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Graded Buchsbaum Algebras and Segre Products

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§0. Introduction.

This paper is devoted to the study of Buchsbaum property in terms of certain spectral sequences and the application of this method to the investigation of Buchsbaum conditions for the Segre product of graded modules. In their paper [4], Goto and Watanabe showed the homogeneous version of Künneth's formula. This gives Cohen-Macaulay condition of the Segre product of graded modules. On the other hand, some sufficient conditions that the Segre product of graded modules have Buchsbaum property has been obtained by Stückrad and Vogel [15], and Schenzel [10]. The difficulty of seeking better sufficient conditions is caused by the difference between Buchsbaum property and quasi-Buchsbaum property. In other words, Buchsbaum property is not completely described in terms of local cohomology group without derived category. Our method is to describe Buchsbaum property in terms of spectral sequence. This approach leads to the three theorems in the end of this section.

In Section 1, we introduce the notion of r-Buchsbaum modules. In (1.8) and (1.11), we give a criterion of r-Buchsbaum property from the viewpoint of spectral sequence. As their corollaries, we have some inequalities concerning extension groups.

Section 2 is the homogeneous version of Section 1. The results of this section are applied in the later sections. Theorems 2.2 and 2.3 are the main theorems of this section. In (2.6) these give another proof of Stückrad and Vogel [16, Proposition 3.10]. Also, the theorems give how to calculate the dimension of some extension groups, for example, in (2.9).

Section 3 is, in some sense, an introduction to Section 4. In this section, we investigate some examples. In fact, the study of Example 3.4 gives the motivation to obtain our main theorems.

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The proofs of Theorems A, B and C appear in Section 4. In this section, we make use of the notations of Goto and Watanabe [4, Chapter 4]. Let $R=\bigoplus_{d\geq 0} R_d$ and $S=\bigoplus_{d\geq 0} S_d$ be finitely generated graded algebras over an infinite field $k=R_0=S_0$. We call such algebras, simply, graded rings over k. We always assume that each algebra is generated by R_1 or S_1 respectively, that is, $R=k[R_1]$ and $S=k[S_1]$. For a graded ring R, its homogeneous maximal ideal is represented by $\mathfrak{m}_R$, or often simply by $\mathfrak{m}$. For a graded R-module M and a graded S-module N, we define $M\# N=\bigoplus_{d\in \mathbf{Z}}(M_d\otimes_k N_d)$ and call this module the Segre product of M and N. In particular, $R\# S$ is a graded ring over k and is called the Segre product of R and S. We consider $M\# N$ as a graded $R\# S$-module.

The following are our main theorems:

THEOREM A. *Let R and S be graded rings over an infinite field k. Let M be a finitely generated graded R-module with* $\dim M=m\geqq 2$ *and N a finitely generated graded S-module with* $\dim N=n\geqq 2$. *Suppose that M and N are Cohen-Macaulay modules. Then the following conditions are equivalent:*

(a) *$M\# N$ is a Buchsbaum $(R\# S)$-module.*

(b) *$M\# N$ is a quasi-Buchsbaum $(R\# S)$-module.*

(c) $\mathfrak{m}_{R\# S}(M\# H^n_{\mathfrak{m}_S}(N))=0$ *and* $\mathfrak{m}_{R\# S}(H^m_{\mathfrak{m}_R}(M)\# N)=0$.

THEOREM B. *Let R and S be graded rings over an infinite field k. Let M be a finitely generated graded R-module with* $\dim M=m\geqq 2$ *and N a finitely generated graded S-module with* $\dim N=n\geqq 2$. *Suppose that M is a Cohen-Macaulay module and N is a Buchsbaum module with* depth $N\geqq 2$.

If $M\# H^n_{\mathfrak{m}_S}(N)=0$ and $H^m_{\mathfrak{m}_R}(M)\# N=0$, then $M\# N$ is a Buchsbaum module.

THEOREM C. *Let R and S be graded rings over an infinite field k. Let M be a finitely generated graded R-module with* $\dim M=m\geqq 2$ *and N a finitely generated graded S-module with* $\dim N=n\geqq 2$. *Suppose that M is a Cohen-Macaulay module and N is a Buchsbaum module with* $2\leqq \operatorname{depth} N<n$. *Let l be an integer with $l<n$. If there is a subsystem of parameters $x_1, \cdots, x_l$ in R_1 for M satisfying that $x_1\cdots x_l(H^m_{\mathfrak{m}_R}(M)_d)\neq 0$ and $H^l_{\mathfrak{m}_S}(N)_d\neq 0$ for some d, then $M\# N$ is not a Buchsbaum module.*

§1. r-Buchsbaum modules.

Let $(A, \mathfrak{m})$ be a local ring with residue field k. Let us assume k is an infinite field. Let M be a finitely generated A-module of dimension d.

DEFINITION 1.1. (a) The A-module M is a Buchsbaum module if the difference $l_A(M/\mathfrak{q}M)-e_{\mathfrak{q}}(M)$ is an invariant for any parameter ideal $\mathfrak{q}$ of M, where l, respectively e, denotes length, respectively multiplicity of $\mathfrak{q}$.

(b) The A-module M is a quasi-Buchsbaum module if $\mathfrak{m}H_{\mathfrak{m}}^i(M)=0$ for every $0\leqq i\leqq d$.

DEFINITION 1.2. The A-module M is an r-Buchsbaum module if, for every system $f_1, \cdots, f_d$ of parameters for M, $M/(f_1, \cdots, f_i)M$ is a quasi-Buchsbaum module for $0\leqq i\leqq r-1$.

REMARK 1.3. (a) The A-module M is a 1-Buchsbaum module if and only if M is a quasi-Buchsbaum module.

(b) The A-module M is a d-Buchsbaum module if and only if M is a Buchsbaum module. (See Stückrad-Vogel [13].)

(c) In (1.2), we have only to take elements $f_1, \cdots, f_d$ in $\mathfrak{m}\setminus\mathfrak{m}^2$ as a system of parameters for M. (See Suzuki [17, Theorem 3.6].)

Now let us take a minimal generator $x_1, \cdots, x_n$ of $\mathfrak{m}$ satisfying that, for every $1\leqq i_1<\cdots<i_d\leqq n$, $x_{i_1}, \cdots, x_{i_d}$ is a system of parameters for M. This can be done because k is an infinite field.

Let V_i be an open set $D(x_i)$ for $1\leqq i\leqq n$ in $U=\operatorname{Spec} A-\{\mathfrak{m}\}$. Then $\mathfrak{U}=\{V_i\}_{1\leqq i\leqq n}$ is an open covering of U. Let $C^{\cdot}$ be the Čech complex $C^{\cdot}(\mathfrak{U}; \tilde{M})$ of $\tilde{M}$. Then we will consider the complex $L^{\cdot}=(0\to M\xrightarrow{\varepsilon} C^{\cdot}[-1])$, where $L^i=C^{i-1}$ for $i\neq 0$, $L^0=M$ and ε is the natural map. What is important here is that $H^i(L^{\cdot})=H_{\mathfrak{m}}^i(M)$ for every i.

In this way, we have the double complex $C^{\cdot\cdot}=\operatorname{Hom}_A(K_{\cdot}, L^{\cdot})$, where $K_{\cdot}$ is the Koszul complex $K_{\cdot}((x_1, \cdots, x_n); A)$ and $C^{p,q}=\operatorname{Hom}_A(K_p, L^q)$. We write its differentials as $d'^{p,q}\colon C^{p,q}\to C^{p+1,q}$, and $d''\colon C^{p,q}\to C^{p,q+1}$. Then we take two filtrations $'F_t(C^{\cdot\cdot})=\sum_{p\geqq t} C^{p,q}$ and $''F_t(C^{\cdot\cdot})=\sum_{q\geqq t} C^{p,q}$. The filtrations $'F_t$ and $''F_t$ give spectral sequences $\{'E_1^{p,q}\}$ and $\{''E_1^{p,q}\}$ respectively:

$$\begin{array}{l} 'E_1^{p,q}=\operatorname{Ker} d''^{p,q}/\operatorname{Im} d''^{p,q-1} \searrow \\ ''E_1^{p,q}=\operatorname{Ker} d'^{p,q}/\operatorname{Im} d'^{p-1,q} \nearrow \end{array} H^{p+q}(C^{\cdot\cdot})\,.$$

Now let $\langle e_1^*, \cdots, e_n^*\rangle$ be the basis of $K_1((x_1, \cdots, x_n); A)$. Since $C^{p,q}\cong L^q\otimes_A \wedge^p(\bigoplus_{k=1}^n Ae_k^*)$, we have

$$(1.4)\qquad 'E_1^{p,q}=H_{\mathfrak{m}}^q(M)\otimes_A \wedge^p\Big(\bigoplus_{k=1}^n Ae_k^*\Big)\,.$$

If A is regular, then the Koszul complex $K_{\cdot}((x_1, \cdots, x_n); A)$ is a free resolution of the residue field k. Hence we have $''E_1^{p,0}=\operatorname{Exp}_A^p(k, M)$ and $''E_1^{p,q}=0$ for $q>0$. This implies that $H^{p+q}(C^{\cdot\cdot})=\operatorname{Exp}_A^{p+q}(k, M)$. From now on we treat in most cases the first filtration and write $E_r^{p,q}$ for $'E_r^{p,q}$.

