TOKYO J. MATH.
VOL. 14, NO. 2, 1991

On Some Class Properties of Statistical Experiments under Weak Blackwell Equivalence

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In this paper, the author studies permanence properties of statistical experiments under weak Blackwell equivalence. It turns out that majorization and pivotality are class properties. We give some remarks on LeCam equivalence.

§1. Preliminaries.

Let I denote a nonempty index set. A statistical experiment for the parameter set I is a triplet $\underline{E}=(\underline{X}, \underline{A}, \underline{P})$, where $(\underline{X}, \underline{A})$ is a measurable space and $\underline{P}=\{P_i \mid i\in I\}$ is a family of probability measures.

Throughout this paper, we consider two experiments $\underline{E}=(\underline{X}, \underline{A}, \underline{P})$ and $\underline{F}=(\underline{Y}, \underline{B}, \underline{Q})$ with the same parameter set I.

A weak kernel from $\underline{E}=(\underline{X}, \underline{A}, \underline{P})$ to $(\underline{Y}, \underline{B})$ is a mapping N from $\underline{X}\times\underline{B}$ to $\boldsymbol{R}$ with the following properties:

(w-1) $0\leqq N(\cdot, B)$ [a.e.P_i] for all $i\in I$ and $B\in\underline{B}$.

(w-2) $N(\cdot, B)$ is $\underline{A}$-measurable for all $B\in\underline{B}$.

(w-3) $N(\cdot, \underline{Y})=1$ [a.e.P_i] for all $i\in I$.

(w-4) For every sequence $\{B_n : n=1, 2, \cdots\}$ of pairwise disjoint sets $B_n\in\underline{B}$ we have

$$N(\cdot, \bigcup_{n\geqq 1}B_n)=\sum_{n\geqq 1}N(\cdot, B_n) \qquad \text{[a.e.}P_i\text{]} \quad \text{for all } i\in I\,.$$

The set of weak kernels from $\underline{E}=(\underline{X}, \underline{A}, \underline{P})$ to $(\underline{Y}, \underline{B})$ will be abbreviated by $Weak((\underline{X}, \underline{A}, \underline{P}), (\underline{Y}, \underline{B}))$.

$\underline{E}$ is called more informative than $\underline{F}$ in the weak sense of Blackwell, in symbols $\underline{E}>\underline{F}$ (w.B), if there exists $N\in Weak((\underline{X}, \underline{A}, \underline{P}), (\underline{Y}, \underline{B}))$ with $N(P_i)=Q_i$ for all $i\in I$. Here $N(P_i)$ is a measure such that $N(P_i)(B)=\int N(x, B)dP_i(x)$. $\underline{E}$ and $\underline{F}$ are called weakly Blackwell equivalent if $\underline{E}>\underline{F}$ (w.B) and $\underline{F}>\underline{E}$ (w.B) and we write $\underline{E}\sim\underline{F}$ (w.B).

Received December 10, 1990

By $ca(\underline{A})$, we denote the space of all bounded signed measures on $\underline{A}$. The band $L(\underline{E})$ generated by $\underline{P}$ in $ca(\underline{A})$ is called the L-space of the experiment $\underline{E}$. A linear, positive and positively isometric mapping from $L(\underline{E})$ to $L(\underline{F})$ is called a transition from $\underline{E}$ to $\underline{F}$. By $T(\underline{E}, \underline{F})$, we denote the set of transitions from $\underline{E}$ to $\underline{F}$. $d(\underline{E}, \underline{F}) = inf\{sup_i \|T(P_i) - Q_i\| \mid T \in T(\underline{E}, \underline{F})\}$ is called the deficiency of $\underline{E}$ for $\underline{F}$ ([9]). If $d(\underline{E}, \underline{F}) = 0$, $\underline{E}$ is called more informative than $\underline{F}$ in the sense of LeCam and we write $\underline{E} > \underline{F}$ (L). $\underline{E}$ and $\underline{F}$ are called LeCam equivalent if $\underline{E} > \underline{F}$ (L) and $\underline{F} > \underline{E}$ (L) and we write $\underline{E} \sim \underline{F}$ (L). Obviously we have $\underline{E} > \underline{F}$ (w.B) $\Rightarrow \underline{E} > \underline{F}$ (L) and $\underline{E} \sim \underline{F}$ (w.B) $\Rightarrow \underline{E} \sim \underline{F}$ (L).

