TOKYO J. MATH.
VOL. 13, NO. 2, 1990

Surfaces with Constant Kaehler Angle All of Whose Geodesics Are Circles in a Complex Space Form

Sadahiro MAEDA and Seiichi UDAGAWA

Kumamoto Institute of Technology and Nihon University

(Communicated by K. Ogiue)

Dedicated to Professor Tadashi Nagano on his sixtieth birthday

§0. Introduction.

Let $f: M\to\tilde{M}$ be an isometric immersion of a connected complete Riemannian manifold M into a Riemannian manifold $\tilde{M}$. We call M *a circular geodesic* submanifold of $\tilde{M}$ provided that for every geodesic γ of M the curve $f\circ\gamma$ is a circle in $\tilde{M}$. The following problem is still open: Classify circular geodesic submanifolds M in a complex space form (for details, see [7]).

The purpose of this paper is to consider this problem in the case of $\dim M=2$.

§1. Preliminaries.

A Riemannian manifold of constant curvature is called a *real space form*. Let M be an n-dimensional submanifold of $\tilde{M}^{n+p}$ with the metric g. We denote by ∇ and $\tilde{\nabla}$ the covariant differentiations on M and $\tilde{M}$, respectively. Then, the second fundamental form σ of the immersion is defined by $\sigma(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y$, where X and Y are the vector fields tangent to M. We call $\mu=(1/n)(\text{trace } \sigma)$ the *mean curvature vector* of M in $\tilde{M}$. The *mean curvature* H of M in $\tilde{M}$ is the length of μ. If μ is identically zero, the submanifold is said to be *minimal*. The submanifold M is *totally umbilic* provided that $\sigma(X, Y)=g(X, Y)\mu$ for all vector fields X and Y on M. In particular, if σ vanishes identically, then M is said to be a *totally geodesic* submanifold of $\tilde{M}$. For a vector field ξ normal to M, we write $\tilde{\nabla}_X\xi=-A_\xi X+D_X\xi$, where $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. the normal) component of $\tilde{\nabla}_X\xi$. We call D

Received October 21, 1989

the normal connection on the normal bundle $T^{\perp}M$ of M. A normal vector field ξ is said to be *parallel* if $D_X\xi=0$ for each vector field X tangent to M. We define the covariant differentiation $\bar{\nabla}$ of the second fundamental form σ with respect to the connections in the tangent bundle and normal bundle as:

$$(\bar{\nabla}_X\sigma)(Y, Z)=D_X(\sigma(Y, Z))-\sigma(\nabla_X Y, Z)-\sigma(Y, \nabla_X Z) .$$

The second fundamental form σ is said to be *parallel* if $(\bar{\nabla}_X\sigma)(Y, Z)=0$ for all tangent vector fields X, Y and Z on M. The manifold M is said to be a (λ-)*isotropic* submanifold of $\tilde{M}$ provided that $\|\sigma(X, X)\|$ is equal to a constant $(=\lambda)$ for all unit tangent vectors X at each point. In particular, if the function λ is constant on M then the immersion is said to be (λ-)*constant isotropic*. A *planar geodesic* immersion is an isometric immersion such that every geodesic of M is locally contained in a 2-dimensional totally geodesic submanifold in $\tilde{M}$. We here explain the Frenet formula for a curve $x: I\to M$ parametrized by arc length t. Let $V_1=\dot{x}$ be the unit tangent vector and put $\lambda_1=\|\tilde{\nabla}_{\dot{x}}V_1\|$. If λ_1 vanishes on I, then x is said to be of order 1. If λ_1 is not identically zero, then we define V_2 by $\tilde{\nabla}_{\dot{x}}V_1=\lambda_1V_2$ on the set $I_1=\{t\in I: \lambda_1(t)\neq 0\}$. Put $\lambda_2=\|\tilde{\nabla}_{\dot{x}}V_2+\lambda_1V_1\|$. If $\lambda_2=0$ on I_1, then x is said to be of order 2 on I_1. If λ_2 is not identically zero on I_1, then we define V_3 by $\tilde{\nabla}_{\dot{x}}V_2=-\lambda_1V_1+\lambda_2V_3$ on the set $I_2=\{t\in I_1: \lambda_2(t)\neq 0\}$. Inductively, we put $\lambda_d=\|\tilde{\nabla}_{\dot{x}}V_d+\lambda_{d-1}V_{d-1}\|$ and if $\lambda_d=0$ on $I_{d-1}=\{t\in I_{d-2}: \lambda_{d-1}(t)\neq 0\}$, then x is said to be of order d on I_{d-1}. It follows that if the curve x is of order d, then we have a matrix equation on I_{d-1}