The spectral sequence $\{E_r^{p,q}\}$ does not depend on the choice of minimal generators $x_1, \cdots, x_n$ of $\mathfrak{m}$.

REMARK 1.5. Let $I^\cdot$ be a complex which is quasi-isomorphic to $L^\cdot$. Then the spectral sequence $\{F_r^{p,q}, \bar{d}_r^{p,q}\}$ obtained from the double complex $\mathrm{Hom}_A(K_\cdot, I^\cdot)$ with first filtration is isomorphic to the spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$.

In fact, let $\alpha: I^\cdot \to L^\cdot$ be a morphism of complexes. The map α induces a homomorphism $\alpha_r^{p,q}: F_r^{p,q} \to E_r^{p,q}$ for every p, q, and r such that $\alpha_r^{p+r,q-r+1} \circ \bar{d}_r^{p,q} = d_r^{p,q} \circ \alpha_r^{p,q}$. Assume that α is a quasi-isomorphism. Then the map $\alpha_1^{p,q}: F_1^{p,q} \to E_1^{p,q}$ is an isomorphism for every p and q because $K_p = \wedge^p(\bigoplus_{k=1}^n Ae_k^*)$ is a free A-module. By induction, we see that $\alpha_r^{p,q}: F_r^{p,q} \to E_r^{p,q}$ is an isomorphism for p, q, and r.

By (1.4), we see that $E_r^{p,q} = H_\mathfrak{m}^q(M) \otimes_A \wedge^p(\bigoplus_{k=1}^n Ae_k^*)$ and

$$d_1^{p,q}(z \otimes e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*) = \sum_{j=1}^n x_j z \otimes e_j^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \qquad \text{for} \quad z \in H_\mathfrak{m}^q(M) .$$

Hence we have the following:

PROPOSITION 1.6. *Let M be a finitely generated A-module of dimension d. Let $\{E_r^{p,q}\}$ be the spectral sequence associated to the A-module M. Then the following conditions are equivalent*:

(a) *M is a quasi-Buchsbaum A-module.*

(b) *$d_1^{p,q}: E_1^{p,q} \to E_1^{p+1,q}$ is a zero map for every p and q ($<d$).*

(c) *$d_1^{0,q}: E_1^{0,q} \to E_1^{1,q}$ is a zero map for every q ($<d$).*

Now suppose that M is a quasi-Buchsbaum module. Setting $\bar{M} = M/x_jM$, we have the following exact sequence:

$$0 \longrightarrow [0: x_j]_M \longrightarrow M \xrightarrow{\cdot x_j} M \longrightarrow \bar{M} \longrightarrow 0 .$$

Since $H_\mathfrak{m}^q([0: z_j]_M) = 0$ for $q \geqq 1$ and M is a quasi-Buchsbaum module, we have the following short exact sequence:

$$0 \longrightarrow H_\mathfrak{m}^{q-1}(M) \longrightarrow H_\mathfrak{m}^{q-1}(\bar{M}) \longrightarrow H_\mathfrak{m}^q(M) \longrightarrow 0$$

for $1 \leqq q \leqq d$. Thus we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_\mathfrak{m}^{q-1}(M) & \longrightarrow & H_\mathfrak{m}^{q-1}(\bar{M}) & \xrightarrow{\alpha} & H_\mathfrak{m}^q(M) & \longrightarrow & 0 \\ & & \phi\downarrow & (1) & \bar{\psi}\downarrow & (2) & \phi'\downarrow & & \\ 0 & \longrightarrow & H_\mathfrak{m}^{q-1}(M) & \xrightarrow{\beta} & H_\mathfrak{m}^{q-1}(\bar{M}) & \longrightarrow & H_\mathfrak{m}^q(M) & \longrightarrow & 0 \end{array} \tag{1.7.1}$$

for $1 \leqq q < d$, where ϕ, ϕ' and $\bar{\psi}$ are the mappings induced by the multiplication of x_i $(i \neq j)$. Since M is a quasi-Buchsbaum module, ϕ and ϕ' are zero maps. By snake lemma, we get an A-homomorphism ψ: $H^q_{\mathfrak{m}}(M) \to H^{q-1}_{\mathfrak{m}}(M)$ such that $\beta \circ \psi \circ \alpha = \bar{\psi}$. Note that the conditions $\psi = 0$ and $\bar{\psi} = 0$ are equivalent. Let us write $\psi = (x_i \otimes x_j)$. Thus we see that $\bar{M}$ is a quasi-Buchsbaum module if and only if $(x_i \otimes x_j)$ is a zero map for every i. In other words, $M/x_i M$ is a quasi-Buchsbaum module for every $1 \leqq i \leqq n$ if and only if $(x_i \otimes x_j)$ is a zero map for every i and j.

Next assume that, for any $l < r-1$, $M/(x_{i_1}, \cdots, x_{i_l})M$ is a quasi-Buchsbaum module for every $1 \leqq i_1 < \cdots < i_l \leqq n$. We will define a homomorphism (x_J): $H^q_{\mathfrak{m}}(M) \to H^{q-r+1}_{\mathfrak{m}}(M)$ for $r-1 \leqq q < d$, where $(x_J) = (x_{j_1} \otimes \cdots \otimes x_{j_r})$. When $r=2$, it has been already done. When $r>2$, the hypothesis of induction gives the following diagram with exact rows:

$$\begin{array}{llllllllll}
 & 0 \longrightarrow & H^{q-1}_{\mathfrak{m}}(M) & \longrightarrow & H^{q-1}_{\mathfrak{m}}(\bar{M}) & \xrightarrow{\alpha} & H^q_{\mathfrak{m}}(M) & \longrightarrow 0 \\
(1.7.2) & & \downarrow{\scriptstyle (x_I)} & & \downarrow{\scriptstyle (x_I)} & & \downarrow{\scriptstyle (x_I)} & \\
 & 0 \longrightarrow & H^{q-r+1}_{\mathfrak{m}}(M) & \xrightarrow{\beta} & H^{q-r+1}_{\mathfrak{m}}(\bar{M}) & \longrightarrow & H^{q-r+2}_{\mathfrak{m}}(M) & \longrightarrow 0
\end{array}$$

for $r-1 \leqq q < d$, where $\sharp I = r-1$. If the diagram commutes, we can get a homomorphism $(x_J) = (x_I \otimes x_j)$: $H^q_{\mathfrak{m}}(M) \to H^{q-r+1}_{\mathfrak{m}}(M)$ by snake lemma. What we have to show is that the diagram (1.7.2) is commutative. Now set $\bar{I} = I - \{i\}$. Let $N = M/x_i M$ and $\bar{N} = N/x_j N$. Then we have a diagram (1.7.3) such that each row sequence and each column one are exact, where the broken arrows represent $(x_{\bar{I}})$'s. Here the commutativity of the diagram (1.7.3) together with the broken arrows follows from the hypothesis of induction. Note that three left broken arrows and three right ones in (1.7.3) are zero maps. This gives the maps of snake lemma $H^{q-1}_{\mathfrak{m}}(M) \to H^{q-r+1}_{\mathfrak{m}}(M)$, $H^{q-1}_{\mathfrak{m}}(\bar{M}) \to H^{q-r+1}_{\mathfrak{m}}(M)$, and $H^q_{\mathfrak{m}}(M) \to H^{q-r+2}_{\mathfrak{m}}(M)$. These maps are the column maps (x_I) in (1.7.2). Thus the diagram (1.7.2) commutes. By the diagram (1.7.2), we have, moreover, that (x_I): $H^{q-1}_{\mathfrak{m}}(\bar{M}) \to H^{q-r+1}_{\mathfrak{m}}(\bar{M})$ is a zero map if and only if (x_J): $H^q_{\mathfrak{m}}(M) \to H^{q-r+1}_{\mathfrak{m}}(M)$ is a zero map. Hence we see that M is an r-Buchsbaum module if and only if (x_J): $H^q_{\mathfrak{m}}(M) \to H^{q-s+1}_{\mathfrak{m}}(M)$ is a zero map for $s \leqq r$, $r-1 \leqq q < d$ and $\sharp J = s$. Summarizing the preceding argument, we have the following:

