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A Local Ring Is CM If and Only If Its Residue Field Has a CM Syzygy

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Introduction.

Some homological properties of a Noetherian local ring R can be characterized in terms of syzygies of the residue field k of R. For example, let

$$F: \quad \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow k \longrightarrow 0$$

be a resolution of k consisting of finite R-free modules, $\Omega_i = \mathrm{Ker}(F_{i-1} \to F_{i-2})$ an i-th syzygy of k where $F_{-1} = k$, $F_{-2} = 0$, and set $n = \dim R$. Then the following facts are well-known.

R is regular,
$\Leftrightarrow$ There exists an integer $i \geq n$, such that Ω_i is R-free,
$\Leftrightarrow$ For all $i \geq n$, Ω_i is R-free.

R is a Gorenstein ring,
$\Leftrightarrow$ There exists an integer $i \geq n$, such that $\mathrm{Ext}^1_R(\Omega_i, R) = 0$,
$\Leftrightarrow$ For all $i \geq n$, $\mathrm{Ext}^1_R(\Omega_i, R) = 0$.
(Because $\mathrm{Ext}^1_R(\Omega_i, R) = \mathrm{Ext}^{i+1}_R(k, R)$.)

In this paper, we consider the case of the Cohen-Macaulay (abbreviated to CM) property. Our main results are the following.

THEOREM 9. *For all* $i > 0$, $\mathrm{Supp}\, \Omega_i = \mathrm{Spec}\, R$ *and* $\dim \Omega_i = n$ *unless* $\Omega_i = 0$.

THEOREM 11. $\operatorname{depth} \Omega_i = \begin{cases} i & (\text{if } 0 \leq i \leq \operatorname{depth} R), \\ \operatorname{depth} R & (\text{if } i > \operatorname{depth} R \text{ and } \Omega_i \neq 0). \end{cases}$

COROLLARY 12.
(i) $\Omega_1, \cdots, \Omega_{n-1}$ *are not CM.*
(ii) *R is CM,*
$\Leftrightarrow$ *There exists an integer $i \geq n$, such that Ω_i is CM,*

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$\Leftrightarrow$ *For all* $i \geq n$, Ω_i *is CM unless* $\Omega_i = 0$.

To prove these theorems, we shall study $\dim \Omega_i$ in section 1 and depth Ω_i in section 2. Moreover, it is known that Ω_i's are Buchsbaum modules in more general situation than when R is CM. In section 3, we shall consider relationship between I (Ω_i) and I (R).

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§1. Dimension of syzygies and betti numbers.

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$, and let $k = R/\mathfrak{m}$. For a resolution F of an R-module M, always set $\Omega_i = \mathrm{Ker}(F_{i-1} \to F_{i-2})$ $(i \geq 0)$ where $F_{-1} = M$, $F_{-2} = 0$. The following facts are well-known.

PROPOSITION 1 (cf. e.g. [3]). *Let M be a finite R-module.*

(i) *There exists a minimal free resolution F of M* (*abbreviated to MFR of M*), *and F is unique up to isomorphisms of complexes.*

(ii) *For any resolution F of M consisting of finite R-free modules* (*abbreviated to FR of M*), *there exist an MFR G of M and an exact complex H consisting of finite R-free modules, such that* $F = G \oplus H$.

COROLLARY 2. *In the notation of Proposition* 1 (ii), *put*

$$\Omega_i = \mathrm{Ker}(F_{i-1} \to F_{i-2}), \quad \Omega'_i = \mathrm{Ker}(G_{i-1} \to G_{i-2}) \qquad (i \geq 0),$$

then

$$\Omega_i = \Omega'_i \oplus R^{t_i} \qquad (t_i \geq 0).$$

In particular,

$$\dim \Omega_i = \begin{cases} \dim \Omega'_i & (\text{if } t_i = 0) \\ \dim R & (\text{if } t_i > 0), \end{cases}$$

$$\operatorname{depth} \Omega_i = \begin{cases} \operatorname{depth} \Omega'_i & (\text{if } t_i = 0) \\ \inf(\operatorname{depth} \Omega'_i, \operatorname{depth} R) & (\text{if } t_i > 0). \end{cases}$$

PROOF. Since finite projective modules of local rings are free, the exact complex H of Proposition 1 (ii) is split and

$$\Omega_i = Z_{i+1}(F) = Z_{i+1}(G) \oplus Z_{i+1}(H) = \Omega'_i \oplus R^{t_i}$$ ■

By Corollary 2, our problems are reduced to the case of syzygies of an MFR of k.

Next, we consider dimensions of syzygies. Let β_i be the i-th betti number of k, i.e. $\beta_i = \dim_k \mathrm{Tor}_i^R(k, k) =$ (rank of i-th module of an MFR of k). The betti numbers $\beta_0, \beta_1, \cdots$

play important roles below.

