TOKYO J. MATH.
VOL. 14, NO. 1, 1991

Symmetry of θ_4-Curves

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(Communicated by S. Suzuki)

1. Introduction.

In [2], S. Kinoshita showed that there exists a knotted θ_3-curve in the 3-sphere S^3 such that its all cycles are unknotted. K. Wolcott proved that Kinoshita's θ_3-curve is not amphicheiral (see [7]). In [5], S. Suzuki showed that, for any integer m, there exists a knotted θ_m-curve in S^3 such that its all subgraphs are unknotted, and such knotted θ_m-curves in S^3 are called *almost unknotted.*

In this paper, we give infinitely many almost unknotted θ_4-curves in S^3, and determine their amphicheirality.

Let e_1, e_2, e_3 and e_4 be simple arcs in S^3 with common endpoints v_1, v_2 and mutually disjoint interiors. Then the union of these arcs is called a *θ_4-curve*. Two θ_4-curves θ and θ' are said to be *equivalent* (or *of the same knot type*), denoted by $\theta \cong \theta'$, if there exists an orientation preserving homeomorphism $f: S^3 \to S^3$ such that $f(\theta) = \theta'$. We call a θ_4-curve θ *unkotted* if there exists an embedded S^2 in S^3 with $S^2 \supset \theta$.

Let θ be a θ_4-curve. Let B_1 and B_2 be mutually disjoint regular neighborhoods of v_1 and v_2 in S^3 such that the pairs $(B_i, B_i \cap \theta)$ are as illustrated in Fig. 1 (a). Remove $(B_1, B_1 \cap \theta) \cup (B_2, B_2 \cap \theta)$ from (S^3, θ) and sew back trivial tangles (B_i, T_i) as illustrated in Fig. 1 (b) by some homeomorphisms

$$h_i : (\partial B_i, \partial T_i) \to (\partial B_i, \partial(B_i \cap \theta)),$$

then we obtain a link ℓ in S^3. Note that the link type of ℓ depends on attaching homeomorphisms h_i. By $L(\theta)$, we denote the set of all such knot and link types, and we set

$$K_n(\theta) = \{k \in L(\theta) \mid \mu(k) = 1, b(k) \leq n\},$$

where $\mu(k)$ and $b(k)$ are the number of components and the bridge index of k respectively.

THEOREM 1. *Let θ be a θ_4-curve. Then θ is unknotted if and only if $L(\theta) = \{k \in L(\theta) \mid b(k) \leq 2\}$.*

Received March 1, 1990

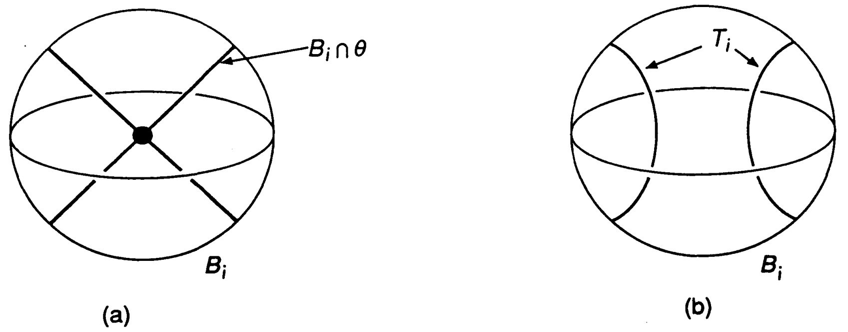
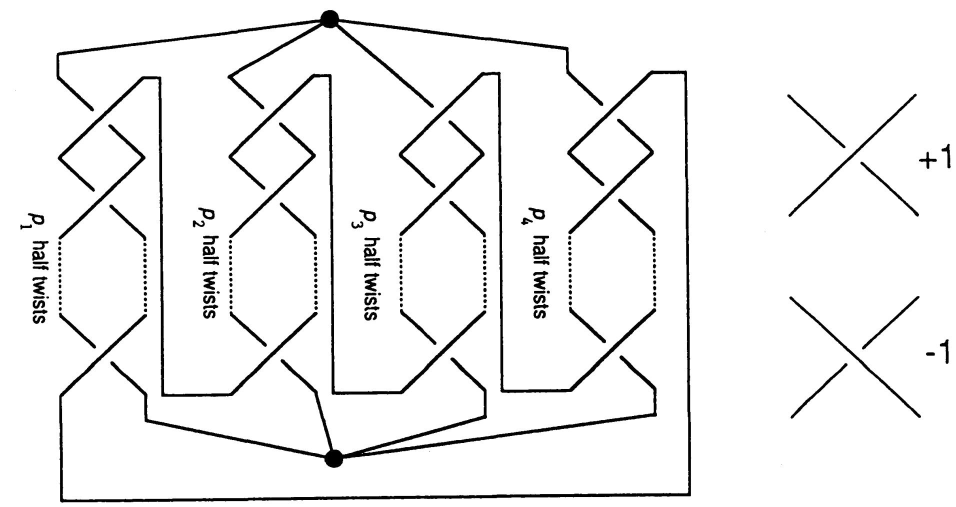