THEOREM 1.8. *Let $(A, \mathfrak{m})$ be a local ring with an infinite residue field k. Let M be a minimal generator of $\mathfrak{m}$ satisfying that, for every $1 \leqq i_1 < \cdots < i_d \leqq n$, $x_{i_1}, \cdots, x_{i_d}$ is a system of parameters for M. Suppose that, for any $l < r-1$, $M/(x_{i_1}, \cdots, x_{i_l})M$ is a quasi-Buchsbaum module for*

(1.7.3)

$$
\begin{tikzcd}
 & & 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & \\
 & 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & & \\
 & 0 \arrow[r] & H^{q-2}_{\mathfrak{m}}(M) \arrow[rr] \arrow[dd] \arrow[dl, dashed] & & H^{q-2}_{\mathfrak{m}}(N) \arrow[rr] \arrow[dd] \arrow[dl, dashed] & & H^{q-1}_{\mathfrak{m}}(M) \arrow[r] \arrow[dd] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & H^{q-r+1}_{\mathfrak{m}}(M) \arrow[rr] \arrow[dd] & & H^{q-r+1}_{\mathfrak{m}}(N) \arrow[rr] \arrow[dd] & & H^{q-r+2}_{\mathfrak{m}}(M) \arrow[r] \arrow[dd] & 0 & \\
 & 0 \arrow[r] & H^{q-2}_{\mathfrak{m}}(\bar{M}) \arrow[rr] \arrow[dd] \arrow[dl, dashed] & & H^{q-2}_{\mathfrak{m}}(\bar{N}) \arrow[rr] \arrow[dd] \arrow[dl, dashed] & & H^{q-1}_{\mathfrak{m}}(\bar{M}) \arrow[r] \arrow[dd] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & H^{q-r+1}_{\mathfrak{m}}(\bar{M}) \arrow[rr] \arrow[dd] & & H^{q-r+1}_{\mathfrak{m}}(\bar{N}) \arrow[rr] \arrow[dd] & & H^{q-r+2}_{\mathfrak{m}}(\bar{M}) \arrow[r] \arrow[dd] & 0 & \\
 & 0 \arrow[r] & H^{q-1}_{\mathfrak{m}}(M) \arrow[rr] \arrow[dd] \arrow[dl, dashed] & & H^{q-1}_{\mathfrak{m}}(N) \arrow[rr] \arrow[dd] \arrow[dl, dashed] & & H^{q}_{\mathfrak{m}}(M) \arrow[r] \arrow[dd] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & H^{q-r+2}_{\mathfrak{m}}(M) \arrow[rr] \arrow[dd] & & H^{q-r+2}_{\mathfrak{m}}(N) \arrow[rr] \arrow[dd] & & H^{q-r+3}_{\mathfrak{m}}(M) \arrow[r] \arrow[dd] & 0 & \\
 & & 0 & & 0 & & 0 & \\
 & 0 & & 0 & & 0 & &
\end{tikzcd}
$$

every $1 \leqq i_1 < \cdots < i_l \leqq n$. *Then*

$$(x_I) = (x_{i_1} \otimes \cdots \otimes x_{i_r}) : \quad H^q_{\mathfrak{m}}(M) \longrightarrow H^{q-r+1}_{\mathfrak{m}}(M)$$

is well-defined.

Furthermore, $M/(x_{i_1}, \cdots, x_{i_{r-1}})M$ *is a quasi-Buchsbaum module for every* $1 \leqq i_1 < \cdots < i_{r-1} \leqq n$ *if and only if* (x_I) *defined as above is a zero map for every subset* $I = \{i_1, \cdots, i_r\}$ *of* $\{1, \cdots, n\}$.

LEMMA 1.9. *Let M be a finitely generated A-module with dimension d. Suppose that, for any $l < r-1$, $M/(x_{i_1}, \cdots, x_{i_l})M$ is a quasi-Buchsbaum module for every $1 \leqq i_1 < \cdots < i_l \leqq n$. Then we have*

(1) $E_r^{p,q} = H^q_{\mathfrak{m}}(M) \otimes_A \wedge^p(\bigoplus_{k=1}^n Ae_k^*)$ *for any* $q \neq d$.

By (1), *we can write* $(d_r^{p,q})_{J,K}\colon H^q_{\mathfrak{m}}(M) \to H^{q-r+1}_{\mathfrak{m}}(M)$ *for the map from e_K^*-component to $e_J^* \wedge e_K^*$-component of the map $d_r^{p,q}$. Then we have*

(2) $(d_r^{p,q})_{J,K} = (-1)^{(r-1)(p+q-r)}(x_J)$, *where (x_J) is the map defined in* (1.8).

PROOF. We will show the lemma by induction on r. When $r=1$, (1) follows from (1.4). Next, if both (1) and (2) hold for $r \leqq s-1$, then we see that (1) holds for $r=s$ by (1.8).

Now let us assume that (1) holds for $r \leqq s$ and (2) for $r \leqq s-1$. Let us consider the double complex $C^{\cdot\cdot}$ associated to M and the double complex $\bar{C}^{\cdot\cdot}$ associated to $\bar{M}=M/x_jM$. Then we have the following diagram with exact rows for $1 \leqq q < d$:

$$(1.10)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & C^{p,q-1} & \longrightarrow & C^{p,q-1} & \xrightarrow{u} & \bar{C}^{p,q-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^{p,q} & \xrightarrow{v} & C^{p,q} & \longrightarrow & \bar{C}^{p,q} & \longrightarrow & 0\,, \end{array}$$

where the column maps are $d''^{p,q-1}$'s. Now recall that

$$C^{p,q}=(\bigoplus M_{x_{i_1}\cdots x_{i_q}})\otimes_A \wedge^p\Big(\bigoplus_{k=1}^{n} Ae_k^*\Big)\,,\qquad \bar{C}^{p,q}=(\bigoplus \bar{M}_{x_{i_1}\cdots x_{i_q}})\otimes_A \wedge^p\Big(\bigoplus_{k=1}^{n} Ae_k^*\Big)\,,$$

and the maps $d''^{p,q-1}$ are the natural map. In (1.7.2), for $\tilde{z} \in H^q_{\mathfrak{m}}(M)$, we take $\tilde{y} \in H^q_{\mathfrak{m}}(\bar{M})$ such that $\alpha(\tilde{y})=\tilde{z}$. Precisely describing the map α in (1.10), we have $y \in C^{p,q-1}$ such that $u(y)=\bar{y}$ and $d''(y)=v(z)$. Here $z=\tilde{z}\otimes e_K^*$ in $C^{p,q}$ and $\bar{y}=\tilde{y}\otimes e_K^*$ in $\bar{C}^{p,q-1}$, where $e_K^*=e_{k_1}^*\wedge\cdots\wedge e_{k_p}^*$ for some fixed $1\leqq k_1<\cdots<k_p\leqq n$. Since $d''d'(y)=-d'd''(y)=-d'v(z)=-vd'(z)=0$, we have $d'(y)$ is a cycle of $C^{p+2,q-1}$. In particular, regarding y as an element of $E_{r-1}^{p+1,q-1}$, we see that $d_{r-1}^{p+1,q-1}(y)=d_{r-1}^{p+1,q-1}(\bar{y})$. By the hypothesis of induction, we have

$$(d_{r-1}^{p+1,q-1})_{I,K}(\bar{y})=(-1)^{(r-2)(p+q-r+1)}(X_I)(\bar{y})\;.$$

Here we should remark that $d''^{p,q-1}$ is the composition of the map of Čech resolution and $(-1)^{p+q}$-multiplication. Thus we have $(d_r^{p,q})_{J,K}(\tilde{z})=(-1)^{p+q}(d_{r-1}^{p+1,q-1})_{I,K}(\tilde{y})=(-1)^{(r-1)(p+q-r)}(X_I)(\tilde{z})$. Hence the assertion is proved.

THEOREM 1.11. *Let $(A, \mathfrak{m})$ be a local ring with an infinite residue field k. Let M be a finitely generated A-module of dimension d. Suppose that M is an $(r-1)$-Buchsbaum module. Then the following conditions are equivalent*:

(a) *M is an r-Buchsbaum A-module.*

(b) *$d_s^{p,q}\colon E_s^{p,q}\to E_s^{p+s,q-s+1}$ is a zero map for $s\leqq r$ and $q<d$.*

(c) *$d_s^{0,q}\colon E_s^{0,q}\to E_s^{s,q-s+1}$ is a zero map for $s\leqq r$ and $q<d$.*

PROOF. It follows immediately from (1.8) and (1.9).

