TOKYO J. MATH.
VOL. 13, No. 2, 1990

On Peak Sets for Certain Function Spaces II

Shinya YAMAGUCHI and Junzo WADA

Waseda University

Introduction.

This paper is the continuation of [15]. In [15], the authors showed that some theorems on function algebras can be generalized to the case of a wider class of function spaces containing the class of function algebras. This class is of function spaces having the condition (A) (see §1). In this paper, we introduce the conditions (B) and (C) which are weaker than (A). In §1, we discuss the conditions (B) and (C), and give some examples connected with them. In §2, we consider the class $\mathscr{B}$ of function spaces having (B) and the class $\mathscr{C}$ of function spaces having (C), and give characterizations to assert that $A=C(X)$ for $A\in\mathscr{B}$ or $A\in\mathscr{C}$. Especially, we establish generalizations of a theorem of Rudin [13] and a theorem of Hoffman and Wermer [11] (Theorems 2.1 and 2.5).

§1. Conditions for function spaces.

Throughout this paper, X will denote a compact Hausdorff space. A is said to be a *function space* (resp. *function algebra*) on X if A is a closed subspace (resp. subalgebra) in $C(X)$ containing constant functions and separating points in X, where $C(X)$ denotes the Banach algebra of complex-valued continuous functions on X with the supremum norm.

Let A be a function space on X. For a subset E in X, we denote

$$A(E)=\{f\in C(E): fg\in A|_E \text{ for any } g\in A|_E\}$$
$$A_R(E)=\{f\in C_R(E): fg\in A|_E \text{ for any } g\in A|_E\}\,,$$

where $A|_E$ is the restriction of A to E and $C_R(E)$ is the set of all real-valued continuous functions on E.

Let E be a subset in X. Then we call E an *antisymmetric set* for A if any function in $A_R(E)$ is constant. We write $\mathscr{K}(A)$ the family

Received November 8, 1989

of maximal antisymmetric sets for A.

Let A be a uniformly closed subspace in $C(X)$ or $C_R(X)$. Then a closed subset F in X is called a *peak set* for A if $f(x)=1$ $(x\in F)$ and $|f(x)|<1$ $(x\in X\setminus F)$ for an $f\in A$. A *p-set* for A is an intersection of peak sets for A. A closed subset F in X is called a *BEP-set* for A if for any $f\in A|_F$ and for any closed subset G in X with $G\cap F=\emptyset$ and any $\varepsilon>0$, there is a $g\in A$ such that $g=f$ on F, $|g(x)|<\varepsilon$ on G and $\|g\|=\|f\|_F$, where $\|g\|=\sup_{x\in X}|g(x)|$ and $\|f\|_F=\sup_{x\in F}|f(x)|$. For a uniformly closed subspace A in $C(X)$, F is a BEP-set for A if and only if $\mu_F\in A^\perp$ for any $\mu\in A^\perp$, where $A^\perp$ denotes the set of measures μ on X such that $\int f d\mu=0$ for any $f\in A$ (cf. [8]).

Let A be a uniformly closed subspace in $C(X)$ or $C_R(X)$. A closed subset F in X is called a *sharp peak set* for A if for any closed subset G in X with $G\cap F=\emptyset$ and for any $\varepsilon>0$, there is an $f\in A$ such that $f(x)=1$ $(x\in F)$, $|f(x)|<\varepsilon$ $(x\in G)$ and $|f(x)|<1$ $(x\in X\setminus F)$ (cf. [7]). We note that if F is a sharp peak set for A, then $\mu(F)=0$ for any $\mu\in A^\perp$.

The Bishop antisymmetric decomposition theorem for function spaces is given as follows. This is a generalization of Bishop's theorem on function algebras ([2], [6], [9]).

THEOREM 1.1. *Let A be a function space on a compact Hausdorff space X. Then X is decomposed by the family $\mathscr{K}(A)$ of maximal antisymmetric sets for A and the following is satisfied.*

(i) *Any $K\in\mathscr{K}(A)$ is a BEP-set for A.*

(ii) *If $f\in C(X)$ and if $f|_K\in A|_K$ for any $K\in\mathscr{K}(A)$, then $f\in A$.*

This theorem was essentially proved in [1]. We also see it in [15].

Now we consider the following three conditions for a function space A on X.

(A) Any peak set for A is a peak set for $A(X)$.