FIGURE 1



θ (p_1, p_2, p_3, p_4)

FIGURE 2

For integers p_1, p_2, p_3 and p_4, the θ_4-curve as shown in Fig. 2 is denoted by $\theta(p_1, p_2, p_3, p_4)$.

THEOREM 2. *Let p_1, p_2, p_3 and p_4 be integers such that $|p_i| \geq 2$ for $i=1, 2, 3, 4$. Then, $\theta(p_1, p_2, p_3, p_4)$ is knotted.*

If $p_1 = p_2 = p_3 = p_4 = 2$, then Theorem 2 is a special case of Suzuki's theorem (see [5]). In this paper, we determine amphicheirality of $\theta(p_1, p_2, p_3, p_4)$.

THEOREM 3. *Let p_1, p_2, p_3 and p_4 be even integers such that $|p_i| \geq 4$ for $i=1, 2, 3, 4$. Then the θ_4-curve $\theta(p_1, p_2, p_3, p_4)$ is amphicheiral if and only if p_1, p_2, p_3 and p_4 satisfy one of the following three conditions.*

(i) $p_1 = -p_2$ *and* $p_3 = -p_4$.

(ii) $p_1 = -p_3$ *and* $p_2 = -p_4$.

(iii) $p_1 = -p_4$ *and* $p_2 = -p_3$.

2. Proof of Theorem 1.

We call an incompressible torus T in a 3-manifold M *essential* if T is not boundary parallel in M.

LEMMA 4. *Let $\ell = k_1 \cup k_2$ be a two-component link in a lens space L. If all 3-manifolds which are obtained by Dehn surgeries along ℓ are lens spaces* (*allowing $S^2 \times S^1$ and S^3 both as a lens space*), *then the exterior of ℓ in L is homeomorphic to $T^2 \times I$.*

PROOF. Let V_1 and V_2 be mutually disjoint regular neighborhoods of k_1 and k_2 respectively. We set $\partial V_1 = T_1$, $\partial V_2 = T_2$ and $M = L - \text{int}(V_1 \cup V_2)$. If ℓ were a split link, then we could obtain non-prime manifolds by some Dehn surgeries along ℓ. Therefore, ℓ is non-splittable and M is irreducible and ∂-irreducible.

Since M is a Haken manifold with torus boundary, M admits a torus decomposition, that is, M contains (possibly empty) mutually disjoint and non-parallel, essential tori $U_1, U_2, \cdots, U_n$ such that, for the closure P (called a *piece*) of each component of $M - (U_1 \cup U_2 \cup \cdots \cup U_n)$, either P is Seifert fibered or int P is a (complete) hyperbolic 3-manifold of finite volume. Let P_1 be the piece containing T_1. By Hyperbolic Dehn Surgery Theorem (see [6, Theorem 5.9]) (resp. the definition of Seifert fibered manifolds), if int P_1 is hyperbolic (resp. P_1 is a Seifert fibered manifold not homeomorphic to $T^2 \times I$), then there exists a homeomorphism $f_1 : T_1 \to T_1$ such that $\text{int}(P_1 \cup_{f_1} V_1)$ is hyperbolic (resp. $P_1 \cup_{f_1} V_1$ is a Seifert fibered manifold with incompressible boundary). Therefore, if $P_1 \neq M$, then $M \cup_{f_1} V_1$ would contain an essential torus. By a similar argument, there would exist a homeomorphism $f_2 : T_2 \to T_2$ such that $(M \cup_{f_1} V_1) \cup_{f_2} V_2$ contains an incompressible torus. This contradicts that any lens space contains no incompressible tori. Hence, $M = P_1$, in other words either int M is hyperbolic or M is Seifert fibered. By Hyperbolic Dehn Surgery Theorem, if int M is hyperbolic, then we can obtain a hyperbolic manifold from M by some Dehn surgery. Therefore, M is Seifert fibered and its base space is either a disk, or an annulus, or a Möbius band. Since ∂M has two components, the base space is an annulus. If M contained exceptional fibers, then we could obtain a Seifert fibered manifold not homeomorphic to a lens space, from M by some Dehn surgery, a contradicton. Therefore, M is homeomorphic to $T^2 \times I$. □