LEMMA 3 (K. Yoshida). *Let F be an MRF of k.*

(i) *If $R \neq k$, then $\sum_{i=0}^{r}(-1)^{r-i}\beta_i \geq 0$ for all $r \geq 0$.*

(ii) *If* $\dim R \geq 1$, *then for all $r \geq 0$ the following conditions are equivalent.*

(a) $\Omega_{r+1} \neq 0$ *and* $\dim \Omega_{r+1} = \dim R$,

(b) $\sum_{i=0}^{r}(-1)^{r-i}\beta_i > 0$.

In this case, $\operatorname{Supp} \Omega_{r+1} = \operatorname{Spec} R$.

PROOF. (i) If $\dim R = 0$, then the exact sequence

$$0 \longrightarrow \Omega_{r+1} \longrightarrow F_r \longrightarrow \cdots \longrightarrow F_0 \longrightarrow k \longrightarrow 0$$

implies that

$$(-1)^r + \text{length}\, \Omega_{r+1} = \sum_{i=0}^{r} (-1)^{r-i} \text{length}\, F_i = \left(\sum_{i=0}^{r} (-1)^{r-i}\beta_i \right) \text{length}\, R\,.$$

Since R is not regular, $\Omega_{r+1} \neq 0$. Thus $\sum_{i=0}^{r}(-1)^{r-i}\beta_i \geq 0$.

If $\dim R \geq 1$, for any prime ideal $\mathfrak{p} \neq \mathfrak{m}$ of R we have an exact sequence

$$0 \longrightarrow (\Omega_{r+1})_{\mathfrak{p}} \longrightarrow (F_r)_{\mathfrak{p}} \longrightarrow \cdots \longrightarrow (F_0)_{\mathfrak{p}} \longrightarrow 0\,.$$

Since this sequence is split, $(\Omega_{r+1})_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-free. Thus

$$\operatorname{rank}(\Omega_{r+1})_{\mathfrak{p}} = \sum_{i=0}^{r} (-1)^{r-i}\beta_i \geq 0\,. \tag{$*$}$$

(ii) If $\sum_{i=0}^{r}(-1)^{r-i}\beta_i > 0$, then the formula ($*$) implies $\mathfrak{p} \in \operatorname{Supp} \Omega_{r+1}$ for all prime ideal $\mathfrak{p} \neq \mathfrak{m}$. Thus $\operatorname{Supp} \Omega_{r+1} = \operatorname{Spec} R$ and $\dim \Omega_{r+1} = \dim R$.

Conversely, if $\dim \Omega_{r+1} = \dim R \geq 1$, then the formula ($*$) also implies $\sum_{i=0}^{r}(-1)^{r-i}\beta_i > 0$. ∎

EXAMPLE 4. Let R be a regular local ring of dimension $n \geq 1$, and $x_1, \cdots, x_n$ a minimal basis of the maximal ideal $\mathfrak{m}$. Since the Koszul complex $F = K(x_1, \cdots, x_n)$ is an MFR of $k = R/\mathfrak{m}$, we have $\beta_r = \binom{n}{r}$ where $\binom{n}{r} = 0$ for $r > n$. By induction on r, $\sum_{i=0}^{r}(-1)^{r-i}\beta_i = \binom{n-1}{r}$. By the previous lemma, $\dim \Omega_r = n$ for $0 < r \leq n$.

But we can also prove this result straightforwardly. Assume R is an integral domain, or more generally, assume $\dim R/\mathfrak{p} = \dim R$ for any $\mathfrak{p} \in \operatorname{Ass} R$. Since every syzygy Ω_r is a submodule of the R-free module F_{r-1}, we get $\operatorname{Ass} \Omega_r \subset \operatorname{Ass} F_{r-1} = \operatorname{Ass} R$, hence $\dim \Omega_r = \dim R$, unless $\Omega_r = 0$.

Conversely, if a local ring R has a prime $\mathfrak{p} \in \operatorname{Ass} R$ such that $\dim R/\mathfrak{p} < \dim R$, then there exists a non-zero syzygy of a *finite module* which has dimension less than $\dim R$; for example, the second syzygy $\operatorname{Hom}_R(R/x, R) = \operatorname{Ann}_R(x)$ of $R/(x)$ where x is an element of $\mathfrak{p}$ not lying in any minimal prime ideal $\mathfrak{q}$ with $\dim R/\mathfrak{q} = \dim R$.

Next, we shall calculate lower bounds of β_i and of $\sum_{i=0}^{r}(-1)^{r-i}\beta_i$, using the following theorem.