COROLLARY 1.12. *Let $(A, \mathfrak{m})$ be a local ring with an infinite residue field k. Let M be a finitely generated A-module of dimension d. Let $x_1, \cdots, x_n$ be a minimal generator of $\mathfrak{m}$ satisfying that, for every $1\leqq i_1<\cdots<i_d\leqq n$, $x_{i_1}, \cdots, x_{i_d}$ is a system of parameters for M. Then M is*

an r-Buchsbaum module if and only if, for any $l<r-1$, $M/(x_{i_1}, \cdots, x_{i_l})M$ is a quasi-Buchsbaum module for every $1 \leqq i_1 < \cdots < i_l \leqq n$.

PROOF. It follows immediately from (1.8), (1.9), and (1.11).

REMARK 1.13. Under the above conditions, assume that M is an $(r-1)$-Buchsbaum module. Even without the assumption that, for every $1 \leqq i_1 < \cdots < i_d \leqq n$, $x_{i_1}, \cdots, x_{i_d}$ is a system of parameters for M, the map

$$(x_I)=(x_{i_1} \otimes \cdots \otimes x_{i_r}) : \ H_{\mathfrak{m}}^q(M) \longrightarrow H_{\mathfrak{m}}^{q-r+1}(M)$$

is well-defined by (1.9), (1.11), and (1.12).

COROLLARY 1.14. *Let $(A, \mathfrak{m})$ be a regular local ring of dimension n with an infinite residue field k. Let M be a finitely generated A-module of dimension d. Assume that $M_{\mathfrak{p}}$ is Cohen-Macaulay for every prime ideal $\mathfrak{p}\,(\neq \mathfrak{m})$ of A. Then*

$$\dim \operatorname{Exp}_A^p(k, M) \leqq \sum_{j=0}^{p} \binom{n}{p-j} l_A(H_{\mathfrak{m}}^j(M))$$

for $0 \leqq p < d$.

Furthermore, equality holds for $0 \leqq p \leqq r$ if M is an $(r+1)$-Buchsbaum module.

PROOF. The spectral sequence $\{E_r^{p,q}\}$ associated to M converges to $\{\operatorname{Ext}_A^{p+q}(k, M)\}$. Since $E_1^{p,q}=H_{\mathfrak{m}}^q(M) \otimes_A \wedge^p(\bigoplus_{k=1}^n Ae_k^*)$, we see $l_A(E_1^{p,q})= \binom{n}{p} l_A(H_{\mathfrak{m}}^q(M))$. Thus the inequality is proved. Further, in case M is an $(r+1)$-Buchsbaum module, by (1.11), the equality holds.

COROLLARY 1.15. *Under the above conditions, let I be an ideal of A. Assume that A/I is a Buchsbaum ring of dimension d. Then*

$$l_A(\operatorname{Ext}_A^d(k, A/I)) \leqq \sum_{j=0}^{n} \binom{n}{d-j} l_A(H_{\mathfrak{m}}^j(A/I)) + \mu(K_{A/I})$$

where $\mu(K_{A/I})$ is the number of minimal generators of the canonical module $K_{A/I}$ of A/I. If, furthermore, A contains a field, then

$$l_A(\operatorname{Ext}_A^d(k, A/I)) \geqq \sum_{j=0}^{n} \binom{n}{d-j} l_A(H_{\mathfrak{m}}^j(A/I)) + 1 \,.$$

PROOF. Let us consider the spectral sequence $\{E_r^{p,q}\}$ associated to the A-module A/I. We see that $E_2^{0,d} \cong [0: \mathfrak{m}]_{H_{\mathfrak{m}}^d(A)}$. By local duality, we have $E_2^{0,d} \cong \operatorname{Hom}_A(K_{A/I}/\mathfrak{m}K_{A/I}, k)$, where $K_{A/I}=\operatorname{Ext}_A^{n-d}(A/I, A)$. Thus the former inequality holds. On the other hand, since $\mathrm{Ext}^d_{A/I}(k, A/I) \to H^d_{\mathfrak{m}}(A/I)$ is not a zero map by Hochster [8], we see $E^{0,d}_\infty \neq 0$. Hence the latter inequality holds.

§2. Graded r-Buchsbaum modules.

Let $R=k[x_1, \cdots, x_a]$ be a graded ring over an infinite field k, where $x_i \in R_1$ $(i=1, \cdots, a)$. Let us write $\mathfrak{m}$ for the graded maximal ideal $R_+=\bigoplus_{d\geq 1} R_d=(x_1, \cdots, x_a)$. Let M be a finitely generated graded R-module of dimension m.

DEFINITION 2.1. The graded R-module M is a graded r-Buchsbaum module if, for every homogeneous system $f_1, \cdots, f_m$ of parameters in R_1 for M, $M/(f_1, \cdots, f_i)M$ is a quasi-Buchsbaum module for $0\leqq i\leqq r-1$.

Parallel to §1, first, let us define the double complex $C^{\cdot\cdot}$. Let $\mathfrak{U}=\{D_+(x_i)\}_{i=1,\cdots,a}$ be an affine open covering of $\mathrm{Proj}\, R$. Let $L^{\cdot}=(0\to M \xrightarrow{\varepsilon} (\bigoplus_{d\in Z} C^{\cdot}(\mathfrak{U}; \tilde{M}(d)))[-1])$, where ε is the natural map. Note that $L^p=\bigoplus_{d\in Z} C^{p-1}(\mathfrak{U}; \tilde{M}(d))$ for $p>1$. Now we define $C^{p,q}=\mathrm{Hom}_R(K_p(\mathfrak{m}; A), L^q)$. Then the complex $C^{\cdot\cdot}=(C^{p,q})$ is a double complex of graded R-modules whose boundary homomorphisms are homogeneous maps. The filtration $F_t(C^{\cdot\cdot})=\sum_{p\geq t} C^{p,q}$ of the double complex $C^{\cdot\cdot}$ gives a spectral sequence $\{E^{p,q}_r\}$. The following theorems are proved in the same way as in §1.

THEOREM 2.2. *Under the above conditions, assume that, for every $1\leqq i_1<\cdots<i_m\leqq a$, $x_{i_1}, \cdots, x_{i_m}$ is a homogeneous system of parameters for M. Suppose that, for any $l<r-1$, $M/(x_{i_1}, \cdots, x_{i_l})M$ is a quasi-Buchsbaum module for every $1\leqq i_1<\cdots<i_l\leqq a$. Then*

$$(x_I)=(x_{i_1}\otimes\cdots\otimes x_{i_r}):\quad H^q_{\mathfrak{m}}(M)\longrightarrow H^{q-r+1}_{\mathfrak{m}}(M)$$

is well-defined.

Furthermore, $M/(x_{i_1}, \cdots, x_{i_{r-1}})M$ is a quasi-Buchsbaum module for every $1\leqq i_1<\cdots<i_{r-1}\leqq a$ if and only if (x_I) defined as above is a zero map for every subset $I=\{i_1, \cdots i_r\}$ of $\{1, \cdots, a\}$.

THEOREM 2.3. *Under the above conditions, suppose that M is an $(r-1)$-Buchsbaum module. Then the following conditions are equivalent:*

(a) *M is an r-Buchsbaum R-module.*

(b) *$d^{p,q}_s$: $E^{p,q}_s \to E^{p+s,q-s+1}_s$ is a zero map for $s\leqq r$ and $q<m$.*

(c) *$d^{0,q}_s$: $E^{0,q}_s \to E^{s,q-s+1}_s$ is a zero map for $s\leqq r$ and $q<m$.*

REMARK 2.4. Under the above conditions, assume that M is an

$(r-1)$-Buchsbaum module. Even without the assumption that, for every $1 \leqq i_1 < \cdots < i_d \leqq a$, $x_{i_1}, \cdots, x_{i_d}$ is a homogeneous system of parameters for M, the map

$$(x_I) = (x_{i_1} \otimes \cdots \otimes x_{i_r}) : \quad H^q_{\mathfrak{m}}(M) \longrightarrow H^{q-r+1}_{\mathfrak{m}}(M)$$

is well-defined by (2.2) and (2.3).

COROLLARY 2.5. *Under the above conditions, M is an r-Buchsbaum graded R-module if and only if $M_{\mathfrak{m}}$ is an r-Buchsbaum $R_{\mathfrak{m}}$-module.*

PROOF. It follows immediately from (1.8) and (2.3).

COROLLARY 2.6. *Under the above conditions, let us define*

$$\mathfrak{S} = \{(i, l) \mid 0 \leqq i \leqq m, \ l \in \boldsymbol{Z}, \ H^i_{\mathfrak{m}}(M)_l \neq 0\} .$$

Suppose that $\mathfrak{S}$ satisfies the following condition:

For any (i, l) and (j, n) of $\mathfrak{S}$, if $i \geqq j$, then $i+l+1 \neq j+n$.

Then M is a Buchsbaum module.