$$\tilde{\nabla}_{\dot{x}}(V_1, V_2, \cdots, V_d)=(V_1, V_2, \cdots, V_d)\Lambda , \tag{1.1}$$

where Λ is a (d, d)-matrix defined by

$$\Lambda=\begin{pmatrix} 0 & -\lambda_1 & & & & 0 \\ \lambda_1 & 0 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 0 & -\lambda_{d-1} \\ 0 & & & & \lambda_{d-1} & 0 \end{pmatrix}. \tag{1.2}$$

Equation (1.1) is known as the *Frenet formula*. When each λ_i $(1\leqq i\leqq d-1)$ is constant, the curve x is called a *helix* of order d. In particular, when $d=2$, the curve x is called a *circle*.

Now, let M be an oriented surface in a Kaehler manifold $\tilde{M}$ with the complex structure J. We define $\cos\theta = g(e_1, Je_2)$, where $\{e_1, e_2\}$ is a local field of orthonormal frames on M. We call θ the *Kaehler angle* of M in $\tilde{M}$. Let M be a Riemannian submanifold of a Kaehler manifold $\tilde{M}$ with the complex structure J. The submanifold M is called a *Kaehler* submanifold (resp. a *totally real* submanifold) of $\tilde{M}$ if each tangent space of M is mapped into the tangent space of M (resp. the normal space of M) by the complex structure J. A Kaehler manifold of constant holomorphic sectional curvature is called a *complex space form*. Let $\tilde{M}^N(c)$ be an N-dimensional complex space form (with complex structure J) of constant holomorphic sectional curvature c. Let M be an n-dimensional submanifold of $\tilde{M}^N(c)$. For later use, we write the following fundamental equations which are called the equations of Gauss and Codazzi, respectively:

$$
\begin{aligned}
(1.3)\quad & g(R(X,Y)Z,W) \\
&= (c/4)\{g(Y, Z)g(X,W) - g(X, Z)g(Y,W) \\
&\quad + g(JY, Z)g(JX,W) - g(JX, Z)g(JY,W) + 2g(X, JY)g(JZ,W)\} \\
&\quad + g(\sigma(Y, Z), \sigma(X,W)) - g(\sigma(X, Z), \sigma(Y,W))
\end{aligned}
$$

$$
\begin{aligned}
(1.4)\quad & (c/4)\{g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ\}^{\perp} \\
&= (\bar{\nabla}_X\sigma)(Y, Z) - (\bar{\nabla}_Y\sigma)(X, Z)\ ,
\end{aligned}
$$

where R is the curvature tensor of M and $\{*\}^{\perp}$ means the normal component of $\{*\}$.

Finally, we prepare the following without proof in order to prove our theorems:

PROPOSITION 1 ([5]). *Let M be a submanifold in a Riemannian manifold $\tilde{M}$. Then, the following two conditions are equivalent*:

(i) *The submanifold M is nonzero constant (λ-)isotropic and the second fundamental form σ of M in $\tilde{M}$ satisfies $(\bar{\nabla}_X\sigma)(X, X) = 0$ for all vector fields X tangent to M.*

(ii) *M is a circular geodesic submanifold of $\tilde{M}$.*

PROPOSITION 2 ([5]). *Let M be a submanifold in a complex space form $\tilde{M}(c)$ of constant holomorphic sectional curvature c with the complex structure J. Then, the following are equivalent*:

(i) *$(\bar{\nabla}_X\sigma)(X, X) = 0$ for all vector fields X tangent to M.*

(ii) *$(\bar{\nabla}_X\sigma)(Y, Z) = (c/4)\{g(X, JY)JZ + g(X, JZ)JY\}^{\perp}$ for all vector fields X, Y and Z tangent to M.*

§ 2. Results.