(B) Any peak set for A is a BEP-set for A.

(C) Any peak set for A is a sharp peak set for A.

THEOREM 1.2. *Let A be a function space on X. Then the following are satisfied:*

(i) *If A has* (A), *then it has* (B).

(ii) *If A has* (B), *then it has* (C).

PROOF. (i) Let A have (A) and let F be a peak set for A. Then F is a peak set for $A(X)$, that is, there is an $f_0\in A(X)$ such that $f_0(x)=1$ $(x\in F)$ and $|f_0(x)|<1$ $(x\in X\setminus F)$. By the definition of $A(X)$,

$ff_0^n \in A$ for any $f \in A$ and for any $n \in \boldsymbol{N}$. For any $\mu \in A^\perp$, $\int_F f d\mu = \int_F ff_0^n d\mu = -\int_{X \setminus F} ff_0^n d\mu \to 0$ $(n \to \infty)$. From this $\mu_F \in A^\perp$ and so (B) holds.

(ii) Let A have (B) and let F be a peak set for A. Then F is a BEP-set for A and a G_δ-set. It follows that F is a sharp peak set for A.

But, implications (C)$\to$(B) and (B)$\to$(A) are not true in general.

EXAMPLES. (1) The following function space A has (C), but it does not satisfy (B): Let D, $\bar{D}$ and ∂D be $\{z \in \boldsymbol{C}:\ 1/2 < |z| < 1\}$, $\{z \in \boldsymbol{C}:\ 1/2 \leqq |z| \leqq 1\}$ and $\{z \in \boldsymbol{C}:\ |z| = 1/2 \text{ or } |z| = 1\}$ respectively. Let B be the restriction of $A(\bar{D})$ to ∂D, where $A(\bar{D})$ is the function algebra of continuous functions on $\bar{D}$ which are analytic on D. We put $A = \boldsymbol{C}f + B = \{\lambda f + g :\ \lambda \in \boldsymbol{C},\ g \in B\}$, where f is the following function:

$$f = \begin{cases} 0 \text{ on } X_1 = \{z \in \boldsymbol{C}:\ |z| = 1/2\} \\ 1 \text{ on } X_2 = \{z \in \boldsymbol{C}:\ |z| = 1\}\ . \end{cases}$$

Then A is a function space on ∂D. It is not hard to see that a peak set F $(F \neq \partial D)$ for A is a closed subset on ∂D of Lebesgue measure 0 or X_1 or X_2. It implies that A has (C), because such a subset of measure 0 is a sharp peak set for B (cf. [12]). But A has not (B). For, X_1 is a peak set for A but not a BEP-set for A.

(2) An example of function spaces which have (B) but not (A). Let $X = \{z \in \boldsymbol{C}:\ |z| = 1\}$ and let B be a disc algebra on X. We define a continuous function f on X as follows:

$$f(e^{it}) = \begin{cases} \varphi(t/\pi): & 0 \leqq t \leqq \pi \\ \varphi(2 - t/\pi): & \pi < t \leqq 2\pi\ , \end{cases}$$

where φ is the Cantor function on [0, 1] (cf. [10], p. 83). We put $A = \boldsymbol{C}f + B$. Then any peak set F $(F \neq X)$ for A is a closed subset on X of Lebesgue measure 0. It follows that A has (B). But A has not (A) since $A(X) = \boldsymbol{C}$.

§2. Characterizations assert that $A = C(X)$.

In [15] we gave some characterizations to assert that $A = C(X)$ for function spaces A having the condition (A). After this, we moreover give characterizations to assert that $A = C(X)$ for function spaces A having the condition (B) or (C) which is weaker than (A).

We denote by $\partial(A)$ and $\mathrm{Ch}(A)$ the Shilov boundary and the Choquet boundary for a function space A respectively.

Our first goal is to generalize the Rudin's theorem ([13]) to the case of function spaces.

THEOREM 2.1. *Let A be a function space on X having* (C). *If X contains no non-void perfect subset, then $A=C(X)$.*

We begin with the following lemma.