PROOF OF THEOREM 1. Since the "only if" part is clear, we prove the "if" part. Let v_1 and v_2 be vertices of θ. We denote mutually disjoint regular neighborhoods of v_1 and v_2 by B_1 and B_2 respectively. Let ℓ be an element of $L(\theta)$. Since $b(\ell) \leq 2$, the double cover of S^3 branched over ℓ is a lens space L, and the preimage of B_i $(i = 1, 2)$ in L is a solid torus V_i. For any closed 3-manifold L' obtained by Dehn Surgery on $M = L - \text{int}(V_1 \cup V_2)$, there exists a link ℓ' in $L(\theta)$ whose double branched covering space is homeomorphic to L'. Since $b(\ell') \leq 2$, L' is a lens space. By Lemma 4, we have $M \cong T^2 \times I$. Hence, θ is unknotted. □

3. Knots obtained from $\theta(p_1, p_2, p_3, p_4)$.

PROOF OF THEOREM 2. If $\theta(p_1, p_2, p_3, p_4)$ is unknotted, then $L(\theta(p_1, p_2, p_3, p_4))$ is the set of all two-bridge links and trivial knots. But $L(\theta(p_1, p_2, p_3, p_4))$ contains the three-bridge link as shown in Fig. 3. □

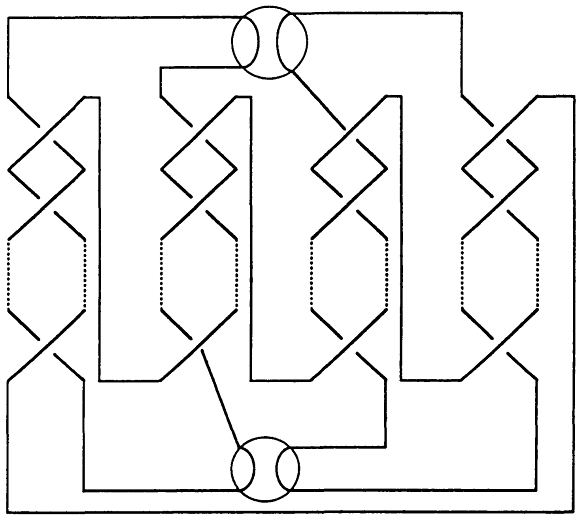


FIGURE 3

The two-bridge knot whose double cover is a lens space $L(s, t)$ is denoted by $C_{t/s}$. For integers $a_1, a_2, \cdots, a_n$, we set

$$[a_1, a_2, \cdots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

and set $C_{t/s} = C[a_1, a_2, \cdots, a_n]$ if $t/s = [a_1, a_2, \cdots, a_n]$.

PROPOSITION 5. *Let p_1, p_2, p_3 and p_4 be even integers such that $|p_i| \geq 4$ for $i = 1$, 2, 3, 4. Then $K_2(\theta(p_1, p_2, p_3, p_4))$ is equal to the union $\mathscr{C}$ defined by*

$$\bigcup_{x \in \mathbf{Z}} \{C[p_1, -p_2, 2x+1, -p_4, p_3], C[p_2, -p_1, 2x+1, -p_3, p_4],$$
$$C[p_1, -p_4, 2x+1, -p_2, p_3], C[p_2, -p_3, 2x+1, -p_1, p_4]\}\,.$$

PROOF. Any element of $L(\theta(p_1, p_2, p_3, p_4))$ is the link which has the diagram as shown in Fig. 4, where α and β are rational numbers and R_γ is a rational tangle diagram of type γ. We denote this link by $\ell(\alpha, \beta; p_1, p_2, p_3, p_4)$. Let $M(\alpha, \beta; p_1, p_2, p_3, p_4)$ be the