First of all, we prove the following:

THEOREM 1. *Let M be a circular geodesic surface in a complex space form $\tilde{M}(c)$ with $c\neq 0$. If the Kaehler angle θ of M (in $\tilde{M}(c)$) is constant, then the second fundamental form of M is parallel.*

PROOF. We choose a local field of orthonormal frames e_1, e_2 around an arbitrary fixed point $p\in M$ in such a way that $\nabla e_1=\nabla e_2=0$ at p. Here and in the following we suppose that M is not a totally real surface in $\tilde{M}(c)$, that is, $g(e_1, Je_2)\neq 0$. Our aim here is to prove that the surface M must be Kaehler: From Proposition 1, we see that

$$g(\sigma(X, X), \sigma(X, X))=\lambda^2 g(X, X)g(X, X) \qquad \text{for any} \quad X\in TM\,,$$

which is equivalent to

$$\begin{aligned} &g(\sigma(X, Y), \sigma(Z, W))+g(\sigma(X, Z), \sigma(Y, W))+g(\sigma(X, W), \sigma(Y, Z)) \\ &\quad =\lambda^2(g(X, Y)g(Z, W)+g(X, Z)g(Y, W)+g(X, W)g(Y, Z)) \end{aligned}$$

for any $X, Y, Z, W\in TM$. Therefore, in particular, we have

$$g(\sigma(e_1, e_1), \sigma(e_2, e_2))+2g(\sigma(e_1, e_2), \sigma(e_1, e_2))=\lambda^2\,.$$

Since λ is constant, the following holds:

$$e_1(g(\sigma(e_1, e_1), \sigma(e_2, e_2)))+2e_1(g(\sigma(e_1, e_2), \sigma(e_1, e_2)))=0\,,$$

which, together with Proposition 2, yields

$$(c/2)g(e_1, Je_2)g(\sigma(e_1, e_1), Je_2)+c\cdot g(e_1, Je_2)g(Je_1, \sigma(e_1, e_2))=0$$

so that

$$g(\sigma(e_1, e_1), Je_2)+2g(\sigma(e_1, e_2), Je_1)=0 \qquad \text{at} \quad p\,. \tag{2.1}$$

On the other hand, from the hypothesis that the Kaehler angle θ is constant, we get

$$0=e_1(g(e_1, Je_2))=g(\sigma(e_1, e_1), Je_2)+g(e_1, J\sigma(e_1, e_2)) \qquad \text{at} \quad p\,,$$

that is,

$$g(\sigma(e_1, e_1), Je_2)-g(\sigma(e_1, e_2), Je_1)=0 \qquad \text{at} \quad p\,. \tag{2.2}$$

From (2.1) and (2.2), we find

$$g(\sigma(e_1, e_1), Je_2)=g(\sigma(e_1, e_2), Je_1)=0 \quad \text{at} \quad p\,. \tag{2.3}$$

Similarly, we obtain

$$g(\sigma(e_2, e_2), Je_1)=g(\sigma(e_1, e_2), Je_2)=0 \quad \text{at} \quad p\,. \tag{2.4}$$

Moreover, we have

$$\begin{aligned} 0=e_2\lambda^2 &= e_2(g(\sigma(e_1, e_1), \sigma(e_1, e_1))) \\ &= 2g((\bar{\nabla}_{e_2}\sigma)(e_1, e_1), \sigma(e_1, e_1)) \\ &= cg(e_2, Je_1)g(Je_1, \sigma(e_1, e_1)) \quad \text{at} \quad p\,, \end{aligned}$$

that is

$$g(\sigma(e_1, e_1), Je_1)=0 \quad \text{at} \quad p\,. \tag{2.5}$$

Similarly, we see that

$$g(\sigma(e_2, e_2), Je_2)=0 \quad \text{at} \quad p\,. \tag{2.6}$$

Hence the equations (2.3)~(2.6) yield the following

$$g(\sigma(X, Y), JZ)=0 \quad \text{for any} \quad X, Y, Z\in TM\,. \tag{2.7}$$

For simplicity, in the following we put $\sigma_{ij}=\sigma(e_i, e_j)$ $(1\leqq i, j\leqq 2)$ and $a=g(e_1, Je_2)$. Differentiating (2.7) with respect to $W\in TM$, we see that

$$\begin{aligned} 0 &= g(\tilde{\nabla}_W(\sigma(X, Y)), JZ)+g(\sigma(X, Y), J\tilde{\nabla}_W Z) \\ &= g(-A_{\sigma(X,Y)}W+D_W(\sigma(X, Y)), JZ)+g(\sigma(X, Y), J(\nabla_W Z+\sigma(W, Z)))\,. \end{aligned}$$