LEMMA 2.2. *Let A be a function space on X having* (C) *and let $K\in\mathscr{K}(A)$. If $x_0\in\partial(A|_K)$ is an isolated point in K, then $\{x_0\}$ is a sharp peak set for $A|_K$.*

PROOF. Since x_0 is an isolated point in $\partial(A|_K)$, we have $x_0\in\mathrm{Ch}(A|_K)$. Hence there is an $f\in A|_K$ such that $\mathrm{Re}\,f\leqq 0$, $\mathrm{Re}\,f(x_0)>-\alpha$ and $\mathrm{Re}\,f(x)<-\beta$ $(x\in K\smallsetminus\{x_0\})$ for some α, β $(0<\alpha<\beta)$ (cf. [5]). By adding a sufficiently large number c to f if necessary, $f+c$ becomes a scalar multiple of a peaking function of $\{x_0\}$ in $A|_K$. That is, $\{x_0\}$ is a peak point for $A|_K$. It follows that $\{x_0\}$ is a sharp peak set for $A|_K$. For, since A has (C) and K is a BEP-set for A, it implies that $A|_K$ has (C).

PROOF OF THEOREM 2.1. If $A\neq C(X)$, by Theorem 1.1, K is not a singleton for some $K\in\mathscr{K}(A)$. We put $I=\{x\in\partial(A|_K):x$ is an isolated point in $\partial(A|_K)\}$ and $J=\{x\in K: x$ is an isolated point in $K\}$. Then we assert that $J\cap\partial(A|_K)=\varnothing$. For, if a point $x_0\in J\cap\partial(A|_K)$, by Lemma 2.2, $\{x_0\}$ is a sharp peak set for $A|_K$. Since $\{x_0\}$ is open in K, there is an $f\in A|_K$ such that $f(x_0)=1$, $f(x)=0$ $(x\in K\smallsetminus\{x_0\})$. Hence $f\in A_R(K)$ and it contradicts that $K\in\mathscr{K}(A)$. It shows that $J\cap\partial(A|_K)=\varnothing$. Next since $I\neq\varnothing$, we put $x_1\in I$. Then $x_1\in I\subset\mathrm{Ch}(A|_K)\subset\mathrm{Ch}(A|_{\partial(A|_K)})$. By a similar way as in the proof of Lemma 2.2, we observe that $\{x_1\}$ is a peak point for $A|_{\partial(A|_K)}$. Hence $\{x_1\}=F\cap\partial(A|_K)$ for a peak set F for $A|_K$. Since $A|_K$ has (C), F is a sharp peak set for $A|_K$. So it is easy to see that $\{x_1\}$ is a sharp peak set for $A|_{\partial(A|_K)}$. We here assert that $J\cap F=\varnothing$. For, if $x\in J\cap F$, there is a representing measure μ_x for x on $\partial(A|_K)$. Since $\mu_x-\delta_x\in(A|_K)^\perp$ and F is a sharp peak set for $A|_K$, $\mu_x(F)-\delta_x(F)=(\mu_x-\delta_x)(F)=0$, where δ_x is the Dirac measure for x. But $\mu_x(F)=\int_F d\mu_x=\int_{F\cap\partial(A|_K)}d\mu_x=\mu_x(\{x_1\})$. From this, $\mu_x(\{x_1\})=1$, that is, $\mu_x=\delta_{x_1}$ and $x=x_1$. This is a contradiction since $x_1\in\partial(A|_K)$, $x_1=x\in J$ and $J\cap\partial(A|_K)=\varnothing$. So we have $J\cap F=\varnothing$. Now, since $\{x_1\}$ is a sharp peak set for $A|_{\partial(A|_K)}$ and $\{x_1\}$ is open in $\partial(A|_K)$, it implies that there is a $g\in A|_{\partial(A|_K)}$ such that $g(x_1)=1$ and $g=0$ on $\partial(A|_K)\smallsetminus\{x_1\}$. For any x in J, $\mu_x-\delta_x\in(A|_K)^\perp$ for a representing measure μ_x for x on $\partial(A|_K)$. Hence $\mu_x(F)-\delta_x(F)=(\mu_x-\delta_x)(F)=0$. Since $J\cap F=\emptyset$ and $x\in J$, we have $x\notin F$ and so $\mu_x(\{x_1\})=\mu_x(F)=0$. Considering g as a function in $A|_K$, $g(x)=\int_{\partial(A|_K)} g d\mu_x=\mu_x(\{x_1\})=0$ for any $x\in J$. Since $\bar{J}=K$, we have $g=0$ on K. Since $g(x_1)=1$, it is a contradiction, concluding the proof.

