$

It follows from the definition of $g_{m,\mu}(T)$ that

$$\sum_{\gamma\in\mathrm{Gal}(R_{\chi_1}(E_{\bar{\mathfrak{p}}^{m+1}})/R_{\chi_1}(E_{\bar{\mathfrak{p}}^m}))} g_{m+1,\mu}{}^{\gamma}(T)=g_{m,\mu}(T) .$$

Hence, we deduce our assertion.

We write $i: G_m \xrightarrow{\sim} \hat{E}$ for the inverse of $\eta: \hat{E}\xrightarrow{\sim} G_m$, and put

$$h_\mu(T_1,T_2)=\sum_{\sigma\in\mathrm{Gal}(R_{\chi_1}/K)}\chi_1{}^{-1}(\sigma)g_{\sigma,\mu}(i(T_1),T_2) .$$

It is known that each $f(T_1, T_2)\in I[[T_1, T_2]]$ corresponds to an I-valued measure ν_f on $\boldsymbol{Z}_p{}^2$ ([7] §6), and for each $(j_1, j_2)\in(\boldsymbol{Z}/(p-1)\boldsymbol{Z})^2$, a Γ-transform $\Gamma_f^{(j_1,j_2)}: \boldsymbol{Z}_p{}^2\to I$ is defined by

$$\Gamma_f^{(j_1,j_2)}(s_1,s_2)=\int_{\boldsymbol{Z}_p{}^\times\times\boldsymbol{Z}_p{}^\times}\langle x_1\rangle^{s_1}\langle x_2\rangle^{s_2}\omega^{t_1}(x_1)\omega^{t_2}(x_2)d\nu_f ,$$

where we put $\langle x\rangle=x/\omega(x)$ for any $x\in\boldsymbol{Z}_p{}^\times$. Fix a topological generator

u of $1+p\boldsymbol{Z}_p$. Then, there exists a power series $f^{(j_1,j_2)}(T_1, T_2)\in I[[T_1, T_2]]$ such that

$$\Gamma_f^{(j_1,j_2)}(s_1, s_2)=f^{(j_1,j_2)}(u^{s_1}-1, u^{s_2}-1) .$$

If ν_1 and ν_2 are Dirichlet characters with conductors p^{l_1} and p^{l_2} respectively, we put

$$f_{(\nu_1,\nu_2)}(T_1, T_2)=\tau(\nu_1^{-1}, \zeta_{p^{l_1}})^{-1}\tau(\nu_2^{-1}, \zeta_{p^{l_2}})^{-1}\sum_{a=1}^{p^{l_1}}\sum_{b=1}^{p^{l_2}} \nu_1^{-1}(a)\nu_2^{-1}(b)f(\zeta_{p^{l_1}}{}^a(1+T_1)-1, \zeta_{p^{l_2}}{}^b(1+T_2)-1) , \tag{2.1}$$

where $\zeta_{p^{l_j}}$ is an arbitrary primitive p^{l_j}-th root of unity and $\tau(\nu_j^{-1}, \zeta_{p^{l_j}})=\sum_{a=1}^{p^{l_j}}\nu_j^{-1}(a)\zeta_{p^{l_j}}{}^a$ $(j=1, 2)$. The right hand side of (2.1) is independent of the choice of $\zeta_{p^{l_1}}$ and $\zeta_{p^{l_2}}$, and it belongs to $I[[T_1, T_2]]$. We have

$$\Gamma_{f_{(\varphi,\varphi')}}^{(j_1,j_2)}(s_1, s_2)=f^{(j_1,j_2)}(\varphi(u)u^{s_1}-1, \varphi'(u)u^{s_2}-1)$$

and

$$\Gamma_{f_{(\omega^{i_1},\omega^{i_2})}}^{(j_1,j_2)}(s_1, s_2)=\Gamma_f^{(j_1+i_1,j_2+i_2)}(s_1, s_2) .$$

We put

$$G_{\mu,\chi_1}^{(i_1,i_2)}(T_1, T_2)=h_\mu{}^{(i_1-1,-i_2)}(u^{-1}(1+T_1)-1, (1+T_2)^{-1}-1) .$$

Then, we have

$$G_{\mu,\chi_1}^{(i_1,i_2)}(\varphi(u)u^{s_1}-1, \varphi'(u)u^{s_2}-1)=\Gamma_{(h_\mu)_{(\nu_\chi,\nu_\chi'^{-1})}}^{(-1,0)}(s_1-1, -s_2) .$$

In what follows, we assume that for each positive integer n, a primitive p^n-th root of unity ζ_{p^n} is chosen to satisfy

$$i(\zeta_{p^n}-1)=-2P(\Omega_\infty/\psi(\mathfrak{p}^n))/P'(\Omega_\infty/\psi(\mathfrak{p}^n)) . \tag{2.2}$$

PROPOSITION 2.2. *For all integers k_1, k_2 such that $k_1>-k_2\geqq 0$ and $k_1\equiv k_2\equiv 0 \pmod{p-1}$, we have*

$$\begin{aligned} &G_{\mu,\chi_1}^{(i_1,i_2)}(\varphi(u)u^{k_1}-1, \varphi'(u)u^{k_2}-1)\\ &\quad =-12g^{k_1}\nu_\chi(g)e_\mathfrak{g}[K(E_\mathfrak{g})\colon R_\mathfrak{g}]\sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})(N\mathfrak{a}-\psi^{k_1}(\mathfrak{a})\bar\psi^{k_2}(\mathfrak{a})\chi(\mathfrak{a}))\\ &\quad\quad \times(\chi_1(\nu_\chi')_{\bar{\mathfrak{p}}})(\mathfrak{p}^{n_\chi})\psi^{k_1}(\mathfrak{p}^{n_\chi})\bar\psi^{k_2}(\mathfrak{p}^{n_\chi})\tau(\nu_\chi^{-1}, \zeta_{p^{n_\chi}})^{-1}p^{r_\mathfrak{p}k_2}\Omega_\mathfrak{p}{}^{1-k_1+k_2}(k_1-1)!\\ &\quad\quad \times L_\infty(\overline{\psi^{k_1-k_2}\chi}, k_1) . \end{aligned}$$

For each $f(T_1, T_2)\in I[[T_1, T_2]]$, we put

$$D_jf(T_1, T_2)=(1+T_j)d/dT_jf(T_1, T_2) \qquad (j=1, 2) ,$$

$$U_1 f(T_1, T_2) = f(T_1, T_2) - 1/p \sum_{\zeta^p=1} f(\zeta(1+T_1)-1, T_2)$$

and

$$U_2 f(T_1, T_2) = f(T_1, T_2) - 1/p \sum_{\zeta^p=1} f(T_1, \zeta(1+T_2)-1) .$$

For integers n, $m \geqq 0$, we choose ε_n, $\alpha_{n,m} \in O$ such that $\varepsilon_n \psi(\bar{\mathfrak{p}}) \equiv 1 \pmod{\mathfrak{p}^n}$, $\varepsilon_n \equiv 0 \pmod{\mathfrak{f}}$, $\alpha_{n,m} \equiv \psi(\mathfrak{p}^n) \pmod{\mathfrak{f} g \bar{\mathfrak{p}}^m}$, $\alpha_{n,m} \equiv g \pmod{\mathfrak{p}^n}$.