Here, again by (2.7) we find

$$-g(A_{\sigma(X,Y)}W, JZ)+g((\bar{\nabla}_W\sigma)(X, Y), JZ)+g(\sigma(X, Y), J\sigma(Z, W))=0 \tag{2.8}$$

for any $X, Y, Z, W\in TM$. Now, Setting $X=Y=Z=e_1$ and $W=e_2$ in (2.8), from Propositions 1 and 2 we have

$$g(\sigma_{11}, \sigma_{22})a+(c/2)(-a+a^3)+g(\sigma_{11}, J\sigma_{12})=0\,. \tag{2.9}$$

Similarly, from (2.8) we get the following:

$$g(\sigma_{11}, J\sigma_{22})=0\,, \tag{2.10}$$

$$\lambda^2 a+g(\sigma_{22}, J\sigma_{12})=0\,, \tag{2.11}$$

$$-\lambda^2 a+g(\sigma_{11}, J\sigma_{12})=0\,, \tag{2.12}$$

$$g(\sigma_{12}, \sigma_{12})a+(c/4)(a-a^3)-g(\sigma_{11}, J\sigma_{12})=0\,. \tag{2.13}$$

Thus, from (2.9)～(2.13) we obtain the following:

$$(2.14)\qquad \begin{cases} g(\sigma_{12}, \sigma_{12})=\lambda^2-(c/4)(1-a^2)\ , \\ g(\sigma_{11}, \sigma_{22})=(c/2)(1-a^2)-\lambda^2\ , \\ g(\sigma_{11}, J\sigma_{22})=0\ , \\ g(\sigma_{11}, J\sigma_{12})=\lambda^2 a\ , \\ g(\sigma_{22}, J\sigma_{12})=-\lambda^2 a\ . \end{cases}$$

Now, we denote by K the Gaussian curvature of the surface M. It follows from (2.14) and the Gauss equation (1.3) that

$$(2.15)\qquad K=c-2\lambda^2\ ,$$

which implies that K is constant. Next, we shall compute $\tilde{R}(e_1, e_2)\sigma_{11}$ and $\tilde{R}(e_1, e_2)\sigma_{12}$, where $\tilde{R}$ is the curvature tensor of $\tilde{M}(c)$. From Propositions 1, 2 and (2.14) we have

$$\begin{aligned} \tilde{\nabla}_{e_1}\sigma_{11} &= -A_{\sigma_{11}}e_1+D_{e_1}\sigma_{11} \\ &= -\lambda^2 e_1+(\bar{\nabla}_{e_1}\sigma)(e_1, e_1)+2\sigma(\nabla_{e_1}e_1, e_1) \\ &= -\lambda^2 e_1+2\sigma(\nabla_{e_1}e_1, e_1) \end{aligned}$$

so that

$$(2.16)\qquad \tilde{\nabla}_{e_2}(\tilde{\nabla}_{e_1}\sigma_{11})=-\lambda^2\sigma_{12}+2\sigma(\nabla_{e_2}(\nabla_{e_1}e_1), e_1) \quad \text{at}\ \ p\ .$$

Similarly, from Propositions 1, 2 and (2.14) we obtain

$$(2.17)\qquad \begin{aligned} &\tilde{\nabla}_{e_1}(\tilde{\nabla}_{e_2}\sigma_{11}) \\ &=(\lambda^2-(c/2))\sigma_{12}-(c/2)aJ\sigma_{11}+2\sigma(\nabla_{e_1}(\nabla_{e_2}e_1), e_1) \quad \text{at}\ \ p\ . \end{aligned}$$

Hence, from (2.16) and (2.17), we find

$$(2.18)\qquad \tilde{R}(e_1, e_2)\sigma_{11}=(6\lambda^2-(5c/2))\sigma_{12}-(c/2)aJ\sigma_{11}\ .$$

Similarly, we have

$$(2.19)\qquad \tilde{R}(e_1, e_2)\sigma_{12}=(5c/4-3\lambda^2)(\sigma_{11}-\sigma_{22})-(c/2)aJ\sigma_{12}\ .$$