PROOF. Since $(E_1^{p,q})_l = \bigoplus H^q_{\mathfrak{m}}(M)_l$, $(E_1^{p+s,q-s+1})_{l+s} = \bigoplus H^{q-s+1}_{\mathfrak{m}}(M)_{l+s}$ and $d_s^{p,q}$ is a graded homomorphism of degree s, the assumption gives that $d_s^{p,q}$ is zero for any p, q $(<n)$ and s. By (2.3), M is a Buchsbaum module.

COROLLARY 2.7. *Let R be the polynomial ring of dimension a over an infinite field k. Let M be a finitely generated R-module of dimension m. Assume that $M_{\mathfrak{p}}$ is Cohen-Macaulay for every graded prime ideals $\mathfrak{p}\,(\neq \mathfrak{m})$ of R. Then*

$$\dim \operatorname{Ext}^p_R(k, M) \leqq \sum_{j=0}^{p} \binom{n}{p-j} l_R(H^j_{\mathfrak{m}}(M))$$

for $0 \leqq p < m$.

Furthermore, equality holds if M is an $(r+1)$-Buchsbaum module.

PROOF. The proof is the same as (1.14).

COROLLARY 2.8. *Under the above conditions, let I be a graded ideal of R. Assume that R/I is a Buchsbaum ring of dimension d. Then*

$$\sum_{j=0}^{d} \binom{n}{d-j} l_A(H^j_{\mathfrak{m}}(R/I)) + 1 \leqq l_R(\operatorname{Ext}^d_R(k, R/I)) \leqq \sum_{j=0}^{d} \binom{n}{d-j} l_R(H^j_{\mathfrak{m}}(R/I)) + \mu(K_{R/I})$$

where $\mu(K_{R/I})$ is the number of minimal generators of the canonical module $K_{R/I}$ of R/I.

PROOF. The proof is the same as (1.15).

EXAMPLE 2.9. Let $(X, \mathcal{L})$ be a polarized Abelian variety with $\dim X=n$ and $\dim \Gamma(X, \mathcal{L})=N+1$ such that $\mathcal{L}$ is very ample. Let us consider its coordinate ring $A=\bigoplus_{l\geq 0}\Gamma(X, \mathcal{L}^{\otimes l})$. Let $S=\bigoplus_{d\geq 0}S^d\Gamma(X, \mathcal{L})$. It is well-known that A is a Buchsbaum ring with $\mu(K_A)=1$. Hence we have

$$\dim_k \operatorname{Ext}_S^{n+1}(k, A)=\sum_{j=2}^{n}\binom{N+1}{n+1-j}\dim \bigoplus_{l\geq 0} H^{j-1}(X, \mathcal{L}^{\otimes l})+1=\sum_{j=1}^{n}\binom{N+1}{n-j}\binom{n}{j}.$$

Considering its grading in the step of calculating $\dim_k \operatorname{Ext}_S^{n+1}(k, A)$, moreover, we have that $\dim_k \operatorname{Ext}_S^{n+1}(k, A)_d=\binom{N+1}{n+1-d}\binom{n}{d-1}$ if $2\leq d\leq n+1$, and $\dim_k \operatorname{Ext}_S^{n+1}(k, A)_d=0$ otherwise.

§3. Divisors on Segre products.

Let k be an infinite field. Let X be an arithmetically Cohen-Macaulay subscheme of $\boldsymbol{P}_k^N=\operatorname{Proj} S$, that is, its affine cone $C(X)=\operatorname{Spec} S/J$ is locally Cohen-Macaulay, where $J=\bigoplus_{l\in \boldsymbol{Z}}\Gamma(\mathcal{I}_{X/\boldsymbol{P}}(l))$. Let V be a subscheme of X such that $0<\dim V=n<\dim X$. Let A be the homogeneous coordinate ring of V.

PROPOSITION 3.1 (cf. Schenzel [10]). *Under the above conditions, we have*

$$\tau_0^{n+1}\boldsymbol{R}\Gamma(\bigoplus_{l\in \boldsymbol{Z}}\mathcal{I}_{V/X}(l))\cong \tau^{n+1}\boldsymbol{R}\Gamma_{\mathfrak{m}}(A)$$

in the derived category $D_k^+(S)$ of complexes bounded below of graded S-modules.

EXAMPLE 3.2. *Let $X=\boldsymbol{P}_k^r\times \boldsymbol{P}_k^s$ be Segre embedding in $\boldsymbol{P}=\boldsymbol{P}_k^{rs+r+s}$. Let V be a divisor of X corresponding to $\mathcal{O}_X(a, b)=p_1^*\mathcal{O}_{\boldsymbol{P}_k^r}(a)\otimes p_2^*\mathcal{O}_{\boldsymbol{P}_k^s}(b)$, that is, $\mathcal{O}_X[V]\cong \mathcal{O}_X(a, b)$.*

(1) *V is an arithmetically Cohen-Macaulay subscheme of $\boldsymbol{P}$ if and only if $a-r\leq b\leq a+s$.*

(2) *The following conditions are equivalent:*

(a) *V is an arithmetically Buchsbaum subscheme of $\boldsymbol{P}$.*

(b) *V is an arithmetically quasi-Buchsbaum subscheme of $\boldsymbol{P}$.*

(c) *$a-r-1\leq b\leq a+s+1$.*

PROOF. The proof is given, for example, by Goto-Watanabe [4], Stückrad-Vogel [15] or Schenzel [10, Proposition 5.1]. We will prove this, however, because our proof indicates the motivation of the next example and the theorems in § 4.

Now let us assume $a \geqq b$. First of all, let us find the numbers $1 \leqq i < r+s$ and $l \in \boldsymbol{Z}$ satisfying $H^i(\mathscr{I}_{V/X}(l))=0$. Recall that $\mathscr{I}_{V/X}$ is isomorphic to $\mathscr{O}_X(-a, -b)$. By Künneth's formula, we have $H^i(\mathscr{I}_{V/X}(l))=0$ if and only if $i=r$ and $b \leqq l \leqq a-r-1$. This shows (1) and the implication from (c) to (a) in (2). On the other hand it is also clear that (a) implies (b). Thus it remains to prove that if $a-r-1>b$, then V is not an arithmetically quasi-Buchsbaum subscheme of $\boldsymbol{P}$.

Let us write $\boldsymbol{P}^r \times \boldsymbol{P}^s = \operatorname{Proj} k[X_0, \cdots, X_r] \times \operatorname{Proj} k[Y_0, \cdots, Y_s]$. Then we see that

$$\cdot X_j : \quad H^r(\mathscr{O}_{\boldsymbol{P}^r}(l-a)) \longrightarrow H^r(\mathscr{O}_{\boldsymbol{P}^r}(l+1-a))$$

is surjective and

$$\cdot Y_j : \quad H^0(\mathscr{O}_{\boldsymbol{P}^s}(l-b)) \longrightarrow H^0(\mathscr{O}_{\boldsymbol{P}^s}(l+1-b))$$

is injective. On the other hand we see

$$H^r(\mathscr{I}_{V/X}(l)) \cong H^r(\mathscr{O}_{\boldsymbol{P}^r}(l-a)) \oplus H^0(\mathscr{O}_{\boldsymbol{P}^s}(l-b)) .$$

Hence we have that $\bigoplus_{l \in \boldsymbol{Z}} H^i(\mathscr{I}_{V/X}(l))$ is not a k-vector space if $a-r-1>b$. This gives that V is not an arithmetically Buchsbaum subscheme of $\boldsymbol{P}$.

Remark 3.3 (cf. Watanabe [20, (3.3)]). In the above example, assuming, in addition, V is smooth, $r \geqq 2$ and $s \geqq 2$, we see that its affine cone $Y = \operatorname{Spec} \bigoplus_{l \geqq 0} H^0(V, \mathscr{O}_V(l))$ has only one normal isolated singularity. In his paper [9], Ishida proved that a normal isolated Du Bois singularity is a Buchsbaum singularity in the case $\operatorname{char} k=0$. Here we describe the condition that Y is a Du Bois (or rational) singularity.

(1) Y has only a rational singularity if and only if $b-s \leqq a \leqq r$ or $a-r \leqq b \leqq s$.

(2) Y has only a Du Bois singularity if and only if $b-s \leqq a \leqq r+1$ or $a-r \leqq b \leqq s+1$.

In fact, let us take a resolution $\phi: W \to Y$, where $W = \mathbf{Spec}_V \bigoplus_{l \geqq 0} \mathscr{O}_V(l)$. Recall that Y has only a rational singularity if and only if $\boldsymbol{R}\phi_* \mathscr{O}_W \cong \mathscr{O}_Y$ and that Y has only a Du Bois singularity if and only if $\boldsymbol{R}\phi_* \mathscr{O}_W[-E] \cong \mathscr{M}_p$, where E is the exceptional divisor and $\mathscr{M}_p$ is the maximal ideal of the singular point of W. We see that $R^q\phi_*(\mathscr{O}_Y) \cong \bigoplus_{l \geqq 0} H^q(V, \mathscr{O}_V(l))$ and $R^q\phi_*(\mathscr{O}_Y[-E]) \cong \bigoplus_{l \geqq 1} H^q(V, \mathscr{O}_V(l))$. Since $H^q(V, \mathscr{O}_V(l)) \cong H^{q+1}(X, \mathscr{I}_{V/X}(l)) \cong H^{q+1}(X, \mathscr{O}_X(l-a, l-b))$ for $q \geqq 1$ and $l \geqq 0$, the assertion follows.