In order to prove Proposition 2.2, we show the following

PROPOSITION 2.3. *Let n, m, $a \in \boldsymbol{Z}$ with m, $a > 0$, $n \geqq 0$ and $(a, p) = (1)$. Then, for each $k > 0$ and $\tau \in \mathrm{Gal}(K(E_{g_{n,m}})/K(E_{\mathfrak{p}^n}))$,*

$$\begin{aligned} &D_1^k \log_p c_{m,\mu}^{\tau}(i(\zeta_{p^n}^a(1+T_1)-1))|_{T_1=0} \\ &= 12(-1)^{k-1} \Omega_{\mathfrak{p}}^{-k} \psi(\bar{\mathfrak{p}})^{-mk} \sum_{\mathfrak{a} \in I_{\chi_1}} \mu(\mathfrak{a}) \sum_{\gamma \in \mathrm{Gal}(K(E_{g_{n,m}})/R_{\chi_1}(E_{\mathfrak{p}^n \bar{\mathfrak{p}}^m}))} \{N\mathfrak{a} E_k(\Omega_\infty / g_{n,m}) \\ &\qquad - \psi(\mathfrak{a})^k E_k(\psi(\mathfrak{a}) \Omega_\infty / g_{n,m})\}^{\gamma \tau \sigma(\alpha_{n,m}) \tau_a} , \end{aligned}$$

where $\sigma_{(\alpha_{n,m})} = ((\alpha_{n,m}), K(E_{g_{n,m}})/K)$ and τ_a is an element of $\mathrm{Gal}(K(E_{g_{n,m}})/K(E_{g_m}))$ such that $\kappa_1(\tau_a) \equiv a \pmod{p^m}$.

PROOF. From (2.2) and the definition of $c_{m,\mu}(T)$, we see

$$c_{m,\mu}^{\tau}(i(\zeta_{p^n}^a(1+T)-1)) = \Lambda_m^{\tau}(\varepsilon_n^m a \Omega_\infty / \psi(\mathfrak{p}^n) + \psi(\bar{\mathfrak{p}})^{-m} \lambda \circ i(T); \mu) .$$

Hence

$$\begin{aligned} &D_1^k \log_p c_{m,\mu}^{\tau}(i(\zeta_{p^n}^a(1+T)-1))|_{T=0} \\ &= \Omega_{\mathfrak{p}}^{-k} \psi(\bar{\mathfrak{p}})^{-mk} (d/dz)^k \log \Lambda_m^{\tau}(z + \varepsilon_n^m a \Omega_\infty / \psi(\mathfrak{p}^n); \mu)|_{z=0} \\ &= \Omega_{\mathfrak{p}}^{-k} \psi(\bar{\mathfrak{p}})^{-mk} \sum_{\mathfrak{a} \in I_{\chi_1}} \mu(\mathfrak{a}) \sum_{\gamma \in \mathrm{Gal}(K(E_{g_{n,m}})/R_{\chi_1}(E_{\mathfrak{p}^n \bar{\mathfrak{p}}^m}))} (d/dz)^k \log_p \Theta(z + \\ &\qquad (\varepsilon_n^m \Omega_\infty / \psi(\mathfrak{p}^n)) + \Omega_\infty / g_m, \mathfrak{a})^{\tau \gamma \tau_a}|_{z=0} . \end{aligned}$$

Now, $\varepsilon_n^m / \psi(\mathfrak{p}^n) + 1/g_m = (\varepsilon_n^m g_m + \psi(\mathfrak{p}^n))/g_{n,m}$ and it is easy to see that $\psi((\varepsilon_n^m g_m + \psi(\mathfrak{p}^n)) = \varepsilon_n^m g_m + \psi(\mathfrak{p}^n)$ and that $((\varepsilon_n^m g_m + \psi(\mathfrak{p}^n)), K(E_{g_{n,m}})/K) = ((\alpha_{n,m}), K(E_{g_{n,m}})/K)$. Hence by (1.1) and (1.4), we obtain our assertion.

PROOF OF PROPOSITION 2.2. We first have

$$\begin{aligned} G_{\mu,\chi_1}^{(t_1,t_2)}(\varphi(u)u^{k_1}-1, \varphi'(u)u^{k_2}-1) &= \int_{\boldsymbol{Z}_p^\times \times \boldsymbol{Z}_p^\times} x_1^{k_1-1} x_2^{-k_2} d\nu_{(h_\mu)(\nu_\chi, \nu_\chi'^{-1})} \\ &= (D_1^{k_1-1} D_2^{-k_2})((U_1 U_2)(h_\mu)_{(\nu_\chi, \nu_\chi'^{-1})})(0, 0) \\ &= (D_1^{k_1-1} D_2^{-k_2})(U_1 (h_\mu)_{(\nu_\chi, \nu_\chi'^{-1})})(0, 0) . \end{aligned}$$

If $\nu_\chi \neq 1$, then,

$$U_1(h_\mu)_{(\nu_\chi,\nu'_\chi{}^{-1})}(T_1,\ T_2)=(h_\mu)_{(\nu_\chi,\nu'_\chi{}^{-1})}(T_1,\ T_2)\ ,$$

and

$$\begin{aligned}
&(D_1^{k_1-1}D_2^{-k_2})((h_\mu)_{(\nu_\chi,\nu'_\chi{}^{-1})})(0,\ 0)\\
&=\lim_{m\to\infty}\sum_{\sigma\in\mathrm{Gal}(R_{\chi_1}/K)}\chi_1^{-1}(\sigma)\sum_{\substack{\tau\in\mathrm{Gal}(R_{\chi_1}(E_{\bar{\mathfrak p}^m})/K)\\ \tau|R_{\chi_1}=\sigma}}D_1^{k_1-1}(g_{m,\mu}{}^\tau(i(T_1))_{(\nu_\chi,1)})|_{T_1=0}\\
&\qquad\times D_2^{-k_2}(((1+T_2)^{\kappa_2(\tau)})_{(1,\nu'_\chi{}^{-1})})|_{T_2=0}\\
&=\tau(\nu_\chi^{-1},\ \zeta_{p^{n_\chi}})^{-1}\lim_{m\to\infty}\sum_{\tau\in\mathrm{Gal}(R_{\chi_1}(E_{\bar{\mathfrak p}^m})/K)}(\chi_1(\nu'_\chi)_{\bar{\mathfrak p}}\kappa_2^{k_2})^{-1}(\tau)\sum_{a=1}^{p^{n_\chi}}\nu_\chi^{-1}(a)\\
&\qquad\times D_1^{k_1-1}(g_{m,\mu}{}^\tau(i(\zeta_{p^{n_\chi}}{}^a(1+T_1)-1)))|_{T_1=0}\\
&=\tau(\nu_\chi^{-1},\ \zeta_{p^{n_\chi}})^{-1}\lim_{m\to\infty}\sum_{\tau\in\mathrm{Gal}(R_{\chi_1}(E_{\bar{\mathfrak p}^m})/K)}(\chi_1(\nu'_\chi)_{\bar{\mathfrak p}}\kappa_2^{k_2})^{-1}(\tau)\sum_{a=1}^{p^{n_\chi}}\nu_\chi^{-1}(a)\Omega_{\mathfrak p}\\
&\qquad\times D_1^{k_1}(\log_p c_{m,\mu}{}^\tau(i(\zeta_{p^{n_\chi}}{}^a(1+T_1)-1)))|_{T_1=0}\ .
\end{aligned}$$