On the other hand, since the curvature tensor $\tilde{R}$ of $\tilde{M}(c)$ has a nice form, we get the following:

$$(2.20)\qquad \tilde{R}(e_1, e_2)\sigma_{11}=(c/2)aJ\sigma_{11}\ ,$$

$$(2.21)\qquad \tilde{R}(e_1, e_2)\sigma_{12}=(c/2)aJ\sigma_{12}\ .$$

Therefore, the equations (2.18) and (2.20) yield

$$caJ\sigma_{11}=(6\lambda^2-(5/2)c)\sigma_{12}$$

so that

(2.22) $$cag(J\sigma_{11}, J\sigma_{11})=(6\lambda^2-(5/2)c)g(\sigma_{12}, J\sigma_{11})$$

which, combined with (2.14) and $a\neq 0$, implies that

(2.23) $$c=4\lambda^2 .$$

On the other hand, the equations (2.19) and (2.21) give

(2.24) $$caJ\sigma_{12}=((5/4)c-3\lambda^2)(\sigma_{11}-\sigma_{22})$$

so that

(2.25) $$cag(J\sigma_{12}, J\sigma_{12})=((5/4)c-3\lambda^2)g(\sigma_{11}-\sigma_{22}, J\sigma_{12})$$

which, together with (2.14) and $a\neq 0$, yields

(2.26) $$24\lambda^4-6c\lambda^2-c^2(1-a^2)=0 .$$

As an immediate consequence of (2.23), (2.26) and $c\neq 0$, we see that $a^2=1$, that is, the surface M is Kaehler. Namely, the surface M which satisfies the hypothesis of Theorem 1 must be Kaehler or totally real. Therefore, by virtue of the Codazzi equation (1.4) and Propositions 1, 2 we conclude that the second fundamental form of M is parallel. Q.E.D.

REMARK 1. Sakamoto ([12]) classified planar geodesic submanifolds in a real space form. Due to his work, we find that a planar geodesic submanifold M in a Euclidean sphere $S^n(k)$ of constant curvature k is locally congruent to one of compact symmetric spaces of rank one and the immersion is locally equivalent to the second or the first standard immersion according as M is a sphere or not (see also [13]). For later use, we give the examples of full planar geodesic surface M in a real hyperbolic space $\boldsymbol{RH}^n(c)$ of constant curvature c (<0): We denote by $M^n(k)$ an n-dimensional space form of constant curvature k.

EXAMPLE 1. $f_1: M=M^2(k)\xrightarrow[\text{totally umbilic}]{}\boldsymbol{RH}^3(c)\ , \qquad k>c$.

EXAMPLE 2. $f_2: M=S^2(k/3)\xrightarrow[\text{minimal}]{}S^4(k)\xrightarrow[\text{totally umbilic}]{}\boldsymbol{RH}^5(c)$.

REMARK 2. Naitoh ([8]) and Nomizu ([10]) classified circular geodesic submanifolds with parallel second fundamental form in a complex projective space $\boldsymbol{CP}^n(c)$ of constant holomorphic sectional curvature c. Due to their works, we find that the surface M (in $CP^n(c)$) which satisfies the hypothesis of Theorem 1 is locally congruent to one of the following examples (a)~(e):

We denote by $RP^n(k)$ an n-dimensional real projective space of constant curvature k.

(a) $$M=S^2(k) \xrightarrow[\text{totally umbilic}]{} RP^3(c/4) \xrightarrow[\text{totally geodesic}]{} CP^3(c)\ .$$

M is a totally real surface with constant mean curvature $H=\sqrt{k-(c/4)}$ ($\neq 0$) in $CP^3(c)$.

(b) $$M=S^2(k/3) \xrightarrow[\text{minimal}]{} S^4(k) \xrightarrow[\text{totally umbilic}]{} RP^5(c/4) \xrightarrow[\text{totally geodesic}]{} CP^5(c)\ .$$

M is a totally real surface with constant mean curvature $H=\sqrt{k-(c/4)}$ ($\neq 0$) in $CP^5(c)$.

(c) $$M=S^2(c/12) \xrightarrow[\text{minimal}]{} S^4(c/4) \xrightarrow[\text{covering map}]{} RP^4(c/4) \xrightarrow[\text{totally geodesic}]{} CP^4(c)\ .$$

M is a totally real minimal surface in $CP^4(c)$.

(d) $$\begin{array}{ccc} S^1(2/\sqrt{3c})\times S^1(2/\sqrt{3c})\times S^1(2/\sqrt{3c}) & \xrightarrow[\text{minimal}]{} & S^5(c/4) \\ \downarrow \pi & & \downarrow \pi \\ M=\pi(S^1(2/\sqrt{3c})\times S^1(2/\sqrt{3c})\times S^1(2/\sqrt{3c})) & \longrightarrow & CP^2(c)\ , \end{array}$$

where π is the Hopf fibration and $S^1(2/\sqrt{3c})$ is a circle with radius $2/\sqrt{3c}$. M is a totally real minimal surface in $CP^2(c)$. Note that M is a flat torus.