The study of Buchsbaum property for a divisor of product of two projective spaces was rather simple. However the situation is complicated for a divisor of the product of three or more projective spaces. In fact, we have the following:

EXAMPLE 3.4. *Let* $X=\boldsymbol{P}_k^r\times\boldsymbol{P}_k^r\times\boldsymbol{P}_k^r$ *be Segre embedding in* $\boldsymbol{P}=\boldsymbol{P}_k^{(r+1)^3-1}$. *Let* V *be a divisor of* X *corresponding to* $\mathcal{O}_X(a-r-1,\ a,\ a+r+1)$. *Then* V *is an arithmetically* r*-Buchsbaum subscheme in* $\boldsymbol{P}$ *but not an arithmetically* $(r+1)$*-Buchsbaum subscheme.*

In fact, let $\langle X_0,\cdots,X_r\rangle\times\langle Y_0,\cdots,Y_r\rangle\times\langle Z_0,\cdots,Z_r\rangle$ be a coordinate of $X=\boldsymbol{P}_k^r\times\boldsymbol{P}_k^r\times\boldsymbol{P}_k^r$. We will describe the local cohomology groups $H^q(\bigoplus_{l\in\boldsymbol{Z}}\mathcal{I}_{V/X}(l))$ according to Hartshorne [7, Chapter 3, § 5]. By Künneth's formula, we have

$$H^q(\bigoplus_{l\in\boldsymbol{Z}}\mathcal{I}_{V/X}(l))=0 \qquad \text{for} \quad q\neq 0,\ r,\ \text{or}\ 2r\ ,$$

$$H^{2r}(\bigoplus_{l\in\boldsymbol{Z}}\mathcal{I}_{V/X}(l))=k\cdot 1\otimes k\cdot\frac{1}{Y_0\cdots Y_r}\otimes\Bigl(\sum_{\substack{i_0+\cdots+i_r=2r\\ i_0>0,\cdots,i_r>0}} k\cdot\frac{1}{Z_0^{i_0}\cdots Z_r^{i_r}}\Bigr)\ ,$$

and

$$H^{r}(\bigoplus_{l\in\boldsymbol{Z}}\mathcal{I}_{V/X}(l))=\Bigl(\sum_{\substack{i_0+\cdots+i_r=2r\\ i_0>0,\cdots,i_r>0}} k\cdot X_0^{i_0}\cdots X_r^{i_r}\Bigr)\otimes k\cdot 1\otimes k\cdot\frac{1}{Z_0\cdots Z_r}\ .$$

Thus we see that V is arithmetically r-Buchsbaum. On the other hand, we can show that a homomorphism in our spectral sequence in (2.2)

$$\bigotimes_{0\leq j\leq r}(X_j\otimes Y_j\otimes Z_j)\ :\quad H^{2r}(\bigoplus_{l\in\boldsymbol{Z}}\mathcal{I}_{V/X}(l))\longrightarrow H^{r}(\bigoplus_{l\in\boldsymbol{Z}}\mathcal{I}_{V/X}(l))$$

is not a zero map. Hence V is not arithmetically $(r+1)$-Buchsbaum.
We will give another proof of this fact in (4.7).

§ 4. On Buchsbaum property of Segre product.

The aim of this section is to investigate the condition that the Segre product of two graded modules has Buchsbaum property. We will use the notations of Goto-Watanabe [4, Chapter 4].

Let $R=k[R_1]$ and $S=k[S_1]$ be graded rings over an infinite field k. Let us put $R=k[x_1,\cdots,x_a]$ and $S=k[y_1,\cdots,y_b]$, where $x_i\in R_1$ $(i=1,\cdots,a)$ and $y_j\in S_1$ $(j=1,\cdots b)$. Let M be a finitely generated graded R-module with $\dim M=m\geqq 2$ and let N be a finitely generated graded S-module with $\dim N=n\geqq 2$.

REMARK 4.1. It is well known that $\dim M\# N=m+n-1$ and

$p_1^*(\tilde{M}) \otimes p_2^*(\tilde{N}) \cong (M \# N)^{\sim}$.

$$\begin{array}{ccc} \operatorname{Proj} R \times_k \operatorname{Proj} S & \xrightarrow{p_2} & \operatorname{Proj} S \\ {\scriptstyle p_1}\downarrow & & \\ \operatorname{Proj} R & & \end{array}$$

Let $I^{\cdot}$ be the minimal injective resolution of M in the category $M_H(R)$ of the graded R-modules. For each i, we put $I^i = 'I^i \oplus ''I^i$, where $\operatorname{Ass}_R 'I^i = \{\mathfrak{m}\}$ and $\mathfrak{m} \notin \operatorname{Ass}_R ''I^i$. Let $E^{\cdot}$ be the minimal injective resolution of N in $M_H(S)$ and let us put $E^i = 'E^i \oplus ''E^i$ for each i similarly.

Now let us define the complex $\bar{I}^{\cdot}$ which will be used afterwards.

PROPOSITION 4.2. *Let R be a graded ring. Let M be a finitely generated R-module of* $\operatorname{depth} M \geqq 2$. *Then the following complexes $L^{\cdot}$ and $\bar{I}^{\cdot}$ are isomorphic in the derived category $D_h^+(R)$ of complexes bounded below of graded R-modules*:

$$L^{\cdot} = (0 \longrightarrow M \xrightarrow{\bar{\varepsilon}} \bigoplus_{d \in \mathbf{Z}} C^{\cdot}(\mathfrak{U};\, \tilde{M}(d))[-1]) \quad \textit{and}$$

$$\bar{I}^{\cdot} = (0 \longrightarrow M \xrightarrow{\varepsilon} {}''I^{\cdot}[-1]) ,$$

where $\bar{I}^0 = M$, $\bar{I}^i = {}''I^{i-1}$ for $i \geqq 1$ and both $\bar{\varepsilon}$ and ε are the natural maps.

PROOF. Let $X = \operatorname{Spec} R$ and $Y = V(\mathfrak{m})$. Consider the triangle

$$\begin{array}{ccc} \boldsymbol{R}\Gamma_Y(\tilde{M}) & \longrightarrow & \boldsymbol{R}\Gamma(X, \tilde{M}) \\ & {\scriptstyle +1}\nwarrow \quad \swarrow & \\ & \boldsymbol{R}\Gamma(X-Y, \tilde{M}) . & \end{array}$$

Since $H^i(X, \tilde{M}) = 0$ for $i \geqq 1$ and $\operatorname{depth} M \geqq 2$, we have isomorphisms $\boldsymbol{R}\Gamma_Y(\tilde{M}) \cong \tau_1 \boldsymbol{R}\Gamma_Y(\tilde{M}) \cong (\tau_0 \boldsymbol{R}\Gamma(X-Y, \tilde{M}))[-1]$ in $D_h^+(R)$. Thus we have isomorphisms

$$\boldsymbol{R}\Gamma_{\mathfrak{m}}(M) \cong (\tau_0 \boldsymbol{R}\Gamma(X-Y, \tilde{M}))[-1] \cong (\tau_0(\bigoplus_{d \in \mathbf{Z}} \boldsymbol{R}\Gamma(\operatorname{Proj} R, \tilde{M}(d))))[-1] \cong L^{\cdot}$$

in $D_h^+(R)$. What we have to do is to construct a quasi-isomorphism:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & ''I^0 & \longrightarrow & ''I^1 & \longrightarrow & \cdots \\ & & {\scriptstyle f^0}\downarrow & & {\scriptstyle f^1}\downarrow & & {\scriptstyle f^2}\downarrow & & \\ 0 & \longrightarrow & 'I^0 & \longrightarrow & 'I^1 & \longrightarrow & 'I^2 & \longrightarrow & \cdots . \end{array}$$

We write

$$\begin{bmatrix} \varphi^i & \alpha^i \\ 0 & \psi^i \end{bmatrix} : \ 'I^i \oplus ''I^i \longrightarrow 'I^{i+1} \oplus ''I^{i+1} .$$

Then $f^{\cdot}$ is defined as follows:

$$f^0 : M \longrightarrow 'I^0 \quad \text{(the natural map)} \quad \text{and}$$
$$f^i : ''I^{i-1} \longrightarrow 'I^i \quad \text{by} \quad f^i(x) = (-1)^i \alpha^{i-1}(x) \qquad \text{for} \quad i \geqq 1 .$$

This gives a quasi-isomorphism of the complexes. Hence we see $\bar{I}^{\cdot} \cong L^{\cdot}$ in $D_h^+(R)$.