THEOREM 5 [1, T. H. Guliksen and G. Levin]. (i) *There exist integers* $\varepsilon_0, \varepsilon_1, \cdots \geq 0$ *satisfying the following equation of formal power series in* t:

$$\sum_{i\geq 0} \beta_i t^i = \prod_{i\geq 0} \frac{(1+t^{2i+1})^{\varepsilon_{2i}}}{(1-t^{2i+2})^{\varepsilon_{2i+1}}} .$$

In particular, ε_i*'s are uniquely determined by* R, *and*

$$\varepsilon_0 = \dim_k \mathfrak{m}/\mathfrak{m}^2 = (\textit{embedding dimension of } \mathfrak{m}) ,$$

$$\varepsilon_1 = \dim_k H_1(E) ,$$

where E *is a Koszul complex for a minimal basis of* $\mathfrak{m}$.

(ii) *The following conditions are equivalent.*

(a) R *is a complete intersection ring,*

(b) $\varepsilon_2 = 0$,

(c) $\varepsilon_3 = 0$,

(d) $\varepsilon_i = 0$, *for all* $i \geq 2$.

REMARK (cf. e.g. [2, §21]). In the notation above, it is well-known that

$$\varepsilon_1 \geq \varepsilon_0 - \dim R \geq 0 ,$$

$$\varepsilon_1 = 0 \Longleftrightarrow E \text{ is exact} \Longleftrightarrow R \text{ is regular} .$$

EXAMPLE 6. Let k be a field, and

$$R = \frac{k[\![X_1, \cdots, X_u, Y_1, \cdots, Y_v]\!]}{(X_1^2, \cdots, X_u^2)} .$$

It is clear that R is a complete intersection ring with $\varepsilon_0 = u+v$, $\varepsilon_1 = u$. We construct an MFR of residue field k_R of R, and calculate the betti numbers directly. Set

$$S_i = k[\![X_i]\!]/(X_i^2) , \quad T_j = k[\![Y_j]\!], \quad \text{and} \quad R' = \left(S_1 \underset{k}{\otimes} \cdots \underset{k}{\otimes} S_u \underset{k}{\otimes} T_1 \underset{k}{\otimes} \cdots \underset{k}{\otimes} T_v \right) .$$

Then, $R' \subset R$ and $\hat{R}' = R$, where $\hat{R}'$ is $(X_1, \cdots, X_u, Y_1, \cdots, Y_v)R'$-adic completion of R'. Take MFR's of residue fields of S_i and T_j

$$G^{(i)}: \quad \cdots \longrightarrow S_i g_i^2 \xrightarrow{X_i} S_i g_i^1 \xrightarrow{X_i} S_i g_i^0 \longrightarrow k \longrightarrow 0 ,$$

$$F^{(j)}: \quad 0 \longrightarrow T_j h_j^1 \xrightarrow{Y_j} T_j h_j^0 \longrightarrow k \longrightarrow 0 ,$$

where g_i^l, h_j^m are free bases. Then $F = (G^{(1)} \otimes_k \cdots \otimes_k G^{(u)} \otimes_k H^{(1)} \otimes_k \cdots \otimes_k H^{(v)})\hat{}$ is an MFR of residue field k_R of R. In fact, by Künneth formula, we have

$$\mathrm{H}_l(F)=\begin{cases} k & (l=0)\,, \\ 0 & (l>0)\,. \end{cases}$$

Moreover, since

$$F_l=\bigoplus_{\substack{l=i_1+\cdots+i_u+j_1+\cdots+j_v \\ 0\le i_1,\cdots,i_u \\ 0\le j_1,\cdots,j_v\le 1}} Rg_1^{i_1}\cdots g_u^{i_u}h_1^{j_1}\cdots h_v^{j_v}\,,$$

the betti numbers β_i of k_R are determined by the following equation:

$$\begin{aligned}\sum_{i\ge 0}\beta_i t^i &= (1+t+t^2+\cdots)^u(1+t)^v \\ &= (1+t^2+t^4+\cdots)^u(1+t)^{u+v}=\frac{(1+t)^{\varepsilon_0}}{(1-t^2)^{\varepsilon_1}}\,.\end{aligned}$$

J. Tate has proved in [4] that if R is not regular, then for all $r\ge 0$,

$$\beta_r\ge\binom{\varepsilon_0}{r}+\binom{\varepsilon_0}{r-2}+\binom{\varepsilon_0}{r-4}+\cdots\,.$$

We calculate a slightly improved lower bound of β_r and give a condition that $\{\beta_r\}$ is bounded above.