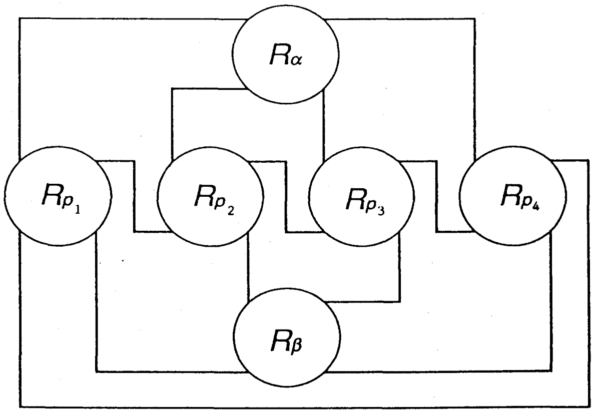


FIGURE 4

double cover of S^3 branched over $\ell(\alpha, \beta; p_1, p_2, p_3, p_4)$ and V_1 the preimage of R_{p_1} in $M(\alpha, \beta; p_1, p_2, p_3, p_4)$.

First we will show that $K_2(\theta(p_1, p_2, p_3, p_4)) \subset \mathscr{C}$. If $\ell(\alpha, \beta; p_1, p_2, p_3, p_4) \in K_2(\theta(p_1, p_2, p_3, p_4))$, then $M(\alpha, \beta; p_1, p_2, p_3, p_4)$ is a lens space. Since $\ell(\alpha, \beta; 0, p_2, p_3, p_4)$ is two-bridge, $M(\alpha, \beta; 0, p_2, p_3, p_4)$ is also a lens space. The latter $M(\alpha, \beta; 0, p_2, p_3, p_4)$ is obtained from the former $M(\alpha, \beta; p_1, p_2, p_3, p_4)$ by a Dehn surgery along a core of V_1. By Cyclic Surgery Theorem in [1], the closure of $M(\alpha, \beta; p_1, p_2, p_3, p_4) - V_1$ is Seifert fibered and its base space is either a disk with at most two exceptional points or a Möbius band with at most one exceptional point. Therefore, $M(\alpha, \beta; 1/0, p_2, p_3, p_4)$ is either the connected sum of two lens spaces, or a Seifert fibered manifold whose base space is a 2-sphere with at most three exceptional points, or a Seifert fibered manifold whose base space is a projective plane with two exceptional points. In particuler, we have

(*) an incompressible separating torus in $M(\alpha, \beta; 1/0, p_2, p_3, p_4)$ bounds a twisted I-bundle over the Klein bottle.

Let S_1 and S_2 be spheres in S^3 which intersect the standard S^2 as shown in Fig. 5, and let T_1 and T_2 be the preimages in $M(\alpha, \beta; 1/0, p_2, p_3, p_4)$ of S_1 and S_2 respectively. Both T_1 and T_2 are separating tori in $M(\alpha, \beta; 1/0, p_2, p_3, p_4)$. We consider the closures of the components of $M(\alpha, \beta; 1/0, p_2, p_3, p_4) - T_1$, one of them contains the preimage of R_α, it is denoted by A, and the other is denoted by B. If α is an integer or $1/0$, then T_1 $(=\partial A)$ is compressible in A. If α is not an integer, then A is a Seifert fibered manifold such that it has two exceptional fibers and one of them is an exceptional fiber of index $|p_4|$. Then, in particular, A is ∂-irreducible and not homeomorphic to a twisted I-bundle over the Klein bottle. If $\beta = 0$, then T_1 $(=\partial B)$ is compressible in B. If $1/\beta$ is an integer, then B is a Seifert fibered manifold such that it has two exceptional fibers and one of them is an exceptional fiber of index $|p_3|$. If $1/\beta$ is not an integer, then B is a ∂-irreducible Haken manifold which contains a separating essential annulus. Therefore, if $\beta \neq 0$, then B is ∂-irreducible and not homeomorphic to a twisted I-bundle over the Klein bottle.