By Proposition 2.3, we obtain

$$\begin{aligned}
&G_{\mu,\chi_1}^{(i_1,i_2)}(\varphi(u)u^{k_1}-1,\ \varphi'(u)u^{k_2}-1)\\
&=-12\Omega_{\mathfrak p}^{1-k_1}\tau(\nu_\chi^{-1},\ \zeta_{p^{n_\chi}})^{-1}\lim_{m\to\infty}\psi(\mathfrak p)^{-mk_1}\sum_{\sigma\in\mathrm{Gal}(K(E_{g_{n_\chi,m}})/K)}\chi^{-1}(\sigma)\kappa_2(\sigma)^{-k_2}\\
&\qquad\times\sum_{\mathfrak a\in I_{\chi_1}}\mu(\mathfrak a)\{N\mathfrak a E_{k_1}(\Omega_\infty/g_{n_\chi,m})-\psi(\mathfrak a)^{k_1}E_{k_1}(\psi(\mathfrak a)\Omega_\infty/g_{n_\chi,m})\}^{\sigma\sigma(\alpha_{n_\chi,m})}\\
&=-12\Omega_{\mathfrak p}^{1-k_1}\tau(\nu_\chi^{-1},\ \zeta_{p^{n_\chi}})^{-1}\sum_{\mathfrak a\in I_{\chi_1}}\mu(\mathfrak a)(N\mathfrak a-\psi(\mathfrak a)^{k_1}\bar\psi(\mathfrak a)^{k_2}\chi(\mathfrak a))\\
&\quad\times\lim_{m\to\infty}\{\psi(\bar{\mathfrak p})^{-mk_1}\chi((\alpha_{n_\chi,m}))\kappa_2(\sigma_{(\alpha_{n_\chi,m})})^{-k_2}\\
&\qquad\times\sum_{\tau\in\mathrm{Gal}(K(E_{g_{n_\chi,m}})/K)}\chi^{-1}(\tau)\kappa_2(\tau)^{-k_2}E_{k_1}(\Omega_\infty/g_{n_\chi,m})^\tau\}\ .
\end{aligned}$$

By Propositions 1.2 and 1.3, and equations (1.5), (1.7) and (1.9), we see

$$\begin{aligned}
&\lim_{m\to\infty}\psi(\bar{\mathfrak p})^{-mk_1}\sum_{\tau\in\mathrm{Gal}(K(E_{g_{n_\chi,m}})/K)}\chi^{-1}(\tau)\kappa_2(\tau)^{-k_2}E_{k_1}(\Omega_\infty/g_{n_\chi,m})^\tau\\
&=\lim_{m\to\infty}\psi(\bar{\mathfrak p})^{-mk_1}\sum_{\tau\in\mathrm{Gal}(K(E_{g_{n_\chi,m}})/K)}\{\chi^{-1}(\tau)\kappa_2(\tau)^{-k_2}(p^{r_{\mathfrak p}}\kappa_2(\tau)\Omega_{\mathfrak p})^{k_2}\\
&\qquad\times\overline{g_{n_\chi,m}}^{-k_2}E_{-k_2,k_1}(\Omega_\infty/g_{n_\chi,m})^\tau\}\\
&=p^{r_{\mathfrak p}k_2}\Omega_{\mathfrak p}^{k_2}\lim_{m\to\infty}\psi(\bar{\mathfrak p})^{-mk_1}\overline{g_{n_\chi,m}}^{-k_2}[K(E_{g_{n_\chi,m}}):R_{(g_{n_\chi,m})}]\\
&\quad\times\sum_{\tau\in\mathrm{Gal}(R_{(g_{n_\chi,m})}/K)}\{\chi^{-1}(\tau)(k_1-1)!(2\pi/\sqrt{d_K})^{-k_2}\Omega_\infty^{-(k_1-k_2)}\\
&\qquad\times(g_{n_\chi,m}{}^{k_1-k_2}/Ng_{n_\chi,m}{}^{-k_2})e_{(g_{n_\chi,m})}L_{(g_{n_\chi,m})}(\tau,\ \overline{\psi^{k_1-k_2}},\ k_1)\}\\
&=p^{r_{\mathfrak p}k_2}\Omega_{\mathfrak p}^{k_2}g^{k_1}\psi(\mathfrak p^{n_\chi})^{k_1}e_{\mathfrak g}[K(E_{\mathfrak g}):R_{\mathfrak g}](k_1-1)!L_\infty(\overline{\psi^{k_1-k_2}}\chi,\ k_1)\ .
\end{aligned}$$

Now, it is easy to see $\chi((\alpha_{n_\chi,m}))=(\chi_1(\nu'_\chi)_{\bar{\mathfrak p}})(\mathfrak p^{n_\chi})\nu_\chi(g)$ for $m\geqq 2$, and $\kappa_2(\sigma_{(\alpha_{n_\chi,m})})\equiv\bar\psi(\mathfrak p)^{n_\chi}\ (\mathrm{mod}\ \mathfrak p^m)$. Hence, we obtain our assertion in the case $\nu_\chi\neq 1$.