The Stone-Weierstrass theorem is stated as follows (see [15] for the case of function spaces having (A)).

Theorem 2.3. *If a function space A on X having* (C) *is self-adjoint, then $A=C(X)$.*

Proof. If A is self-adjoint, the real part $\operatorname{Re} A$ of A is a real-valued function space. If F is a peak set for $\operatorname{Re} A$, it is a peak set for A since $\operatorname{Re} A\subset A$. By (C) F is a sharp peak set, that is, for any $\varepsilon>0$ and any closed subset G with $F\cap G=\emptyset$, there is an $f\in A$ such that $f=1$ on F, $\|f\|=1$, $|f|<1$ on $X\setminus F$ and $|f|<\varepsilon$ on G. It implies that $\operatorname{Re} f=1$ on F, $\|\operatorname{Re} f\|=1$, $|\operatorname{Re} f|<1$ on $X\setminus F$ and $|\operatorname{Re} f|<\varepsilon$ on G. This shows that F is a sharp peak set for $\operatorname{Re} A$. Hence a theorem of Briem ([3]) guarantees that $\operatorname{Re} A=C_R(X)$. From this $A=C(X)$ since $A=\operatorname{Re} A+i\operatorname{Re} A$ and this completes the proof.

Next we generalize a theorem of Briem ([4]) as follows (see [15] for the case of function spaces having (A)).

Theorem 2.4. *Suppose that a function space A on X has* (B). *If any peak set for* $\operatorname{Re} A$ *is a peak set for A, then $A=C(X)$.*

Proof. If A has (B), any peak set for A is a BEP-set for A. By the hypothesis, any peak set for $\operatorname{Re} A$ is a BEP-set for A. It follows that $A=C(X)$ by [14] Theorem 2.2.

Finally, we consider a generalization of the Hoffman-Wermer theorem ([11]) to the case of function spaces.

Theorem 2.5. *Assume that a function space A on X has* (B). *If* $\operatorname{Re} A$ *is closed in $C(X)$, then $A=C(X)$.*

We begin with the following lemmas.

Lemma 2.6. *Let A be a function space on a compact Hausdorff space X and let $f\in C_R(X)$. If $F_r=\{x\in X: f(x)\leqq r\}$ is a BEP-set for A for any $r\in \mathbf{R}$, then $f\in A_R(X)$.*

PROOF. We can assume that $0 \leqq f \leqq 1$. Put $E_{ni} = \{x \in X : 2^{-n}(i-1) < f(x) \leqq 2^{-n}i\}$ $(i=0, 1, \cdots, 2^n)$. Then E_{ni} is a difference between two BEP-sets. Hence if $\mu \in A^{\perp}$ then $\mu_{E_{ni}} \in A^{\perp}$. For any $g \in A$, we put $h_n = \sum_{i=1}^{2^n} i/2^n \chi_{E_{ni}} g$. Then $\int h_n d\mu \to \int fg\, d\mu$ for any $\mu \in A^{\perp}$ since $h_n \to fg$ boundedly. Since $\int h_n d\mu = 0$ $(n=1, 2, 3, \cdots)$, it implies that $\int fg\, d\mu = 0$ for any $\mu \in A^{\perp}$ and so $fg \in A$. It shows that $f \in A_R(X)$.

REMARK. Let X be a compact Hausdorff space and let A be a function space on X. Let $\mathscr{F}$ be the family of all BEP-sets for A. Then there exists a topology $(\mathscr{T}, X)$ on X such that the family of closed subsets in $(\mathscr{T}, X)$ is $\mathscr{F}$. We can here prove the following: Let $f \in C_R(X)$. Then f is continuous on $(\mathscr{T}, X)$ if and only if $f \in A_R(X)$.