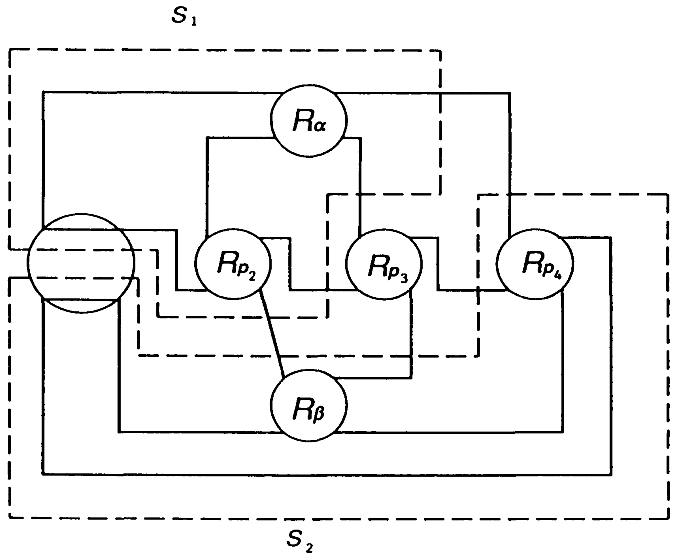


FIGURE 5

By $(*)$, if $\beta \neq 0$, then α is either an integer or 1/0. By the similar argument for T_2, if $\alpha \neq 0$, then β is either an integer or 1/0. Thus either at least one of α and β is equal to 0 or 1/0, or both α and β are non-zero integers.

If both α and β are non-zero integers, then by the similar argument for R_{p_2}, we have

$$|\alpha| = |\beta| = 1 .$$

Then $\ell(1, 1; p_1, p_2, p_3, p_4)$ is a two-component link. Therefore, at least one of α and β is equal to 0 or 1/0, and $\ell(\alpha, \beta; p_1, p_2, p_3, p_4)$ is a two bridge Montesinos knot.

If $\alpha = 0$, then $1/\beta$ must be an odd integer r and

$$\ell(0, 1/r, p_1, p_2, p_3, p_4) \cong C[p_1, -p_2, r, -p_4, p_3] .$$

If $\alpha = 1/0$, then β must be an odd integer r and

$$\ell(1/0, r, p_1, p_2, p_3, p_4) \cong C[p_2, -p_3, r, -p_1, p_4] .$$

If $\beta = 0$, then $1/\alpha$ must be an odd integer r and

$$\ell(1/r, 0, p_1, p_2, p_3, p_4) \cong C[p_2, -p_1, r, -p_3, p_4] .$$

If $\beta = 1/0$, then α must be an odd integer r and

$$\ell(r, 1/0, p_1, p_2, p_3, p_4) \cong C[p_1, -p_4, r, -p_2, p_3] .$$

Therefore, we have $K_2(\theta(p_1, p_2, p_3, p_4)) \subset \mathscr{C}$.

For any odd integer r, $\ell(\alpha, \beta, p_1, p_2, p_3, p_4)$ is an element of $K_2(\theta(p_1, p_2, p_3, p_4))$ if $\{\alpha, \beta\}$ is equal to $\{0, 1/r\}$ or $\{1/0, r\}$. Hence, $\mathscr{C}$ is contained in $K_2(\theta(p_1, p_2, p_3, p_4))$. □

Let k be a knot and ∇_k the Conway polynomial of k. When $\nabla_k(z)=\sum_{i=0}^{n} c_i z^i$ $(c_n \neq 0)$, we denote n, c_n and c_2 by $\deg k$, $a(k)$ and $\lambda(k)$ respectively.

LEMMA 6. *Let a_1, a_2, a_3 and a_4 be even integers and r an odd integer. Then*

(1)
$$\nabla_{C[a_1,a_2,r,a_2,a_3]} = T_{a_1+a_4+r}+\frac{a_2z}{2}T_{a_1}T_{a_4+r}+\frac{a_3z}{2}T_{a_4}T_{a_1+r}+\frac{a_2a_3z^2}{4}T_{a_1}T_{a_4}T_r,$$

where T_s is the Conway polynomial of a $(2, s)$-torus link oriented as shown in Fig. 6. Moreover

(2)
$$\lambda(C[a_1, a_2, r, a_3, a_4])=\frac{a_1a_2+a_3a_4}{4}+\frac{(a_1+a_4+r)^2-1}{8}.$$