If $\nu_\chi=1$, then

$$G_{\mu,\chi_1}^{(i_1,i_2)}(\varphi(u)u^{k_1}-1,\ \varphi'(u)u^{k_2}-1)$$
$$=\lim_{m\to\infty}\sum_{\tau\in\mathrm{Gal}(R_{\chi_1}(E_{\bar{\mathfrak{p}}^m})/K)}\chi^{-1}(\tau)\kappa_2^{-k_2}(\tau)D_1^{k_1-1}\Big\{(1-1/p)g_{m,\mu}{}^{\tau}(i(T_1))$$
$$-1/p\sum_{a=1}^{p-1}g_{m,\mu}{}^{\tau}(i(\zeta_p{}^a(1+T_1)-1))\Big\}\Big|_{T_1=0}\,.$$

As in the previous case, we deduce

$$\lim_{m\to\infty}\sum_{\tau\in\mathrm{Gal}(R_{\chi_1}(E_{\bar{\mathfrak{p}}^m})/K)}\chi^{-1}(\tau)\kappa_2^{-k_2}(\tau)D_1^{k_1-1}(g_{m,\mu}{}^{\tau}(i(T_1)))|_{T_1=0}$$
$$=-12p^{r_p k_2}\Omega_p{}^{1-k_1+k_2}g^{k_1}e_8[K(E_g)\colon R_8]\sum_{\mathfrak{a}\in I_{\chi_1}}\{\mu(\mathfrak{a})(N\mathfrak{a}-\psi(\mathfrak{a})^{k_1}\bar{\psi}(\mathfrak{a})^{k_2}\chi(\mathfrak{a}))$$
$$\times(k_1-1)!(2\pi/\sqrt{d_K})^{-k_2}\Omega_\infty{}^{-(k_1-k_2)}(1-\overline{\psi^{k_1-k_2}\chi}(\bar{\mathfrak{p}})/N\bar{\mathfrak{p}}^{k_1})L(\overline{\psi^{k_1-k_2}\chi},\ k_1)\}\ ,$$

and

$$\lim_{m\to\infty}\sum_{\tau\in\mathrm{Gal}(R_{\chi_1}(E_{\bar{\mathfrak{p}}^m})/K)}\chi^{-1}(\tau)\sum_{a=1}^{p-1}D_1^{k_1-1}(g_{m,\mu}{}^{\tau}(i(\zeta_p{}^a(1+T_1)-1)))|_{T_1=0}$$
$$=-12p^{r_p k_2}\Omega_p{}^{1-k_1+k_2}g^{k_1}e_8[K(E_g)\colon R_8]\sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})(N\mathfrak{a}-\psi(\mathfrak{a})^{k_1}\bar{\psi}(\mathfrak{a})^{k_2}\chi(\mathfrak{a}))$$
$$\times((k_1-1)!(2\pi/\sqrt{d_K})^{-k_2}\Omega_\infty{}^{-(k_1-k_2)}\psi(\mathfrak{p})^{k_1}\bar{\psi}(\mathfrak{p})^{k_2}\chi(\mathfrak{p})$$
$$\times(1-\overline{\psi^{k_1-k_2}\chi}(\mathfrak{p})/N\mathfrak{p}^{k_1})(1-\overline{\psi^{k_1-k_2}\chi}(\bar{\mathfrak{p}})/N\bar{\mathfrak{p}}^{k_1})L(\overline{\psi^{k_1-k_2}\chi},\ k_1)\ .$$

Hence, we see that our assertion holds also in the case $\nu_\chi=1$.

For each $x\in \boldsymbol{Z}_p{}^\times$, let $l(x)\in\boldsymbol{Z}_p$ be such that $\langle x\rangle=u^{l(x)}$.
We put

$$A_{\mu,\chi_1}^{(i_1,i_2)}(T_1,\ T_2)=-12\Omega_p\omega^{i_1}(g)[K(E_g)\colon R_8](1+T_1)^{l(g)}$$
$$\times\sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})\{N\mathfrak{a}-\omega^{i_1}(\psi(\mathfrak{a}))\omega^{i_2}(\bar{\psi}(\mathfrak{a}))\chi_1(\mathfrak{a})(1+T_1)^{l(\psi(\mathfrak{a}))}(1+T_2)^{l(\bar{\psi}(\mathfrak{a}))}\}\ .$$

Then, for all integers k_1, k_2,

$$A_{\mu,\chi_1}^{(i_1,i_2)}(\varphi(u)u^{k_1}-1,\ \varphi'(u)u^{k_2}-1)=-12(\nu_\chi\omega^{-k_1})(g)g^{k_1}e_8[K(E_g)\colon R_8]$$
$$\times\sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})\{N\mathfrak{a}-\psi(\mathfrak{a})^{k_1}\bar{\psi}(\mathfrak{a})^{k_2}(\chi\omega_{\mathfrak{p}}{}^{-k_1}\omega_{\bar{\mathfrak{p}}}{}^{-k_2})(\mathfrak{a})\}\ . \tag{2.3}$$

Let $\delta_{0,\chi_1}=\{\mu\in\delta_{\chi_1}\mid \mu(\mathfrak{a})=0$ for all $\mathfrak{a}\in I_{\chi_1}$ such that $\chi_1(\mathfrak{a})\neq1\}$ and denote by $H_{\chi_1}^{(i_1,i_2)}$ the ideal of $\boldsymbol{Z}_p[[T_1,\ T_2]]$ generated by $\{A_{\mu,\chi_1}^{(i_1,i_2)}(T_1,\ T_2)\mid \mu\in\delta_{0,\chi_1}\}$. Then, by using a method similar to the proof of Lemma 28 of [7], we obtain the following

PROPOSITION 2.4. *Unless* $(i_1,\ i_2)=(0,\ 0)$ *or* $(1,\ 1)$, $H_{\chi_1}^{(i_1,i_2)}=\boldsymbol{Z}_p[[T_1,\ T_2]]$, *and* $H_{\chi_1}^{(0,0)}$ *is the ideal of* $\boldsymbol{Z}_p[[T_1,\ T_2]]$ *generated by* T_1 *and* T_2, *and* $H_{\chi_1}^{(1,1)}$ *is the ideal generated by* $1+T_1-u$ *and* $1+T_2-u$.

THEOREM 2.5. *There exists a power series* $G_\chi(T_1, T_2) \in I[[T_1, T_2]]$ *such that for all integers* k_1, k_2 *with* $k_1 > -k_2 \geqq 0$ *and* $k_1 \equiv k_2 \equiv 0 \pmod{p-1}$,

$$G_\chi(u^{k_1}-1, u^{k_2}-1) = p^{r_p k_2} \psi^{k_1}(\mathfrak{p}^{n_\chi}) \tau(\nu_\chi^{-1}, \zeta_{p^{n_\chi}})^{-1} (k_1-1)!\, \Omega_p^{-(k_1-k_2)} \times L_\infty(\overline{\psi^{k_1-k_2}\chi}, k_1) . \tag{2.4}$$

PROOF. Put $G_{\chi_1}^{(i_1,i_2)}(T_1, T_2) = G_{\mu,\chi_1}^{(i_1,i_2)}(T_1, T_2)/\Omega_p A_{\mu,\chi_1}^{(i_1,i_2)}(T_1, T_2)$. From Proposition 2.2 and equation (2.3), we see that $G_{\chi_1}^{(i_1,i_2)}(T_1, T_2)$ is independent of $\mu \in \delta$. We put further

$$G_\chi(T_1, T_2) = (\chi_1(\nu_\chi)_{\bar{\mathfrak{p}}})^{-1}(\mathfrak{p}^{n_\chi})(1+T_2)^{-l(\psi(\bar{\mathfrak{p}}^{n_\chi}))} \times G_{\chi_1}^{(i_1,i_2)}(\varphi(u)(1+T_1)-1, \varphi'(u)(1+T_2)-1) .$$

Then, equation (2.4) holds. It remains to prove $G_{\chi_1}^{(i_1,i_2)}(T_1, T_2)$ lies in $I[[T_1, T_2]]$. Unless $(i_1, i_2) = (0, 0)$ or $(1, 1)$, it is obvious from Proposition 2.4 that $G_{\chi_1}^{(i_1,i_2)}(T_1, T_2) \in I[[T_1, T_2]]$.