REMARK 4.3. In this way, we can use $\bar{I}^{\cdot}$ instead of $L^{\cdot}$ by (1.5) and (4.2). Similarly we write $\bar{E}^{\cdot}$ for the complex $(0 \to N \to ''E^{\cdot}[-1])$.

From now on we will write $T = R \# S$ and $z_{ij} = x_i \otimes y_j \in T$. Now let us consider a complex $L^{\cdot} = (F^q)$, where $F^q = \bigoplus_{i+j=q} I^i \# E^j$. Then the complex $F^{\cdot}$ is a resolution of $M \# N$ by Goto-Watanabe [4, (4.1.4)]. Thus we have the isomorphism $\Gamma_{\mathfrak{m}_T}(F^{\cdot}) \cong \boldsymbol{R}\Gamma_{\mathfrak{m}_T}(M \# N)$ in $D_h^+(T)$. Let $J^{\cdot} = (0 \to M \# N \to \bigoplus_{i+j=q-1} ''I^i \# ''E^j)$. By the proof of (4.2), we have the isomorphism $J^{\cdot} \cong \boldsymbol{R}\Gamma_{\mathfrak{m}_T}(M \# N)$ in $D_h^+(T)$. Thus we can use the complex $J^{\cdot}$ for the Čech complex $\check{C}^{\cdot}(\mathfrak{U}; (M \# N)^{\sim})$ by (1.5). Now we will apply the theory of §1 to the double complex $C^{\cdot\cdot} = \mathrm{Hom}_T(K_{\cdot}(\{z_{ij}\}; T), J^{\cdot})$ and its spectral sequence $\{E_r^{p,q}\}$. Recall that

$$C^{p,q} = \bigoplus_{i+j=q-1} (''I^i \# ''E^j) \otimes_T (\wedge^p \bigoplus_{\substack{1 \leqq t \leqq a \\ 1 \leqq u \leqq b}} Te_{t,u}^*) \qquad \text{for} \quad q \geqq 1 .$$

Putting

$$C^{p,(i,j)} = (''I^i \# ''E^j) \otimes_T (\wedge^p \bigoplus_{\substack{1 \leqq t \leqq a \\ 1 \leqq u \leqq b}} Te_{t,u}^*) \qquad \text{for every } (i, j) \neq (0, 0) \text{ and}$$

$$C^{p,(0,0)} = (M \# N) \otimes_T (\wedge^p \bigoplus_{\substack{1 \leqq t \leqq a \\ 1 \leqq u \leqq b}} Te_{t,u}^*) ,$$

we write $C^{p,q} = \bigoplus_{i+j=q-1} C^{p,(i,j)}$.

PROPOSITION 4.4. *Suppose that M is a graded quasi-Buchsbaum R-module with* $\dim M = m$ *and* $\operatorname{depth} M \geqq 2$ *and N is a graded quasi-Buchsbaum S-module with* $\dim N = n$ *and* $\operatorname{depth} N \geqq 2$. *Then the graded $(R \# S)$-module $M \# N$ is a quasi-Buchsbaum module if and only if* $\mathfrak{m}_{R\#S}(M \# H_{\mathfrak{m}_S}^n(N)) = 0$ *and* $\mathfrak{m}_{R\#S}(H_{\mathfrak{m}_R}^m(M) \# N) = 0$.

PROOF. It follows immediately from Künneth's formula (in Goto-Watanabe [4, (4.1.5)]).

Now we will discuss r-Buchsbaum property. From now on we assume that M is a Cohen-Macaulay module and N is a Buchsbaum module with depth $N\geqq 2$. Note that depth $M\# N\geqq 2$. Let us calculate the local cohomology groups $H^q_{\mathfrak{m}_T}(M\# N)$. Since M is a Cohen-Macaulay module, we see that

$$H^q_{\mathfrak{m}_T}(M\# N)\cong \bigoplus_{i+j=q-1} H^i(''I^{\cdot})\# H^j(''E^{\cdot})$$
$$\cong \begin{cases} (M\# H^q_{\mathfrak{m}_S}(N))\oplus(H^q_{\mathfrak{m}_R}(M)\# H^{q-m+1}_{\mathfrak{m}_S}(N)) & \text{if} \quad q\neq m \\ (M\# H^m_{\mathfrak{m}_S}(N))\oplus(H^m_{\mathfrak{m}_R}(M)\# N) & \text{if} \quad q=m\,. \end{cases}$$

Suppose that $M\# N$ is an $(r-1)$-Buchsbaum module with $r\geqq 2$. By (1.4) and (2.4), we see that

$$E^{p,q}_r=H^q_{\mathfrak{m}_T}(M\# N)\otimes_T(\wedge^p \bigoplus_{\substack{1\leqq t\leqq a\\ 1\leqq u\leqq b}} Te^*_{t,u}) \qquad \text{for} \quad q<m+n-1\,.$$

Let us put

$$H^{p,(i,j)}=(H^i(''I^{\cdot})\# H^j(''E^{\cdot}))\otimes_T(\wedge^p \bigoplus_{\substack{1\leqq t\leqq a\\ 1\leqq u\leqq b}} Te^*_{t,u}) \qquad \text{for every } (i,\ j)\neq(0,\ 0)$$

and $H^{p,(0,0)}=0$. Then we see $E^{p,q}_r=H^{p,(0,q-1)}\oplus H^{p,(m-1,q-m)}$ for $q<m+n-1$. Let us write d_r: $H^{p,(i,j)}\to H^{p+r,(l,i+j-l-r+1)}$ for the map induced by the map $d^{p,q}_r$: $E^{p,q}_r\to E^{p+r,q-r+1}_r$.

LEMMA 4.5. *Under the above conditions, all the maps as below are zero*:

(a) d_r: $H^{p,(0,q-1)}\to H^{p+r,(0,q-r)}$ *for* $r<q<n$,
(b) d_r: $H^{p,(0,q-1)}\to H^{p+r,(m-1,q-m-r+1)}$ *for* $q<m+n-1$,
(c) d_r: $H^{p,(m-1,q-m)}\to H^{p+r,(0,q-r)}$ *for* $q<m+n-1$, *and*
(d) d_r: $H^{p,(m-1,q-m)}\to H^{p+r,(m-1,q-m-r+1)}$ *for* $m+r\leqq q<m+n-1$.

PROOF. First, we will prove the case (a). Notice that

$$H^{p,(0,q-1)}\cong(M\# H^q_{\mathfrak{m}_S}(N))\otimes_T(\wedge^p \bigoplus_{\substack{1\leqq t\leqq a\\ 1\leqq u\leqq b}} Te^*_{t,u}) \quad \text{and}$$
$$H^{p+r,(0,q-r)}\cong(M\# H^{q-r+1}_{\mathfrak{m}_S}(N))\otimes_T(\wedge^{p+r} \bigoplus_{\substack{1\leqq t\leqq a\\ 1\leqq u\leqq b}} Te^*_{t,u})\,.$$

By (2.3), we may assume $p=0$. The T-module $H^{0,(0,q-1)}$ is generated by the homomorphic images of the elements $v\otimes w$ in $C^{0,(0,q-1)}$, where v is a cocycle of $''I^0$ and w is a cocycle of $''E^{q-1}$. Hence we can consider v as an element of M. According to §1 and §2, let us explicitly describe the maps in the spectral sequence. We will show that $(z_{ij}\otimes z_{kl})(v\otimes w)=(x_ix_kv)\otimes((y_j\otimes y_l)w)$ for every v in M and every w in $H^q_{\mathfrak{m}_S}(N)$, where

$z_{ij}\otimes z_{kl}$: $H^q_{\mathfrak{m}_T}(M\#N)\to H^{q-1}_{\mathfrak{m}_T}(M\#N)$ and $y_j\otimes y_l$: $H^q_{\mathfrak{m}_S}(N)\to H^{q-1}_{\mathfrak{m}_S}(N)$ are defined in (2.4). In fact, let us consider w as an element of the double complex $\mathrm{Hom}_S(K_0(\{y_j\}; S), ''E^{q-1})$. Since N is a quasi-Buchsbaum module, we can take elements u_j for $1\leqq j\leqq b$ in $''E^{q-2}$ such that $d''(\sum_{1\leqq j\leqq b} u_j e^*_j)=\sum_{1\leqq j\leqq b} y_j w e^*_j$ in $\mathrm{Hom}_S(K_1(\{y_j\}; S), ''E^{q-1})$. This gives that

$$d''(\sum_{\substack{1\leqq i\leqq a\\ 1\leqq j\leqq b}}((x_i v)\otimes u_j)e^*_{,j})=\sum_{\substack{1\leqq i\leqq a\\ 1\leqq j\leqq b}}((x_i v)\otimes(y_j w))e^*_{i,j}$$

in $C^{1,q}$ because v is a cocycle. On the other hand we see that

$$d'(\sum((x_i v)\otimes u_j)e^*_{i,j})=\sum((x_i x_k v)\otimes(y_l u_j-y_j u_l))e^*_{i,j}\wedge e^*_{k,l}$$

in $C^{2,(0,q-2)}$. Hence we infer that $(z_{ij}\otimes z_{kl})(v\otimes w)=((x_i x_k v)\otimes(y_j\otimes y_l)w)$. Similarly we can inductively get

$$(z_{i_1 j_1}\otimes\cdots\otimes z_{i_r j_r})(v\otimes w)=(x_{i_1}\cdots x_{i_r} v)\otimes((y_{j_1}\otimes\cdots\otimes y_{j_r})w)\ .$$

Since N is a Buchsbaum module, we have $(y_{j_1}\otimes\cdots\otimes y_{j_r})w=0$ by (2.2). Thus $(z_{i_1 j_1}\otimes\cdots\otimes z_{i_r j_r})(v\otimes w)=0$. Hence the assertion is proved.