LEMMA 2.7. *Suppose that a function space A on X has* (B). *If A_R is an algebra, then $A_R = A_R(X)$, where $A_R = A \cap C_R(X)$.*

PROOF. It is clear that $A_R(X) \subset A_R$. We first introduce a relation $\sim$ in X as follows.

$$x \sim y \iff f(x) = f(y) \text{ for any } f \in A_R .$$

We set $\tilde{x} = \{y \in X : y \sim x\}$ for $x \in X$ and $\tilde{X} = \{\tilde{x} : x \in X\}$. By defining a topology in $\tilde{X}$ such that the mapping $\varphi : x \to \tilde{x}$ from X to $\tilde{X}$ is continuous, $\tilde{X}$ becomes a compact Hausdorff space. Put $\tilde{f}(\tilde{x}) = f(x)$ $(f \in A_R)$ and $\tilde{A}_R = \{\tilde{f} : f \in A_R\}$. Then $\tilde{A}_R$ is a closed subalgebra in $C_R(\tilde{X})$ containing 1 and separating points in $\tilde{X}$. Hence $\tilde{A}_R = C_R(\tilde{X})$. For $f \in A_R$ and any $r \in \boldsymbol{R}$, put $F_r = \{x \in X : f(x) \leqq r\}$. Then $\varphi(F_r) = \{\tilde{x} \in \tilde{X} : \tilde{f}(\tilde{x}) \leqq r\}$ is closed in $\tilde{X}$. It follows that $\varphi(F_r)$ is a p-set for $\tilde{A}_R = C_R(\tilde{X})$. Hence F_r is a p-set for A_R and so it is a p-set for A. Since A has (B), F_r is a BEP-set for A. By Lemma 2.6, $f \in A_R(X)$. Thus we have that $A_R \subset A_R(X)$ and so the lemma is proved.

LEMMA 2.8. *Let A be a function space on X having* (C). *If* Re A *is closed in $C_R(X)$ and $A_R = A_R(X)$, then $A = C(X)$.*

PROOF. For $x \in X$, we put $E_x = \{y \in X : f(y) = f(x)$ for any $f \in A_R\}$. Since A_R is an algebra, the function $g = -\varepsilon(f - f(x))^2 + 1$ is contained in A_R for $\varepsilon \in \boldsymbol{R}$, $f \in A_R$. It is not hard to see that E_x is a p-set for A_R by taking sufficiently small $\varepsilon > 0$ for each function g above. Since $A_R = A_R(X)$, E_x is a BEP-set for A. For, since E_x is a p-set for A_R, it is a p-set for $A_R(X)$ and so we see that E_x is a BEP-set for A by a similar argument as in the proof of Theorem 1.2 (i). We here assert that E_x

is a singleton. Suppose otherwise. Then there is a peak set F for $A|_{E_x}$ with $F\subsetneqq E_x$. Since E_x is a BEP-set for A, F is a p-set for A. By (C), F is an intersection of sharp peak sets for A. Since $\operatorname{Re} A$ is closed, we have that $(A\cap\bar{A})^{\perp}=A^{\perp}+\bar{A}^{\perp}$ ([5]). Clearly, $A\cap\bar{A}=A_R+iA_R$ and we here put $A_0=A\cap\bar{A}$. Take $p\in F$ and $q\in E_x\setminus F$. Then $\delta_p-\delta_q\in A_0^{\perp}=A^{\perp}+\bar{A}^{\perp}$. Hence $\delta_p-\delta_q=\mu+\nu_1-i\nu_2$, where $\mu\in A^{\perp}$ and $\nu_1+i\nu_2\in A^{\perp}$ (ν_1, ν_2: real measures). Since F is an intersection of sharp peak sets for A, there is a sharp peak set F_1 for A such that $F\subset F_1$, $p\in F_1$ and $q\notin F_1$. Since $\mu(F_1)=0$ and $\nu_1(F_1)+i\nu_2(F_1)=(\nu_1+i\nu_2)(F_1)=0$, it implies that $(\delta_p-\delta_q)(F_1)=\mu(F_1)+\nu_1(F_1)-i\nu_2(F_1)=0$. On the other hand, $(\delta_p-\delta_q)(F_1)=\delta_p(F_1)-\delta_q(F_1)=1$, since $p\in F_1$ and $q\notin F_1$. This contradiction shows that $E_x=\{x\}$ for any $x\in X$. Hence A_R is a closed subalgebra in $C_R(X)$ containing 1 and separating points in X. It follows that $A_R=C_R(X)$ and so $A=C(X)$.