If $(i_1, i_2) = (0, 0)$, then, Proposition 2.4 shows that both $T_1 G_{\chi_1}^{(0,0)}(T_1, T_2)$ and $T_2 G_{\chi_1}^{(0,0)}(T_1, T_2)$ are in $I[[T_1, T_2]]$. Hence, $G_{\chi_1}^{(0,0)}(T_1, T_2)$ itself belongs to $I[[T_1, T_2]]$.

In a similar way, we see that $G_{\chi_1}^{(1,1)}(T_1, T_2) \in I[[T_1, T_2]]$.

REMARK. The case $\chi = 1$ is already treated in [4] and [7]. See also [2].

In what follows, we put

$$L_p(s_1, s_2; \chi) = G_\chi(u^{1-s_1}-1, u^{1-s_2}-1) .$$

§3. Calculation of $L_p(1, 1; \chi)$.

As in the previous section, let $\chi = \chi_1(\nu_\chi)_{\mathfrak{p}}(\nu'_\chi)_{\bar{\mathfrak{p}}}$, $\nu_\chi = \varphi\omega^{i_1}$, $\nu'_\chi = \varphi'\omega^{i_2}$, $\mathfrak{f}_\chi = \mathfrak{g}\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^{n'_\chi}$ and $(\mathfrak{f}_{\chi_1}, p) = (1)$.

For any integral ideal $\mathfrak{a}$ of K, let $Cl(\mathfrak{a})$ denote the group of ray classes modulo $\mathfrak{a}$ of K. We write $k_\mathfrak{a}$ for the smallest positive integer in $\mathfrak{a}$. For each $C \in Cl(\mathfrak{a})$, let $\varphi_\mathfrak{a}(C) \in R_\mathfrak{a}$ be the ray class invariant defined in [5] §2. If $\mathfrak{f}_\chi$ divides $\mathfrak{a}$, we put

$$S_\mathfrak{a}^{(p)}(\chi) = \sum_{C \in Cl(\mathfrak{a})} \chi^{-1}(C) \log_p \varphi_\mathfrak{a}(C) \in C_p .$$

We put further $S^{(p)}(\chi) = S_{\mathfrak{f}_\chi}^{(p)}(\chi)$.

THEOREM 3.1. *If* $\chi \neq 1$, *then*

$$L_p(1, 1; \chi) = -(1/12 k_{\mathfrak{f}_\chi} \tau(\nu_\chi^{-1}, \zeta_{p^{n_\chi}}))(1-\chi(\mathfrak{p})/p)(1-\chi^{-1}(\bar{\mathfrak{p}}))S^{(p)}(\chi) .$$

Proof. We first have

$$G_{\mu,\chi_1}^{(i_1,i_2)}(\varphi(u)-1,\ \varphi'(u)-1)=\Gamma_{(h_\mu)(\nu_\chi,\nu_\chi'^{-1})}^{(-1,0)}(-1,\ 0)$$
$$=\int_{Z_p^\times\times Z_p^\times} x_1^{-1}d\nu_{(h_\mu)(\nu_\chi,\nu_\chi'^{-1})}\ .$$

From the definition of $h_\mu(T_1,\ T_2)$ and Lemma 4.1 of [4], we deduce

$$G_{\mu,\chi_1}^{(i_1,i_2)}(\varphi(u)-1,\ \varphi'(u)-1)$$
$$=\Omega_p\tau(\nu_\chi^{-1},\ \zeta_{p^{n_\chi}})^{-1}\lim_{m\to\infty}\sum_{\tau\in \mathrm{Gal}(R_{\chi_1}(E\bar{\mathfrak{p}}^m)/K)}(\chi_1(\nu_\chi')_{\bar{\mathfrak{p}}})^{-1}(\tau)\Big\{U_1\sum_{a=1}^{p^{n_\chi}}\nu_\chi^{-1}(a)$$
$$\times\log_p c_{m,\mu}{}^{\tau}(i(\zeta_{p^{n_\chi}}{}^{a}(1+T_1)-1))\Big\}\Big|_{T_1=0}\ .$$

If $\nu_\chi\neq 1$, then

$$G_{\mu,\chi_1}^{(i_1,i_2)}(\varphi(u)-1,\ \varphi'(u)-1)$$
$$=\Omega_p\tau(\nu_\chi^{-1},\ \zeta_{p^{n_\chi}})^{-1}\lim_{m\to\infty}\sum_{\tau\in \mathrm{Gal}(R_{\chi_1}(E\bar{\mathfrak{p}}^m)/K)}\sum_{a=1}^{p^{n_\chi}}(\chi_1(\nu_\chi')_{\bar{\mathfrak{p}}})^{-1}(\tau)\nu_\chi^{-1}(a)$$
$$\times\log_p c_{m,\mu}{}^{\tau}(i(\zeta_{p^{n_\chi}}{}^{a}-1))$$
$$=\Omega_p\tau(\nu_\chi^{-1},\ \zeta_{p^{n_\chi}})^{-1}\lim_{m\to\infty}\sum_{\tau\in \mathrm{Gal}(R_{\chi_1}(E_{\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^m})/K)}\chi^{-1}(\tau)$$
$$\times\log_p \Lambda_m(\varepsilon_{n_\chi}{}^{m}\Omega_\infty/\psi(\mathfrak{p}^{n_\chi});\ \mu)^{\tau}$$
$$=\Omega_p\tau(\nu_\chi^{-1},\ \zeta_{p^{n_\chi}})^{-1}\lim_{m\to\infty}\sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})\sum_{\tau\in \mathrm{Gal}(K(E_{\mathfrak{g}_{n_\chi,m}})/K)}\chi^{-1}(\tau)$$
$$\times\log_p \Theta(\varepsilon_{n_\chi}{}^{m}\Omega_\infty/\psi(\mathfrak{p}^{n_\chi})+\Omega_\infty/g_m,\ \mathfrak{a})^{\tau}\ .$$