PROPOSITION 7. (i) *If* $\varepsilon_1=0$, *then*

$$\beta_r=\begin{cases}\dbinom{\varepsilon_0}{r} & (\textit{if } r\le\varepsilon_0)\,, \\ 0 & (\textit{if } r>\varepsilon_0)\,. \end{cases}$$

(ii) *If* $\varepsilon_1=1$, *then R is a complete intersection ring,* $\varepsilon_0=\dim R+1$, *and*

$$\beta_r=\begin{cases}\sum_{i=0}^r\dbinom{\varepsilon_0-1}{i} & (\textit{if } r<\varepsilon_0)\,, \\ 2^{\varepsilon_0-1} & (\textit{if } r\ge\varepsilon_0)\,. \end{cases}$$

(iii) *If* $\varepsilon_1\ge 2$, *then* $\varepsilon_0\ge 2$ *and*

$$\beta_r\ge\sum_{i=0}^r(r-i+1)\binom{\varepsilon_0-2}{i}\,,$$

where $\dbinom{\varepsilon_0-2}{i}=0$ *for* $i>\varepsilon_0-2$.

In particular, the sequence $\{\beta_r\}$ *is bounded above (i.e. there exists an integer N such that for all* $i\ge 0$, $\beta_i\le N$), *if and only if* $\varepsilon_1\le 1$.

PROOF. (i) *If* $\varepsilon_0=0$, then R is regular, and then assertion is clear.

(ii) If $\varepsilon_1=1$, then R is not regular, and $1=\varepsilon_1\geq\varepsilon_0-\dim R>0$. Thus $\varepsilon_0=\dim R+1$, and R is a complete intersection ring. By Theorem 5, we have $\varepsilon_i=0$ for all $i\geq 2$, and

$$\sum_{i\geq 0}\beta_i t^i=\frac{(1+t)^{\varepsilon_0}}{1-t^2}=(1+t)^{\varepsilon_0-1}(1+t+t^2+\cdots)\,.$$

Comparing coefficients in t^r, we have the desired equation.

(iii) Assume $\varepsilon_1\geq 2$, so that R is not regular. We first show $\varepsilon_0\geq 2$. If $\varepsilon_0=0$, then R would be regular. If $\varepsilon_0=1$, then there exists an element $x\in R$ such that $\mathfrak{m}=xR$. Since $\varepsilon_0>\dim R$, R is Artinian. Thus there exists $t\geq 0$ such that $x^t\neq 0$ and $x^{t+1}=0$. Take the Koszul complex

$$E\colon 0\longrightarrow R\xrightarrow{\ x\ }R\longrightarrow 0\,.$$

Then $\mathrm{H}_1(E)=\mathrm{Ker}(R\xrightarrow{x}R)=Rx^t\cong k$. Thus $\varepsilon_1=\dim_k\mathrm{H}_1(E)=1$. This contradicts our assumption. Therefore, $\varepsilon_0\geq 2$.

By Theorem 5, we have

$$\begin{aligned}\sum_{i\geq 0}\beta_i t^i&=\frac{(1+t)^{\varepsilon_0}}{(1-t^2)^2}(1+a_1t+a_2t^2+\cdots)\qquad(a_i\geq 0)\\&\gg\frac{(1+t)^{\varepsilon_0}}{(1-t^2)^2}=(1+t+t^2+\cdots)^2(1+t)^{\varepsilon_0-2}\\&=(1+2t+3t^2+\cdots)(1+t)^{\varepsilon_0-2}\,,\end{aligned}$$

where $\sum_{i\geq 0}b_it^i\gg\sum_{i\geq 0}c_it^i$ means $b_i\geq c_i$ for all i. Comparing coefficients in t^r, we have the desired inequality. ∎

REMARK. In the notation above, we get

$$(1+a_1t+a_2t^2+\cdots)=\frac{(1+t^3)^{\varepsilon_2}\cdots}{(1-t^2)^{\varepsilon_1-2}\cdots}\,.$$

Thus, the equality of the formula of Proposition 7 (iii) holds for any r, if and only if $\varepsilon_1=2$ and $\varepsilon_i=0$ for any $i\geq 2$. This is equivalent to the condition that R is a complete intersection ring with $\varepsilon_1=2$, by Theorem 5.

Using technique similar to the above, we give a lower bound of $\sum_{i=0}^r(-1)^{r-i}\beta_i$.

PROPOSITION 8. *If R is not regular and $\varepsilon_0\geq 2$, then*

$$\sum_{i=0}^r(-1)^{r-i}\beta_i\geq\sum_{i=0}^r\binom{\varepsilon_0-2}{i}\qquad(r\geq 0)\,.$$

PROOF. Since R is not regular, $\varepsilon_1\geq 1$. By Theorem 5, we have

$$\sum_{i\geq 0}\beta_i t^i=\frac{(1+t)^2}{1-t^2}(1+a_1t+a_2t^2+\cdots)\qquad (a_i\geq 0)\,. \tag{$*$}$$

Since $(1+t)^2/(1-t^2)=(1+2t+2t^2+\cdots)$, it follows that

$$\begin{aligned}\sum_{i\geq 0}^{r}\beta_i t^i=&(1+2t+2t^2+\cdots+2t^r)\\ &+a_1(t+2t^2+\cdots+2t^r)\\ &\cdots\\ &+a_rt^r\,.\end{aligned}$$