PROOF OF THEOREM 2.5. By Lemmas 2.7 and 2.8, to prove the theorem, it remains only to show that A_R is an algebra. As in the proof of Lemma 2.8, if $\operatorname{Re} A$ is closed, $A_0^{\perp}=(A\cap\bar{A})^{\perp}=A^{\perp}+\bar{A}^{\perp}$. For any peak set F for A_0 and for any $\mu\in A_0^{\perp}$, we have $\mu=\mu_1+\bar{\mu}_2$ for some μ_1, $\mu_2\in A^{\perp}$ and F is a peak set for A. By (B), $(\mu_1)_F\in A^{\perp}$ and $(\mu_2)_F\in A^{\perp}$. Hence $\mu_F=(\mu_1)_F+(\bar{\mu}_2)_F\in A^{\perp}+\bar{A}^{\perp}=A_0^{\perp}$. It follows that F is a BEP-set for A_0. Since F is a G_δ-set, F is a sharp peak set for A_0.

We introduce a relation $\sim$ in X as follows:

$$x\sim y \iff f(x)=f(y) \text{ for any } f\in A_0 .$$

We put $\tilde{x}=\{y\in X:\ y\sim x\}$ for $x\in X$ and $\tilde{X}=\{\tilde{x}:\ x\in X\}$. By defining a topology in $\tilde{X}$ such that the mapping $\varphi: x\to\tilde{x}$ from X to $\tilde{X}$ is continuous, $\tilde{X}$ becomes a compact Hausdorff space. Put $\tilde{f}(\tilde{x})=f(x)$ for $f\in A_0$ and $\tilde{A}_0=\{\tilde{f}:\ f\in A_0\}$. For any peak set $\tilde{F}$ for $\tilde{A}_0$, $F=\varphi^{-1}(\tilde{F})$ is a peak set for A_0. By the fact stated above, F is a sharp peak set for A_0. Hence $\tilde{F}$ is a sharp peak set for $\tilde{A}_0$. By putting $\tilde{A}_R=\{\tilde{f}:\ f\in A_R\}$, $\tilde{A}_R$ is a real function space on $\tilde{X}$. Since $A_0=A_R+iA_R$, we have that any peak set $\tilde{F}$ of $\tilde{A}_R$ is a sharp peak set for $\tilde{A}_R$. By a theorem of Briem ([3]), $\tilde{A}_R=C_R(\tilde{X})$. It implies that A_R is an algebra and the proof is finished.

References

[1] E. L. ARENSON, Uniform convergence algebras and regenerate coverings, Math. USSR Sbornik, **8** (1969), 207-233.

[2] E. BISHOP, A generalization of the Stone-Weierstrass theorem, Pacific J. Math., **11** (1961), 777-783.

[3] E. BRIEM, A characterization of simplexes by parallel faces, Bull. London Math. Soc., **12** (1980), 55-59.

[4] E. BRIEM, Peak sets for the real part of a function algebra, Proc. Amer. Math. Soc., **85** (1982), 77–78.
[5] A. BROWDER, *Introduction to Function Algebras*, Benjamin, 1969.
[6] R. B. BURCKEL, *Characterizations of $C(X)$ among Its Subalgebras*, Marcel Dekker, 1972.
[7] P. C. CURTIS, *Topics in Banach Spaces of Continuous Functions*, Lecture Note, **25**, Aarhus Univ., 1970.
[8] T. W. GAMELIN, Restrictions of subspaces of $C(X)$, Trans. Amer. Math. Soc., **111** (1964), 278–286.
[9] I. GLICKSBERG, Measures orthogonal to algebras and sets of antisymmetry, Trans. Amer. Math. Soc., **105** (1962), 415–435.
[10] P. R. HALMOS, *Measure Theory*, Van Nostrand, 1950.
[11] K. HOFFMAN and J. WERMER, A characterization of $C(X)$, Pacific J. Math., **12** (1962), 941–944.
[12] W. RUDIN, Analyticity and the maximum modulus principle, Duke Math. J., **20** (1953), 449–457.
[13] W. RUDIN, Continuous functions on compact spaces without perfect subsets, Proc. Amer. Math. Soc., **8** (1957), 39–42.
[14] S. YAMAGUCHI and J. WADA, On peak sets for the real part of a function space, Tokyo J. Math., **7** (1984), 141–146.
[15] S. YAMAGUCHI and J. WADA, On peak sets for certain function spaces, Tokyo J. Math., **11** (1988), 415–425.

Present Address:
DEPARTMENT OF MATHEMATICS, SCHOOL OF EDUCATION, WASEDA UNIVERSITY
NISHIWASEDA, SHINJUKU-KU, TOKYO 169, JAPAN