From Theorem 1 and Proposition 10 of [5], we see

$$\Theta(\varepsilon_{n_\chi}{}^{m}\Omega_\infty/\psi(\mathfrak{p}^{n_\chi})+\Omega_\infty/g_m,\ \mathfrak{a})^{k_{\mathfrak{g}\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^m}}$$
$$=(\varphi_{\mathfrak{g}\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^m}(C_0)^{N\mathfrak{a}}/\varphi_{\mathfrak{g}\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^m}(C_{\mathfrak{a}}))^{((\alpha_{n_\chi,m}),R_{\mathfrak{g}\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^m}/K)}\ ,$$

where C_0 is the unit ray class of $Cl(\mathfrak{g}\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^m)$ and $C_{\mathfrak{a}}$ is the class of $Cl(\mathfrak{g}\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^m)$ containing $\mathfrak{a}$. By Theorem 2 of [5], we obtain

$$G_{\mu,\chi_1}^{(i_1,i_2)}(\varphi(u)-1,\ \varphi'(u)-1)$$
$$=\Omega_p\tau(\nu_\chi^{-1},\ \zeta_{p^{n_\chi}})^{-1}\sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})(N\mathfrak{a}-\chi(\mathfrak{a}))\lim_{m\to\infty}\chi((\alpha_{n_\chi,m}))/k_{\mathfrak{g}\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^m}$$
$$\times\sum_{\tau\in \mathrm{Gal}(K(E_{\mathfrak{g}_{n_\chi,m}})/K)}\chi^{-1}(\tau)\log_p\varphi_{\mathfrak{g}\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^m}(C_0)^{\tau}$$
$$=\Omega_p\tau(\nu_\chi^{-1},\ \zeta_{p^{n_\chi}})^{-1}(\chi_1(\nu_\chi')_{\bar{\mathfrak{p}}})(\mathfrak{p}^{n_\chi})\nu_\chi(g)\sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})(N\mathfrak{a}-\chi(\mathfrak{a}))$$
$$\times\lim_{m\to\infty}[K(E_{\mathfrak{g}_{n_\chi,m}}):\ R_{\mathfrak{g}\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^m}](1/k_{\mathfrak{g}\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^m})S_{\mathfrak{g}\mathfrak{p}^{n_\chi}\bar{\mathfrak{p}}^m}^{(p)}(\chi)$$
$$=\Omega_p\tau(\nu_\chi^{-1},\ \zeta_{p^{n_\chi}})^{-1}(\chi_1(\nu_\chi')_{\bar{\mathfrak{p}}})(p^{n_\chi})\nu_\chi(g)e_{\mathfrak{g}}[K(E_{\mathfrak{g}}):\ R_{\mathfrak{g}}]\sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})(N\mathfrak{a}-\chi(\mathfrak{a}))$$
$$\times(1/k_{\mathfrak{f}_\chi})(1-\chi^{-1}(\bar{\mathfrak{p}}))S^{(p)}(\chi)\ .$$

Taking account of equation (2.3) in the case $k_1=k_2=0$, and of the construction of the power series $G_\chi(T_1, T_2)$ in the proof of Theorem 2.5, we obtain our assertion in the case $\nu_\chi\neq 1$.

Next, assume that $\nu_\chi=1$. Then,

$$\begin{aligned}&G_{\mu,\chi_1}^{(i_1,i_2)}(\varphi(u)-1,\ \varphi'(u)-1)\\&\quad=\Omega_\mathfrak{p}\lim_{m\to\infty}\{(1-1/p)\sum_{\tau\in\mathrm{Gal}(R_{\chi_1}(E_{\bar{\mathfrak{p}}^m})/K)}\chi^{-1}(\tau)\log_p c_{m,\mu}{}^{\tau}(0)\\&\qquad\qquad-1/p\sum_{\tau\in\mathrm{Gal}(R_{\chi_1}(E_{\mathfrak{p}\bar{\mathfrak{p}}^m})/K)}\chi^{-1}(\tau)\log_p c_{m,\mu}(i(\zeta_p-1))^{\tau}\}\,.\end{aligned}$$

As in the previous case, we deduce

$$\begin{aligned}&\sum_{\tau\in\mathrm{Gal}(R_{\chi_1}(E_{\bar{\mathfrak{p}}^m})/K)}\chi^{-1}(\tau)\log_p c_{m,\mu}{}^{\tau}(0)\\&\quad=e_\mathfrak{g}[K(E_\mathfrak{g})\colon R_\mathfrak{g}]\sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})(N\mathfrak{a}-\chi(\mathfrak{a}))(1/k_{\mathfrak{g}\bar{\mathfrak{p}}^m})S^{(p)}_{\mathfrak{g}\bar{\mathfrak{p}}^m}(\chi)\end{aligned}$$

and

$$\begin{aligned}&\sum_{\tau\in\mathrm{Gal}(R_{\chi_1}(E_{\mathfrak{p}\bar{\mathfrak{p}}^m})/K)}\chi^{-1}(\tau)\log_p c_{m,\mu}(i(\zeta_p-1))^{\tau}\\&\quad=e_\mathfrak{g}[K(E_\mathfrak{g})\colon R_\mathfrak{g}]\sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})(N\mathfrak{a}-\chi(\mathfrak{a}))(\chi(\mathfrak{p})/k_{\mathfrak{g}\mathfrak{p}\bar{\mathfrak{p}}^m})S^{(p)}_{\mathfrak{g}\mathfrak{p}\bar{\mathfrak{p}}^m}(\chi)\,.\end{aligned}$$

By Theorem 2 of [5], we obtain

$$\begin{aligned}G_{\mu,\chi_1}^{(i_1,i_2)}(\varphi(u)-1,\ \varphi'(u)-1)&=\Omega_\mathfrak{p}e_\mathfrak{g}[K(E_\mathfrak{g})\colon R_\mathfrak{g}]\sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})(N\mathfrak{a}-\chi(\mathfrak{a}))\\&\quad\times(1/k_{\mathfrak{f}_\chi})(1-\chi(\mathfrak{p})/p)(1-\chi^{-1}(\bar{\mathfrak{p}}))S^{(p)}(\chi)\,.\end{aligned}$$

Hence, our assertion holds also in this case.

§ 4. One variable p-adic L-functions.

In the next section, we prove a p-adic class number formula using two variable p-adic L-functions. However, for a character of which conductor is prime to $\bar{\mathfrak{p}}$, we must use a one variable p-adic L-function. Hence, in this section, we summarize the construction and basic properties of a one variable p-adic L-function attached to a character χ with $(\mathfrak{f}_\chi, \bar{\mathfrak{p}})=(1)$. All the results in this section can be obtained by using methods similar to § 2 and § 3 ([1], [2], [3]).

As in § 2, we express $\chi=\chi_1(\nu_\chi)_\mathfrak{p}$ with $\nu_\chi=\varphi\omega^i$ and let $\mathfrak{f}_\chi=\mathfrak{g}\mathfrak{p}^{n_\chi}$ and $\mathfrak{g}=(g)$. For any integral ideal $\mathfrak{a}$ of K, put

$$\Lambda(z,\mathfrak{a})=\prod_{\sigma\in\mathrm{Gal}(K(E_\mathfrak{g})/R_{\chi_1})}\Theta(z+\Omega_\infty/g,\mathfrak{a})^{\sigma}\,.$$

For each $\mu\in\delta_{\chi_1}$, we put

$$\Lambda(z;\mu)=\prod_{\mathfrak{a}\in I_{\chi_1}}\Lambda(z,\mathfrak{a})^{\mu(\mathfrak{a})}\ ,\quad c_\mu(T)=\Lambda(\lambda(T);\mu)\ ,$$

$$g_\mu(T)=\lambda'(T)^{-1}d/dT\log_p c_\mu(T)\quad\text{and}\quad h_\mu(T)=\sum_{\sigma\in\mathrm{Gal}(R_{\chi_1}/K)}\chi_1^{-1}(\sigma)g_\mu{}^\sigma(i(T))\ .$$

Then, $c_\mu(T)$, $g_\mu(T)$ and $h_\mu(T)$ belong to $I[[T]]$.