Substituting -1 for t and multiplying both sides by $(-1)^r$, we obtain

$$\sum_{i\geq 0}^{r}(-1)^{r-i}\beta_i=1+a_1+a_2+\cdots+a_r\,.$$

Next we calculate lower bound of the right hand of the formula above. By the formula $(*)$, we have

$$\begin{aligned}(1+a_1t+a_2t^2+\cdots)&=\frac{(1+t)^{\varepsilon_0-2}(1+t^3)^{\varepsilon_2}\cdots}{(1-t^2)^{\varepsilon_1-1}(1-t^4)^{\varepsilon_3}\cdots}\\ &=(1+t)^{\varepsilon_0-2}(1+b_1t+b_2t^2+\cdots)\qquad (b_i\geq 0)\\ &\gg(1+t)^{\varepsilon_0-2}\,.\end{aligned}$$

Thus,

$$1+a_1t+a_2t^2+\cdots+a_rt^r\gg\sum_{i=0}^{r}\binom{\varepsilon_0-2}{i}t^i\,.$$

Substituting 1 for t, we get

$$1+a_1+a_2+\cdots+a_r\geq\sum_{i=0}^{r}\binom{\varepsilon_0-2}{i}.\qquad\blacksquare$$

REMARK. The equality of the formula of Proposition 8 holds for any r, if and only if $\varepsilon_1=1$ and $\varepsilon_i=0$ for any $i\geq 2$. By Theorem 5, this is equivalent to the condition that R is a complete intersection ring with $\varepsilon_1=1$.

THEOREM 9. *Let F be an FR of k, and $\Omega_{r+1}=\operatorname{Ker}(F_r\to F_{r-1})$, then for all $r>0$*

$$\dim\Omega_r=\dim R\quad\text{and}\quad\operatorname{Supp}\Omega_r=\operatorname{Spec}R\,,$$

unless $\Omega_r=0$.

PROOF. By Corollary 2, we may assume that F is an MFR of k. If $\dim R=0$, the theorem is trivial. If R is regular, the theorem follows from Example 4.

Assume that $\dim R \geq 1$ and R is not regular. Then $\varepsilon_0 > \dim R \geq 1$, and the theorem follows from Proposition 8 and Lemma 3. ■

§2. Depth of syzygies.

First of all, we shall study depth of syzygies of *finite R-modules*.

PROPOSITION 10. *Let M be a non-zero finite R-module, and F an FR of M. If $r = \operatorname{depth} M \leq \operatorname{depth} R = s$, then*

$$\operatorname{depth} \Omega_i \begin{cases} = r+i & (0 \leq i \leq s-r) & (1) \\ \geq s & (i = s-r+1) . & (2) \end{cases}$$

Moreover, if $\operatorname{proj.dim} M = \infty$, *then*

$$\operatorname{depth} \Omega_i = s \quad (i \geq s-r+2) \tag{3}$$

REMARK. Auslander and Buchsbaum have proved that if $\operatorname{proj.dim} M < \infty$, then $\operatorname{proj.dim} M = s-r$ in the notation above. In this case, for $i \geq s-r$, Ω_i is R-free and $\operatorname{depth} \Omega_i = s$ unless $\Omega_i = 0$.

PROOF. The exact sequence $0 \to \Omega_{i+1} \to F_i \to \Omega_i \to 0$ implies the following exact sequences:

$$0 \longrightarrow \operatorname{Ext}_R^{s-1}(k, \Omega_i) \longrightarrow \operatorname{Ext}_R^s(k, \Omega_{i+1}) ,$$

$$0 \longrightarrow \operatorname{Ext}_R^{j-1}(k, \Omega_i) \longrightarrow \operatorname{Ext}_R^j(k, \Omega_{i+1}) \longrightarrow 0 \qquad (j < s) .$$

Using induction on i, we can easily derive the formulas (1) and (2) from these sequences.

To prove the formula (3), we may assume that F is an MFR of k by Corollary 2. Since $\operatorname{depth} \Omega_{s-r} = s$ and $\operatorname{depth} \Omega_{s-r+1} \geq s$, it is enough to prove the following claim.

CLAIM. *If* $\operatorname{depth} \Omega_{i-1} \geq s$ *and* $\operatorname{depth} \Omega_i \geq s$, *then* $\operatorname{depth} \Omega_{i+1} = s$.

If $\operatorname{depth} \Omega_i > s$, then

$$0 \longrightarrow \operatorname{Ext}_R^s(k, \Omega_{i+1}) \longrightarrow \operatorname{Ext}_R^s(k, F_i) \longrightarrow 0 .$$

The desired equation follows from the sequences above.