(e) $$\begin{array}{ccc} M=S^2(c/2)\ (=CP^1(c/2)) & \xrightarrow[f]{} & CP^2(c) \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ f:\ (Z_0,\ Z_1) & \longrightarrow & (Z_0^2,\ \sqrt{2}\,Z_0Z_1,\ Z_1^2)\ . \end{array}$$

Of course, the immersion f is Kaehler so that M is a minimal surface in $CP^2(c)$.

When the ambient space is a complex hyperbolic space $CH^n(c)$ of constant holomorphic sectional curvature c (<0), it follows from [9] that a surface M with parallel second fundamental form in $CH^n(c)$ is either of the following (i)~(iii):

(i) M is a totally real surface with parallel second fundamental form in $CH^2(c)$.

(ii) $M \xrightarrow[f]{} RH^s(c/4) \xrightarrow[\text{totally geodesic}]{} CH^n(c)$,

where f is a full immersion with parallel second fundamental form.

(iii) M is a complex curve in $CH^n(c)$.

Assume that M is a circular geodesic surface with parallel second fundamental form in $CH^n(c)$. In the case (i), M is minimal by Ejiri's result ([3]), which states that (λ-)isotropic totally real surface in $CH^2(c)$ is minimal. Then, by Ejiri's result ([4]), we see that M is totally geodesic in $CH^2(c)$. In the case (ii), M is a planar geodesic surface with parallel second fundamental form in $RH^s(c/4)$ for some $s \in N$. Then, we have the following two examples:

(f) $M = M^2(k) \xrightarrow[f_1]{} RH^3(c/4) \xrightarrow[\text{totally geodesic}]{} CH^3(c)$,

where $k > c/4$ and f_1 is given in Example 1.

(g) $M = S^2(k/3) \xrightarrow[f_2]{} RH^5(c/4) \xrightarrow[\text{totally geodesic}]{} CH^5(c)$,

where f_2 is given in Example 2.

In the case (iii), it is known that M is totally geodesic.

Therefore, we see that circular geodesic surface with constant Kaehler angle in a non-flat complex space form must be of constant curvature. Then, motivated by Remark 2, we now prove the following:

THEOREM 2. *Let M be a surface of constant curvature. Assume that M is a circular geodesic surface fully and isometrically immersed in a non-flat complex space form. Then, M is locally congruent to one of the examples* (a), (b), (c), (d), (e), (f) *and* (g).

PROOF. We denote by K the constant Gaussian curvature of M. It follows from the Gauss equation (1.3) that

$$K = (c/4)\{1 + 3(g(e_1, Je_2))^2\} - g(\sigma_{12}, \sigma_{12}) + g(\sigma_{11}, \sigma_{22}) . \tag{2.27}$$

Differentiating (2.27) with respect to e_1, from Propositions 1 and 2, we get

$$\begin{aligned} e_1K &= (3c/4)2g(e_1, Je_2)\{g(\sigma_{11}, Je_2) + g(e_1, J\sigma_{12})\} \\ &\quad - 2g((\bar{\nabla}_{e_1}\sigma)(e_1, e_2), \sigma_{12}) + g(\sigma_{11}, (\bar{\nabla}_{e_1}\sigma)(e_2, e_2)) \\ &= (3c/2)g(e_1, Je_2)\{g(\sigma_{11}, Je_2) + g(e_1, J\sigma_{12})\} \\ &\quad - (c/2)g(e_1, Je_2)g(Je_1, \sigma_{12}) + (c/2)g(e_1, Je_2)g(\sigma_{11}, Je_2) \\ &= 2cg(e_1, Je_2)\{g(\sigma_{11}, Je_2) + g(e_1, J\sigma_{12})\} \\ &= c\{e_1((g(e_1, Je_2))^2)\} . \end{aligned}$$

Similarly, we have

$$e_2K=c\{e_2((g(e_1, Je_2))^2)\}\ .$$

Hence, the Kaehler angle θ of M is constant. Therefore, from Theorem 1 and Remark 2, we get our conclusion. Q.E.D.