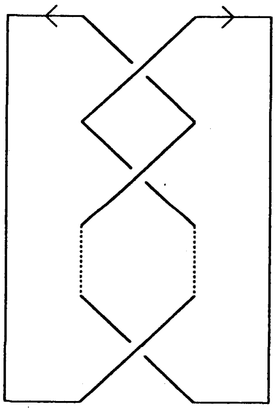


FIGURE 6

PROOF. Equation (1) is proved by induction on $|a_2|+|a_3|$. We prove only (2). For an odd integer s, let k_s be a $(2, s)$-torus knot, then we have

$$\lambda(k_s)=\frac{s^2-1}{8}.$$

For a two-component link ℓ, the coefficient of z of ∇_ℓ is equal to the linking number of ℓ. By (1), we have (2). □

COROLLARY 7. *Let a_1, a_2, a_3 and a_4 be even integers and r an odd integer. If $|a_i|\geq 4$ for $i=1, 2, 3, 4$ and $|a_1+a_4+r|=1$, then*

(3)
$$\deg(C[a_1, a_2, r, a_3, a_4])=|a_1|+|a_4|+|r|-1.$$

Moreover, $a(C[a_1, a_2, r, a_3, a_4])<0$ if and only if $a_2a_3<0$. □

THEOREM 8. *Let p_1, p_2, p_3 and p_4 be even integers such that $|p_i|\geq 4$ for $i=1, 2, 3, 4$ and $p_1p_2+p_3p_4\geq p_1p_4+p_2p_3$. The integer $p_1p_2+p_3p_4$ is a knot type invariant of*

$\theta(p_1, p_2, p_3, p_4)$, and the subset $\Lambda(\theta(p_1, p_2, p_3, p_4))$ of $K_2(\theta(p_1, p_2, p_3, p_4))$ defined by

$$\{C[p_1, -p_2, -p_1-p_3\pm 1, -p_4, p_3], C[p_2, -p_1, -p_2-p_4\pm 1, -p_3, p_4]\}$$

is also a knot type invariant of $\theta(p_1, p_2, p_3, p_4)$.

PROOF. By Proposition 8 and (2), we have

$$\min\{\lambda(k) \mid k\in K_2(\theta(p_1, p_2, p_3, p_4))\} = -\frac{p_1p_2+p_3p_4}{4}.$$

Therefore, $p_1p_2+p_3p_4$ is a knot type invariant of $\theta(p_1, p_2, p_3, p_4)$, and the subset of $K_2(\theta(p_1, p_2, p_3, p_4))$ defined by

$$\left\{k\in K_2(\theta(p_1, p_2, p_3, p_4)) \;\middle|\; \lambda(k) = -\frac{p_1p_2+p_3p_4}{4}\right\}$$

is also a knot type invariant of $\theta(p_1, p_2, p_3, p_4)$. By Proposition 5 and Lemma 6, we obtain

$$\Lambda(\theta(p_1, p_2, p_3, p_4)) = \left\{k\in K_2(\theta(p_1, p_2, p_3, p_4)) \;\middle|\; \lambda(k) = -\frac{p_1p_2+p_3p_4}{4}\right\}. \qquad \square$$

4. Proof of Theorem 3.

LEMMA 9. *Let p_1, p_2, p_3 and p_4 be even integers such that $|p_i|\geq 4$ for $i=1, 2, 3, 4$. Then at least one of the two knots $C[p_1, -p_2, -p_1-p_3\pm 1, -p_4, p_3]$ is not amphicheiral.*

PROOF. For an amphicheiral two-bridge knot k, the writhe of an alternating diagram of k is equal to zero (see [3] and [4]). Since at least one of $C[p_1, -p_2, -p_1-p_3\pm 1, -p_4, p_3]$ has no alternating diagram whose writhe is equal to zero, it is not amphicheiral. $\square$

PROOF OF THEOREM 3. Since $\theta(p_1, p_2, p_3, p_4)\cong\theta(p_2, p_3, p_4, p_1)\cong\theta(p_4, p_3, p_2, p_1)$, the "if" part is clear. We prove the "only if" part. Since $\theta(p_1, p_2, p_3, p_4)\cong\theta(p_2, p_3, p_4, p_1)$, we may assume that $p_1p_2+p_3p_4\geq p_1p_4+p_2p_3$ and $p_1>0$. By Theorem 8, if $\theta(p_1, p_2, p_3, p_4)$ is amphicheiral, then

$$\Lambda(\theta(p_1, p_2, p_3, p_4)) = \Lambda(\theta(-p_1, -p_2, -p_3, -p_4)). \tag{4}$$