Assume $\operatorname{depth} \Omega_i = s$. In the same way as in the proof of formula (2), we have $\operatorname{depth} \Omega_{i+1} \geq s$. If $\operatorname{depth} \Omega_{i+1} > s$, then

$$0 = \operatorname{Ext}_R^s(k, \Omega_{i+1}) \longrightarrow \operatorname{Ext}_R^s(k, F_i) \xrightarrow{f} \operatorname{Ext}_R^s(k, \Omega_i) ,$$

$$0 = \operatorname{Ext}_R^{s-1}(k, \Omega_{i-1}) \longrightarrow \operatorname{Ext}_R^s(k, \Omega_i) \xrightarrow{g} \operatorname{Ext}_R^s(k, F_{i-1}) .$$

The composition gf is induced by the composition $F_i \to \Omega_{i-1} \to F_{i-1}$. Since the map

$F_i = R^{\beta_i} \to F_{i-1} = R^{\beta_{i-1}}$ is expressed by a matrix with entries in $\mathfrak{m}$, and $\mathrm{Ext}_R^s(k, F_i)$ is annihilated by any element of $\mathfrak{m}$, we have $gf = 0$. But both f and g are monomorphisms and $\mathrm{Ext}_R^s(k, F_i) \neq 0$. This is a contradiction. ∎

THEOREM 11. *Let F be an FR of k, $\Omega_i = \mathrm{Ker}(F_{i-1} \to F_{i-2})$, and $s = \mathrm{depth}\, R$, then*

$$\mathrm{depth}\, \Omega_i = \begin{cases} i & (\text{if } 0 \leq i \leq s)\,, \\ s & (\text{if } i > s \text{ and } \Omega_i \neq 0)\,. \end{cases}$$

PROOF. By Proposition 10, it is enough to prove that, if R is not regular, $\mathrm{depth}\, \Omega_{s+1} = s$. We may assume that F is an MFR of k.

Assume $\mathrm{depth}\, R = s = 0$, then

$$\mathrm{Hom}_R(k, \mathfrak{m}) \cong \mathrm{Ann}_{\mathfrak{m}}\, \mathfrak{m} = \mathrm{Ann}_R\, \mathfrak{m} \cong \mathrm{Hom}_R(k, R) \neq 0\,.$$

Thus $\mathrm{depth}\, \Omega_1 = \mathrm{depth}\, \mathfrak{m} = 0$.

Assume $s > 0$. By Proposition 10, we have $\mathrm{depth}\, \Omega_{s+1} \geq s$. If $\mathrm{depth}\, \Omega_{s+1} > s$, then we have the following exact sequences:

$$0 \longrightarrow \mathrm{Ext}_R^s(k, F_s) \xrightarrow{f} \mathrm{Ext}_R^s(k, \Omega_s)\,,$$

$$0 \longrightarrow \mathrm{Ext}_R^{s-1}(k, \Omega_{s-1}) \longrightarrow \mathrm{Ext}_R^s(k, \Omega_s) \xrightarrow{g} \mathrm{Ext}_R^s(k, F_{s-1})\,.$$

Since $gf = 0$ and f is a monomorphism,

$$\mathrm{Ker}\, g \supset \mathrm{Ext}_R^s(k, F_s) = \mathrm{Ext}_R^s(k, R)^{\beta_s}\,.$$

Moreover, the last exact sequence implies that

$$\mathrm{Ker}\, g = \mathrm{Ext}_R^{s-1}(k, \Omega_{s-1}) \cong \mathrm{Hom}_R(k, k) = k\,.$$

Since $\mathrm{Ext}_R^s(k, R) \neq 0$, comparing dimensions of k-vector spaces above, we have $\beta_s \leq 1$. This contradicts the result of Proposition 7, since R is not regular and $\varepsilon_0 > \dim R \geq s > 0$. ∎

COROLLARY 12. *Let R be a local ring with residue field k, $n = \dim R$, F an FR of k, and $\Omega_i = \mathrm{Ker}(F_{i-1} \to F_{i-2})$.*

(i) *$\Omega_1, \cdots, \Omega_{n-1}$ are not CM.*

(ii) *The following conditions are equivalent.*

(a) *R is a CM ring,*

(b) *There exists an integer $i \geq n$, such that Ω_i is CM,*

(c) *For all $i \geq n$, Ω_i is CM unless $\Omega_i \neq 0$.*

PROOF. If $\Omega_i \neq 0$, then by Theorem 9 and Theorem 11, we have

$$\dim \Omega_i = \dim R \qquad (i > 0)\,,$$

$$\operatorname{depth}\Omega_i\begin{cases} =\operatorname{depth}R & (i\geq\operatorname{depth}R)\,,\\ <\operatorname{depth}R & (i<\operatorname{depth}R)\,.\end{cases}$$