Second, we will prove the case (b). Let us take an element $v\otimes w$ in $H^{0,(0,q-1)}$, where v is a cocycle of $''I^0$ and w is a cocycle of $''E^{q-1}$. Hence we can consider v as an element of M. Here we can assume that $v\otimes w$ is a non-zero element. Assume that $d'(v\otimes w)=0$. Since

$$d'(v\otimes w)=\sum_{\substack{1\leqq i\leqq a\\ 1\leqq j\leqq b}}((x_i v)\otimes(y_j w))e^*_{i,j}$$

and $\operatorname{depth} M>0$, we can take elements u_j for $1\leqq j\leqq b$ in $''E^{q-2}$ such that

$$d''(\sum_{1\leqq j\leqq b} u_j e^*_j)=\sum_{1\leqq j\leqq b} y_j w e^*_j \qquad \text{in} \quad \mathrm{Hom}_S(K_1(\{y_j\}; S), ''E^{q-1})\ .$$

This gives that

$$d''(\sum_{\substack{1\leqq i\leqq a\\ 1\leqq j\leqq b}}((x_i v)\otimes u_j)e^*_{i,j})=\sum_{\substack{1\leqq i\leqq a\\ 1\leqq j\leqq b}}((x_i v)\otimes(y_j w))e^*_{i,j} \qquad \text{in} \quad C^{1,q}$$

because v is a cocycle. Thus we have that $d^{0,q}_2(H^{0,(0,q-1)})\subset H^{2,(0,q-2)}$ through the map $d^{0,q}_2$: $E^{0,q-1}_2\to E^{2,q-2}_2$. In the same way, we inductively have that $d^{0,q}_r(H^{0,(0,q-1)})\subset H^{r,(0,q-r)}$. Hence the assertion is proved.

The remaining cases (c) and (d) are proved similarly.

Now we will prove Theorems A, B, and C.

PROOF OF THEOREM A. It is clear that (a) implies (b). By Künneth's formula, we see that

$$\sum_{0\leq q<m+n-1} H^q_{\mathfrak{m}_T}(M\# N)=(M\# H^n_{\mathfrak{m}_S}(N))\oplus(H^m_{\mathfrak{m}_R}(M)\# N)\ . \tag{4.6.1}$$

This gives the equivalence between (b) and (c). Now we will show that (c) implies (a). Let us consider the spectral sequence $\{E_r^{p,q}\}$ associated to $M\# N$. By (4.6.1), we have

$$\begin{aligned}\sum_{i+j<m+n-2} H^{p,(i,j)} &= H^{p,(0,n-1)}\oplus H^{p,(m-1,0)}\\ &=(M\# H^n_{\mathfrak{m}_S}(N))\otimes_T(\wedge^p \bigoplus_{\substack{1\leq t\leq a\\ 1\leq u\leq b}} Te^*_{t,u})\oplus(H^m_{\mathfrak{m}_R}(M)\# N)\otimes_T(\wedge^p \bigoplus_{\substack{1\leq t\leq a\\ 1\leq u\leq b}} Te^*_{t,u})\ .\end{aligned}$$

By (4.5.b) and (4.5.c), all the maps $d_r^{p,q}$: $E_r^{p,q}\to E_r^{p-r,q+r-1}$ for every $q<m+n-1$ are zero maps. By (2.3), the assertion is proved.

PROOF OF THEOREM B. Since M is a Cohen-Macaulay R-module and $M\# H^n_{\mathfrak{m}_S}(N)=H^m_{\mathfrak{m}_R}(M)\# N=0$, we see that

$$\sum_{0\leq q<m+n-1} H^q_{\mathfrak{m}_T}(M\# N)=(\sum_{2\leq q<n} M\# H^q_{\mathfrak{m}_S}(N))\oplus(\sum_{2\leq q<n} H^m_{\mathfrak{m}_R}(M)\# H^q_{\mathfrak{m}_S}(N))\ . \tag{4.6.2}$$

In other words, we have that

$$\sum_{i+j<m+n-2} H^{p,(i,j)}=(\sum_{2\leq q<n} H^{p,(0,q-1)})\oplus(\sum_{2\leq q<n} H^{p,(m-1,q-1)})\ .$$

By (4.5), all the maps of our spectral sequence $d_r^{p,q}$: $E_r^{p,q}\to E_r^{p-r,q+r-1}$ for every $q<m+n-1$ are zero maps. By (2.3), the assertion is proved.

PROOF OF THEOREM C. We may assume that S is a polynomial ring and that $M\# N$ is an $(l-1)$-Buchsbaum T-module. By Künneth's formula, we see that

$$\sum_{0\leq q<m+n-1} H^q_{\mathfrak{m}_T}(M\# N)=(\sum_{2\leq q\leq n} M\# H^q_{\mathfrak{m}_S}(N))\oplus(H^m_{\mathfrak{m}_R}(M)\# N)\ .$$

Now we will show that the map d_l: $H^{0,(m-1,l-1)}\to H^{p,(m-1,0)}$ is a non-zero map. Let us consider the double complex $D^{\cdot\cdot}=\mathrm{Hom}_S(K_\cdot(y_1,\cdots,y_s;S),\bar{E}^\cdot)$ and its spectral sequence $\{F_r^{p,q}\}$. By the hypothesis there is a non-zero element v of degree d in $F_1^{0,l}$. Since N is a Buchsbaum module, we have the isomorphism $F_1^{0,l}\cong F_\infty^{0,l}$. Moreover, since S is regular, this spectral sequence $\{F_r^{p,q}\}$ converges to $\mathrm{Ext}^l_S(k,N)$. Thus we have a non-zero component w of an element of degree $d+l$ in $D^{l,0}$ corresponding to v. On the other hand, there are a subsystem of parameters $x_{i_1},\cdots,x_{i_l}$ in R_1 and an element u in $H^m_{\mathfrak{m}_R}(M)_d$ such that $x_{i_1}\cdots x_{i_l}u\neq 0$ in $H^m_{\mathfrak{m}_R}(M)_{d+l}$. Now we see $d_l(u\otimes v)=(x_{i_1}\cdots x_{i_l}u)\otimes w$ through the map d_l: $H^{0,(m-1,l-1)}\to H^{p,(m-1,0)}$. Since $x_{i_1}\cdots x_{i_l}u\neq 0$ and $w\neq 0$, we have that d_l: $H^{0,(m-1,l-1)}\to H^{p,(m-1,0)}$ is a non-zero map. By (2.3), $M\# N$ is not an l-Buchsbaum module. Thus the assertion is proved.

REMARK 4.7. Theorem C gives another proof of Example 3.4. Let us describe this example in terms of § 4. Let $R=k[X_0, \cdots, X_r]$ be a polynomial ring. We will show that $R(a-r-1)\#R(a)\#R(a+r+1)$ is not an $(r+1)$-Buchsbaum module. We may assume $a=0$. Note that $R(-r-1)$ is a Cohen-Macaulay module and $X_0 \cdots X_r(H_{\mathfrak{m}}^{r+1}(R(-r-1))_{-r-1}) \neq 0$. On the other hand, $R\#R(r+1)$ is a Buchsbaum module of depth $r+1$ and $H_{\mathfrak{m}}^{r+1}(R\#R(r+1))_{-r-1}=(H_{\mathfrak{m}}^{r+1}(R)\#R(r+1))_{-r-1} \neq 0$. By Theorem C and its proof, we see that $R(a-r-1)\#R(a)\#R(a+r+1)$ is not an $(r+1)$-Buchsbaum module.

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