By Lemma 9, for $\varepsilon=\pm 1$, $k=C[p_1, -p_2, -p_1-p_3+\varepsilon, -p_4, p_3]$ is not amphicheiral. By (4), k is equivalent to one of three knots in $\Lambda(\theta(-p_1, -p_2, -p_3, -p_4))$:

$$k_0=C[-p_1, p_2, p_1+p_3+\varepsilon, p_4, -p_3],$$

$$k_\varepsilon=C[-p_2, p_1, p_2+p_4+\varepsilon, p_3, -p_4],$$

$$k_{-\varepsilon}=C[-p_2, p_1, p_2+p_4-\varepsilon, p_3, -p_4].$$

We need consider following three cases.

Case 1. $k \cong k_0$. By (3), we have

$$\deg k=|p_1|+|p_3|+|p_1+p_3-\varepsilon|-1,$$

$$\deg k_0=|p_1|+|p_3|+|p_1+p_3+\varepsilon|-1.$$

Since $\deg k=\deg k_0$, it follows that $p_1+p_3=0$. By (1), we have

$$\nabla_{k_0}-\nabla_k=\frac{z}{2}T_{p_1}(T_{p_1+\varepsilon}+T_{p_1-\varepsilon})(p_2+p_4)=0.$$

Therefore p_1, p_2, p_3 and p_4 satisfy that $p_1=-p_3$ and $p_2=-p_4$.

Case 2. $k \cong k_\varepsilon$. If $p_3>0$, then by Corollary 7, p_2 and p_4 have the same sign. Since $\deg k=\deg k_\varepsilon$, by (3), it follows that

$$2(p_1+p_3)-\varepsilon=\pm(2(p_2+p_4)+\varepsilon).$$

Since p_i is even for $i=1, 2, 3, 4$, we have

(5) $$p_1+p_3=-p_2-p_4.$$

Since $a(k)=a(k_\varepsilon)$, it follows that

(6) $$p_2p_4=p_1p_3.$$

By (5) and (6), p_1, p_2, p_3 and p_4 satisfy that either

$$p_1=-p_2 \quad \text{and} \quad p_3=-p_4, \quad \text{or}$$

$$p_1=-p_4 \quad \text{and} \quad p_2=-p_3.$$

If $p_3<0$, then by Corollary 7, $p_2p_4<0$. Since $p_1p_2+p_3p_4 \geq p_1p_4+p_2p_3$, it follows that $p_2>0$ and $p_4<0$. By $\deg k=\deg k_\varepsilon$ and $a(k)=a(k_\varepsilon)$, we have either

$$\begin{cases} 2p_1-\varepsilon &=-2p_4-\varepsilon, \\ p_2p_4-2p_2 &=p_1p_3+2p_3, \end{cases} \quad \text{or}$$

$$\begin{cases} 2p_3+\varepsilon &=-2p_2+\varepsilon, \\ p_2p_4-2p_4 &=p_1p_3+2p_1. \end{cases}$$

Therefore we obtain

$$p_1=-p_4 \quad \text{and} \quad p_2=-p_3.$$

Case 3. $k \cong k_{-\varepsilon}$. If $p_3>0$, then by th argument similar to that in Case 2, we have either

$$p_1=p_2 \quad \text{and} \quad p_3=p_4, \quad \text{or}$$

$$p_1=p_4 \quad \text{and} \quad p_2=p_3.$$

In both cases, k is amphicheiral. This is a contradiction.

If $p_3<0$, then by the argument similar to that in Case 2, we have either

$$p_1=p_2, \quad p_4=p_3+4, \quad p_1>0 \quad \text{and} \quad p_4<0, \quad \text{or}$$

$$p_3=p_4, \quad p_2=p_1+4, \quad p_1>0 \quad \text{and} \quad p_3<0.$$

In both cases, k and $k_{-\varepsilon}$ have alternating diagrams such that the difference between their crossing numbers is four. Then Theorem A in Murasugi [3] implies that $k\not\cong k_{-\varepsilon}$, a contradiction. Thus Case 3 can not occur. □

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