Moreover always $\Omega_n\neq 0$. Therefore the assertions are clear. ∎

EXAMPLE 13. When R is not CM, syzygies of *finite R-modules* can be CM, and generally do not satisfy Theorem 11. Let k be a field, $R=k[\![X, Y]\!]/(X^2, XY)$, and $x, y\in R$ represent X, Y, respectively. Then $\dim R=1$. Since $x\in \operatorname{Ann}_R \mathfrak{m}\cong \operatorname{Hom}_R(k, R)$, we get $\operatorname{depth}R=0$. Thus R is not CM. Consider the finite R-module $R/(y)$. We get $\dim R/(y)=\operatorname{depth}R/(y)=0$. But the first syzygy (y) of $R/(y)$ is a CM-module with dimension 1. In fact, since $y\in R$ is (y)-regular element, $0<\operatorname{depth}(y)\leq\dim(y)\leq\dim R=1$.

§3. Invariant I( ) of syzygies.

Let $\mathfrak{q}$ be a parameter ideal of R, and $\mathrm{I}_\mathfrak{q}(-)=\operatorname{length}(-\otimes_R R/\mathfrak{q})-e_\mathfrak{q}(-)$, where $e_\mathfrak{q}$ is the multiplicity for $\mathfrak{q}$. When M is a Buchsbaum module (i.e. $\mathrm{I}_\mathfrak{q}(M)$ is independent of $\mathfrak{q}$), we write $\mathrm{I}(M)=\mathrm{I}_\mathfrak{q}(M)$.

PROPOSITION 14. *Let F be an MFR of k. If $\dim R\geq 1$, then for all $r\geq 0$,*

$$\mathrm{I}_\mathfrak{q}(\Omega_r)=\left(\sum_{i=0}^{r-1}(-1)^{r-i-1}\beta_i\right)\mathrm{I}_\mathfrak{q}(R)+\sum_{i=0}^{r}(-1)^{r-i}\dim_k\operatorname{Tor}_i^R(k, R/\mathfrak{q})\,.$$

PROOF. The short exact sequence $0\to\Omega_r\to F_{r-1}\to\Omega_{r-1}\to 0$ implies that $e_\mathfrak{q}(\Omega_r)+e_\mathfrak{q}(\Omega_{r-1})=e_\mathfrak{q}(F_{r-1})=\beta_{r-1}e_\mathfrak{q}(R)$. Since $e_\mathfrak{q}(\Omega_0)=e_\mathfrak{q}(k)=0$, by induction on r,

$$e_\mathfrak{q}(\Omega_r)=\left(\sum_{i=0}^{r-1}(-1)^{r-i-1}\beta_i\right)e_\mathfrak{q}(R)\,.$$

The sequence $0\to\Omega_r\to F_{r-1}\to\Omega_{r-1}\to 0$ also implies the following exact sequence.

$$0\longrightarrow \operatorname{Tor}_1^R(\Omega_{r-1}, R/\mathfrak{q})\longrightarrow \Omega_r\otimes_R R/\mathfrak{q}\longrightarrow (R/\mathfrak{q})^{\beta_{r-1}}\longrightarrow \Omega_{r-1}\otimes_R R/\mathfrak{q}\longrightarrow 0\,.$$

Since each module above has finite length and $\operatorname{Tor}_1^R(\Omega_{r-1}, R/\mathfrak{q})=\operatorname{Tor}_r^R(k, R/\mathfrak{q})$, we have

$$\operatorname{length}(\Omega_r\otimes_R R/\mathfrak{q})+\operatorname{length}(\Omega_{r-1}\otimes_R R/\mathfrak{q})=\beta_{r-1}\operatorname{length}R/\mathfrak{q}+\dim_k\operatorname{Tor}_r^R(k, R/\mathfrak{q})\,.$$

By induction on r,

$$\operatorname{length}(\Omega_r\otimes_R R/\mathfrak{q})=\left(\sum_{i=0}^{r-1}(-1)^{r-i-1}\beta_i\right)\operatorname{length}R/\mathfrak{q}+\sum_{i=0}^{r}(-1)^{r-i}\dim_k\operatorname{Tor}_i^R(k, R/\mathfrak{q})\,.$$

Thus the desired equation holds. ∎

COROLLARY 15. *Let R be a Buchsbaum ring with* $\dim R \geq 1$, *and F an FR of k.*

(a) Ω_r *is a Buchsbaum module if and only if* $\sum_{i=0}^{r}(-1)^{r-i}\dim_k \mathrm{Tor}_i^R(k, R/\mathfrak{q})$ *is independent of the parameter ideal* $\mathfrak{q}$.

(b) Ω_r *is a Buchsbaum module for all* $r>0$, *if and only if* $\dim_k \mathrm{Tor}_r^R(k, R/\mathfrak{q})$ *is independent of the parameter ideal* $\mathfrak{q}$ *for all* $r>0$.