Each $f(T)\in I[[T]]$ corresponds to an I-valued measure ν_f on $\boldsymbol{Z}_p$, and for each $j\in\boldsymbol{Z}/(p-1)\boldsymbol{Z}$, a Γ-transform $\Gamma_f^{(j)}:\boldsymbol{Z}_p\to I$ is defined by $\Gamma_f^{(j)}(s)=\int_{Z_p^\times}\langle x\rangle^s\omega^j(x)d\nu_f$. Moreover, there is $f^{(j)}(T)\in I[[T]]$ such that $\Gamma_f^{(j)}(s)=f^{(j)}(u^s-1)$. For any Dirichlet character ν with conductor p^l, we put

$$f_\nu(T)=\tau(\nu^{-1},\zeta_{p^l})^{-1}\sum_{a=1}^{p^l}\nu^{-1}(a)f(\zeta_{p^l}{}^a(1+T)-1)\ .$$

For each $\mu\in\delta_{\chi_1}$, we put $g_{\mu,\chi_1}^{(i)}(T)=h_\mu{}^{(i-1)}(u^{-1}(1+T)-1)$.

PROPOSITION 4.1. *For all integers $k>0$ with $k\equiv0 \pmod{p-1}$, we have*

$$g_{\mu,\chi_1}^{(i)}(\varphi(u)u^k-1)=-12g^k\chi_1(\mathfrak{p}^{n_\chi})\nu_\chi(g)e_{\mathfrak{g}}[K(E_{\mathfrak{g}}):R_{\mathfrak{g}}]\sum_{\mathfrak{a}\in I_{\chi_1}}\mu(\mathfrak{a})(N\mathfrak{a}-\psi^k(\mathfrak{a})\chi(\mathfrak{a}))$$
$$\times\psi^k(\mathfrak{p}^{n_\chi})\tau(\nu_\chi{}^{-1},\zeta_{p^{n_\chi}})^{-1}\Omega_p{}^{1-k}\Omega_\infty{}^{-k}(k-1)!\,(1-\psi^k\chi(\mathfrak{p})/p)L(\overline{\psi^k\chi},k)\ .$$

THEOREM 4.2. *There is a power series $g_{\chi_1}^{(i)}(T)\in T^{a_{\chi_1}^{(i)}}I[[T]]$ such that for all integers $k>0$ with $k\equiv0 \pmod{p-1}$, we have*

$$g_{\chi_1}^{(i)}(\varphi(u)u^k-1)=\psi^k(\mathfrak{p}^{n_\chi})\chi_1(\mathfrak{p}^{n_\chi})\tau(\nu_\chi{}^{-1},\zeta_{p^{n_\chi}})^{-1}(k-1)!$$
$$\times\Omega_p{}^{-k}\Omega_\infty{}^{-k}(1-\psi^k\chi(\mathfrak{p})/p)L(\overline{\psi^k\chi},k)\ ,$$

where $a_{\chi_1}^{(i)}=-1$ if $\chi_1\omega_p{}^i=1$ and $a_{\chi_1}^{(i)}=0$ otherwise.

We put $L_p(s,\chi)=\chi_1{}^{-1}(\mathfrak{p}^{n_\chi})g_{\chi_1}^{(i)}(\varphi(u)u^{1-s}-1)$.

THEOREM 4.3. *If $\chi\neq1$, then*

$$L_p(1,\chi)=-(1/12k_{\mathfrak{f}_\chi}e_{\mathfrak{f}_\chi}\tau(\nu_\chi{}^{-1},\zeta_{p^{n_\chi}}))(1-\chi(\mathfrak{p})/p)S^{(p)}(\chi)\ .$$

§ 5. *p*-adic class number formula.

Let H be a finite abelian extension of K, and denote by h_H, $R^{(p)}(H)$, $d_{H/K}$, W_H and G_H, the class number of H, the p-adic regulator of H, a generator of the relative discriminant of H over K, the number of the roots of unity in H and the Galois group of H over K, respectively. Each character $\chi\in\hat{G}_H$ defines a primitive class character of K, which we also denote by χ.

For any α, $\beta\in\boldsymbol{C}_p^\times$, let $\alpha\sim\beta$ mean $\alpha/\beta\in I^\times$.

THEOREM 5.1. *We have*

$$\Big(\prod_{\substack{\chi\in\hat{G}_H\\ \mathfrak{p}|\mathfrak{f}_\chi}}(1-\chi(\mathfrak{p})/p)^{-1}L_p(1,1;\chi)\Big)\Big(\prod_{\substack{\chi\in\hat{G}_H-\{1\}\\ \mathfrak{p}\nmid\mathfrak{f}_\chi}}(1-\chi(\mathfrak{p})/p)^{-1}L_p(1,\chi)\Big)$$
$$\sim h_H/W_H\sqrt{d_{H/K}}R^{(p)}(H)\ .$$

PROOF. Let $\mathfrak{f}_H$ be the conductor of H over K and write $\mathfrak{f}_H=\prod_{i=1}^t \mathfrak{l}_i^{e_i}$, where $\mathfrak{l}_1,\cdots,\mathfrak{l}_t$ are distinct prime ideals of K and $e_1,\cdots,e_t$ are positive integers. For each $\varepsilon=(\varepsilon_1,\cdots\varepsilon_t)\in\{0,1\}^t$, we put $\mathfrak{f}_\varepsilon=\prod_{i=1}^t\mathfrak{l}_i^{\varepsilon_i e_i}$ and for each $\sigma\in\mathrm{Gal}(R_{\mathfrak{f}_H}/K)$, we put

$$\xi_\sigma=\Big(\prod_{\varepsilon\in\{0,1\}^t}\varphi_{\mathfrak{f}_\varepsilon}(C_{0,\varepsilon})^\sigma/\varphi_{\mathfrak{f}_\varepsilon}(C_{0,\varepsilon})\Big)^{k_{\mathfrak{f}_H}/k_{\mathfrak{f}_\varepsilon}}\ ,$$