REMARK 3. In the case where the ambient manifold M is a real space form, "circular geodesic" always implies "planar geodesic". However, in general this is not true. In fact, the example (d) is not planar geodesic (for details see [8]).

REMARK 4. The following examples are worth mentioning.

(A) For any non-negative integers n and k with $0\leqq k\leqq n$, there exists an $SU(2)$-equivariant minimal immersion $\psi_{n,k}: S^2(K)\to CP^n(c)$ such that $K=c/(2k(n-k)+n)$ and $\cos\theta=K(n-2k)/c$, where θ is the Kaehler angle of $S^2(K)$ (for details, see [1], [2] and [11]). We here note that $\psi_{n,k}$ is neither a Kaehler immersion nor a totally real immersion in the case where n is odd or n is even but $k\neq n/2$ with $0<k<n$ (so that $n\geqq 3$). Moreover, for any nonnegative integers n and k with $0\leqq k\leqq n$, for every geodesic γ of $S^2(K)$ the curve $\psi_{n,k}\circ\gamma$ is a helix of order n (see, Proposition 3.1 in [6]).

(B) For any nonnegative integers p and q, there exists an $SU(2)$-equivariant immersion (with constant mean curvature $H=\sqrt{pqc}/(p+q)$) $f^1_{p,q}: S^2(K)\to CP^N(c)$ with $N=pq+p+q$ such that $K=c/(p+q)$ and $\cos\theta=(p-q)/(p+q)$, where θ is the Kaehler angle of $S^2(K)$ (for details, see [6]). We here note that $f^1_{p,q}$ is neither a Kaehler immersion nor a totally real immersion in the case where $pq\neq 0$ and $p\neq q$. Moreover, for any nonnegative integers p and q and for every geodesic γ of $S^2(K)$, the curve $f^1_{p,q}\circ\gamma$ is a helix of order $(p+q)$ (see, Theorem 3.1 in [6]).

References

[1] S. BANDO and Y. OHNITA, Minimal 2-spheres with constant curvature in $P_n(C)$, J. Math. Soc. Japan, **39** (1987), 477–487.

[2] J. BOLTON, G. R. JENSEN, M. RIGOLI and L. M. WOODWARD, On conformal minimal immersion of S^2 into CP^n, Math. Ann., **279** (1988), 599–620.

[3] N. EJIRI, Totally real isotropic submanifolds in a complex projective space, preprint (unpublished).

[4] N. EJIRI, Totally real minimal immersions of n-dimensional real space forms into n-dimensional complex space forms, Proc. Amer. Math. Soc., **84** (1982), 243–246.

[5] S. MAEDA and N. SATO, On submanifolds all of whose geodesics are circles in a complex space form, Kodai Math. J., **6** (1983), 157–166.

[6] S. MAEDA and Y. OHNITA, Helical geodesic immersions into complex space forms, Geom. Dedicata, **30** (1989), 93–114.

[7] S. MAEDA, Differential geometry of constant mean curvature submanifolds, Mem. Fac. Gen. Ed. Kumamoto Univ. Nat. Sci., **24** (1989), 7-39.

[8] H. NAITOH, Isotropic submanifolds with parallel second fundamental form in $P^n(C)$, Osaka J. Math., **18** (1981), 427-464.

[9] H. NAITOH, Parallel submanifolds of complex space forms I, Nagoya Math. J., **90** (1983), 85-117.

[10] K. NOMIZU, A characterization of the Veronese varieties, Nagoya Math. J., **60** (1976), 181-188.

[11] Y. OHNITA, Minimal surfaces with constant curvature and Kaehler angle in complex space forms, Tsukuba J. Math., **13** (1989), 191-207.

[12] K. SAKAMOTO, Planar geodesic immersions, Tôhoku Math. J., **29** (1977), 25-56.

[13] M. TAKEUCHI, Parallel submanifolds of space forms, *Manifolds and Lie Groups, in Honor of Yozô Matsushima*, ed. by J. Hano et al., Birkhäuser, 1981, 429-447.

Present Address:

SADAHIRO MAEDA
DEPARTMENT OF MATHEMATICS, KUMAMOTO INSTITUTE OF TECHNOLOGY
IKEDA 4-22-1, KUMAMOTO 860, JAPAN

SEIICHI UDAGAWA
DEPARTMENT OF MATHEMATICS, SCHOOL OF MEDICINE, NIHON UNIVERSITY
ITABASHI, TOKYO 173, JAPAN