PROOF. We may assume that F is an MFR of k. It is enough to show that, for any $r>0$, every parameter ideal of Ω_r is one of R unless $\Omega_r = 0$. But this is clear, since $\dim \Omega_r = \dim R$ and

$$\begin{aligned}\{\mathfrak{m}\} &= \mathrm{Supp}\, \Omega_r \otimes_R R/\mathfrak{q} = \mathrm{Supp}\, \Omega_r \cap \mathrm{Supp}\, R/\mathfrak{q} \\ &= \mathrm{Spec}\, R \cap \mathrm{Supp}\, R/\mathfrak{q} = \mathrm{Supp}\, R/\mathfrak{q}. \end{aligned}$$ ■

J. Stückrad and W. Vogel have proved the following useful criterion of Buchsbaum modules.

THEOREM 16 [5, J. Stückrad and W. Vogel]. (a) *Let M be a finite R-module with* $r = \mathrm{depth}\, M < \dim M = d$ *and* $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ *for all* $i \neq r, d$. *Then M is a Buchsbaum module if and only if* $\mathfrak{m}\mathrm{H}^r_{\mathfrak{m}}(M) = 0$.

(b) *Let M be a Buchsbaum module with* $d = \dim M$, *then*

$$\mathrm{I}(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \mathrm{length}\, \mathrm{H}^i_{\mathfrak{m}}(M).$$

We shall apply Proposition 14 and Theorem 16 to the following examples, and compare results.

EXAMPLE 17. (a) Let R be a Buchsbaum ring with $\dim R = 1$, and F an MFR of k, then every syzygy Ω_r is a Buchsbaum module. In fact, for any parameter ideal $\mathfrak{q} = (x)$, the composition of the exact sequence

$$0 \longrightarrow (0 :_R x) \longrightarrow R \xrightarrow{x} R \longrightarrow R/\mathfrak{q} \longrightarrow 0$$

and an MFR of $(0 :_R x) = (0 :_R \mathfrak{m})$ is an MFR of $R/\mathfrak{q}$. Thus

$$\dim_k \mathrm{Tor}_r^R(k, R/\mathfrak{q}) = (\text{rank of } r\text{-th module of an MFR of } R/\mathfrak{q})$$

is independent of $\mathfrak{q}$ for any $r>0$. By Corollary 15, every Ω_r is a Buchsbaum module.

Theorem 16 (a) also implies this result, since $\mathrm{H}^0_{\mathfrak{m}}(\Omega_r)$ is a submodule of $\mathrm{H}^0_{\mathfrak{m}}(F_{r-1}) = \mathrm{H}^0_{\mathfrak{m}}(R)^{\beta_{r-1}}$. Moreover we have

$$\begin{aligned}\text{length}\, \mathrm{H}^0_{\mathfrak{m}}(\Omega_r) &= \mathrm{I}(\Omega_r)\\ &= \left(\sum_{i=0}^{r-1} (-1)^{r-i-1}\beta_i\right)\mathrm{I}(R) + \sum_{i=0}^{r} (-1)^{r-i} \dim_k \mathrm{Tor}_i^R(k, R/\mathfrak{q})\,.\end{aligned}$$

(b) Let R be a CM ring with $\dim R = n \geq 1$, and F an MFR of k, then any syzygy Ω_r of k is a Buchsbaum module with $\mathrm{I}(\Omega_r) = \binom{n-1}{r}$. Indeed, since any parameter ideal $\mathfrak{q}$ is generated by an R-regular sequence $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, a Koszul complex $E = K(\boldsymbol{x})$ is an MFR of $R/\mathfrak{q}$ and

$$\dim_k \mathrm{Tor}_i^R(k, R/\mathfrak{q}) = \text{rank}\, E_i = \binom{n}{r}\,.$$

By Proposition 14,

$$\mathrm{I}(\Omega_r) = \sum_{i=0}^{r} (-1)^{r-i} \binom{n}{i} = \binom{n-1}{r}\,.$$

Theorem 16 gives the same result. For, by induction on i, we have

$$\mathrm{H}^i_{\mathfrak{m}}(\Omega_r) = \begin{cases} k & (\text{if } i = r < n) \\ 0 & (\text{if } i, r < n \text{ and } i \neq r) \\ 0 & (\text{if } i < n \text{ and } r \geq n)\,. \end{cases}$$

REMARK. K. Yamagishi [6] has shown the Buchsbaum property of syzygies in more general situation; *Let R be an n-dimensional Buchsbaum ring with* $\mathrm{H}^p_{\mathfrak{m}}(R) = 0$ *for* $p \neq 1$, n, *and F an FR of the residue field k. Then every syzygy* Ω_i *is a Buchsbaum module.*

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