where $C_{0,\varepsilon}$ is the unit ray class of $Cl(\mathfrak{f}_\varepsilon)$. For each $\tau\in G_H$, choose an extension $\tilde{\tau}$ of τ to $R_{\mathfrak{f}_H}$. Then, $N_{R_{\mathfrak{f}_H}/H}\xi_{\tilde{\tau}}$ is independent of the choice of $\tilde{\tau}$. We put $\eta_\tau=N_{R_{\mathfrak{f}_H}/H}\xi_{\tilde{\tau}}$. Let ε_H be the group generated by $\{\eta_\tau|\tau\in G_H\}$. Then, ε_H is a subgroup of the unit group E_H of H, and the regulator $R(\varepsilon_H)$ of ε_H satisfies

$$\begin{aligned}R(\varepsilon_H)&=\det(\log|\eta_\tau{}^\gamma|^2)_{\tau,\gamma\in G_H-\{1\}}\\
&=\prod_{\chi\in\hat{G}_H-\{1\}}\sum_{\gamma\in G_H}\chi^{-1}(\gamma)\log|(N_{R_{\mathfrak{f}_H}/H}\prod_{\varepsilon\in\{0,1\}^t}\varphi_{\mathfrak{f}_\varepsilon}(C_{0,\varepsilon})^{2k_{\mathfrak{f}_H}/k_{\mathfrak{f}_\varepsilon}})^\gamma|\\
&=\prod_{\chi\in\hat{G}_H-\{1\}}\sum_{\varepsilon\in\{0,1\}^t}2k_{\mathfrak{f}_H}/k_{\mathfrak{f}_\varepsilon}\sum_{\sigma\in\mathrm{Gal}(R_{\mathfrak{f}_H}/K)}\chi^{-1}(\sigma)\log|\varphi_{\mathfrak{f}_\varepsilon}(C_{0,\varepsilon})^\sigma|\ .\end{aligned}$$

If $\mathfrak{f}_\chi\nmid\mathfrak{f}_\varepsilon$, a simple calculation shows

$$\sum_{\sigma\in\mathrm{Gal}(R_{\mathfrak{f}_H}/K)}\chi^{-1}(\sigma)\log|\varphi_{\mathfrak{f}_\varepsilon}(C_{0,\varepsilon})^\sigma|=0\ .$$

Hence, by Theorems 1 and 2 of [5], we see

$$\begin{aligned}R(\varepsilon_H)&=\prod_{\chi\in\hat{G}_H-\{1\}}2k_{\mathfrak{f}_H}\sum_{\substack{\varepsilon\in\{0,1\}^t\\ \mathfrak{f}_\chi|\mathfrak{f}_\varepsilon}}(e_{\mathfrak{f}_\varepsilon}/k_{\mathfrak{f}_\chi}e_{\mathfrak{f}_\chi})[R_{\mathfrak{f}_H}:R_{\mathfrak{f}_\varepsilon}]\prod_{\substack{1\le i\le t\\ \varepsilon_i=1}}(1-\chi^{-1}(\mathfrak{l}_i))\\
&\qquad\times\sum_{C\in Cl(\mathfrak{f}_\chi)}\chi^{-1}(C)\log|\varphi_{\mathfrak{f}_\chi}(C)|\\
&=\prod_{\chi\in\hat{G}_H-\{1\}}2k_{\mathfrak{f}_H}e_{\mathfrak{f}_H}/k_{\mathfrak{f}_\chi}e_{\mathfrak{f}_\chi}\sum_{\substack{\varepsilon\in\{0,1\}^t\\ \mathfrak{f}_\chi|\mathfrak{f}_\varepsilon}}\varphi_K(\mathfrak{f}_H)/\varphi_K(\mathfrak{f}_\varepsilon)\prod_{\substack{1\le i\le t\\ \varepsilon_i=1}}(1-\chi^{-1}(\mathfrak{l}_i))\\
&\qquad\times\sum_{C\in Cl(\mathfrak{f}_\chi)}\chi^{-1}(C)\log|\varphi_{\mathfrak{f}_\chi}(C)|\\
&=\prod_{\chi\in\hat{G}_H-\{1\}}2k_{\mathfrak{f}_H}e_{\mathfrak{f}_H}/k_{\mathfrak{f}_\chi}e_{\mathfrak{f}_\chi}\Big(\sum_{C\in Cl(\mathfrak{f}_\chi)}\chi^{-1}(C)\log|\varphi_{\mathfrak{f}_\chi}(C)|\Big)\\
&\qquad\times\prod_{\mathfrak{l}_i\nmid\mathfrak{f}_\chi}(\varphi_K(\mathfrak{l}_i^{e_i})+1-\chi^{-1}(\mathfrak{l}_i))\end{aligned}$$

where φ_K is the Euler function of K. A similar argument shows that the p-adic regulator $R^{(p)}(\varepsilon_H)$ of ε_H satisfies

$$R^{(p)}(\varepsilon_H)=\prod_{\chi\in\hat{G}_H-\{1\}} k_{\mathfrak{f}_H}e_{\mathfrak{f}_H}/k_{\mathfrak{f}_\chi}e_{\mathfrak{f}_\chi}\cdot S^{(p)}(\chi)\prod_{\mathfrak{l}_i\nmid\mathfrak{f}_\chi}(\varphi_K(\mathfrak{l}_i^{e_i})+1-\chi^{-1}(\mathfrak{l}_i))\,,$$

and the equation $R(\varepsilon_H)/R(H)=R^{(p)}(\varepsilon_H)/R^{(p)}(H)$ holds, where $R(H)$ is the regulator of H. By Theorem 3 of [5],

$$(5.1)\qquad (\prod_{\chi\in\hat{G}_H-\{1\}}\sum_{C\in Cl(\mathfrak{f}_\chi)}\chi^{-1}(C)\log|\varphi_{\mathfrak{f}_\chi}(C)|/k_{\mathfrak{f}_\chi})/R(H)$$
$$\sim(\prod_{\chi\in\hat{G}_H-\{1\}}S^{(p)}(\chi)/k_{\mathfrak{f}_\chi})/R^{(p)}(H)\sim h_H/W_H\,.$$

For each $\chi\in\hat{G}_H$, regarding χ as a class character of K, we write $\chi=\chi_1(\nu_\chi)_p(\nu'_\chi)_{\bar{p}}$ as in § 2. Choose a generator f_χ of $\mathfrak{f}_\chi$. Then,

$$(5.2)\qquad \prod_{\chi\in\hat{G}_H-\{1\}}\tau(\nu_\chi,\zeta_{p^{n_\chi}})^2\sim\prod_{\chi\in\hat{G}_H}\tau(\nu_\chi,\zeta_{p^{n_\chi}})\tau(\nu_\chi^{-1},\zeta_{p^{n_\chi}})\sim\prod_{\chi\in\hat{G}_H}p^{n_\chi}$$
$$\sim\prod_{\chi\in\hat{G}_H}\mathfrak{f}_\chi\sim d_{H/K}\,.$$

Combining (5.1), (5.2) and Theorems 3.1 and 4.3, we obtain our assertion.

REMARK. We note that the index $(E_H:\varepsilon_H)$ is finite, because $R(\varepsilon_H)\neq 0$ as is shown above.

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