An experiment $\underline{E}(\underline{X}, \underline{A}, \underline{P})$ is said to be majorized if there exists a measure m on $\underline{A}$ with respect to which each $P \in \underline{P}$ has an $\underline{A}$-measurable density dP/dm and $m \equiv \underline{P}$. Such an m is called a dominating measure or a majorizing measure for $\underline{E}$. A majorizing measure m for $\underline{E}$ is called a pivotal measure for $\underline{E}$ if the following condition is satisfied: a sub-σ-field $\underline{C}$ of $\underline{A}$ is PSS (*pairwise sufficient containing supports*) for $\underline{E}$ if and only if each $P \in \underline{P}$ has a $\underline{C}$-measurable dP/dm ([3]).

$N \in Weak((\underline{X}, \underline{A}, \underline{P}), (\underline{Y}, \underline{B}))$ is said to be weakly sufficient for $\underline{E}$ if, to every bounded $\underline{A}$-measurable f, there exists a $\underline{B}$-measurable $E^N(f)$ with $E^N_p(f) = E^N(f)$ [a.e.$N(P)$] for all $P \in \underline{P}$, where $E^N_P(f) := dN(f \cdot P)/dN(P)$. $N \in Weak((\underline{X}, \underline{A}, \underline{P}), (\underline{Y}, \underline{B}))$ is called weakly Blackwell sufficient if there exists $N' \in Weak((\underline{Y}, \underline{B}, N(\underline{P})), (\underline{X}, \underline{A}))$ such that $N'N(P_i) = P_i$ for all $i \in I$.

Throughout this paper we denote by $I(A)$ the defining function of a set A.

§2. Majorization.

THEOREM 1. *Suppose that* $\underline{E} \sim \underline{F}$ (w.B). *If* $\underline{E}$ *is a majorized experiment, then* $\underline{F}$ *is also majorized.*

PROOF. A set $S(P_i)$ is called an $\underline{E}$-support of P_i if (1) $P_i(S(P_i)) = 1$ and (2) if $B \subset S(P_i)$ and $P_i(B) = 0$ then $P_j(B) = 0$ for all $j \in I$. Since $\underline{E}$ is majorized, each P_i has its support. By the assumption, there exists $N \in Weak((\underline{Y}, \underline{B}, \underline{Q}), (\underline{X}, \underline{A}))$ such that $P_i = N(Q_i)$ for all $i \in I$. In order to prove $\underline{F}$ is majorized, it is sufficient to prove each Q_i has its support ([1]). Put $D_i = \{y \mid N(y, S(P_i)) = 1\}$. We shall show D_i is an $\underline{F}$-support of Q_i.

Since $1 = P_i(S(P_i)) = \int N(y, S(P_i)) dQ_i$, we have $1 = Q_i(\underline{Y}) = Q_i(D_i)$. Since $\underline{F}_{i,j} = (\underline{Y}, \underline{B}, \{Q_i, Q_j\})$ is a dominated experiment for any $j \in I$ and N is weakly Blackwell sufficient by our assumption, N is weakly sufficient for $\underline{F}_{i,j}$ ([4], [8]). (In [4] Theorem 22.11, it is proved when N is a stochastic kernel. But, if we slightly modify the proof, we can easily prove the theorem when N is a weak kernel). Hence, if $C \subset D_i$, $Q_i(C) = 0$, we have

$$0 = \int I(C) N(y, S(P_i)) dQ_i = \int_{S(P_i)} E^N_{i,j}(I(C)) dN(Q_i)$$

$$=\int_{S(P_i)} E^N_{i,j}(I(C))dP_i\,.$$

Here $E^N_{i,j}(I(C))$ is $E^N(I(C))$ common to P_i and P_j. So we have $P_i(\{E^N_{i,j}(I(C))>0\}\cap S(P_i))=0$. Hence we get $P_j(\{E^N_{i,j}(I(C))>0\}\cap S(P_i))=0$. Consequently

$$0=\int_{S(P_i)} E^N_{i,j}(I(C))dP_j=\int_{S(P_i)} E^N_{i,j}(I(C))dN(Q_j)$$

$$=\int I(C)N(y,\,S(P_i))dQ_j=\int_C N(y,\,S(P_i))dQ_j\,.$$

From this equality and since $C\subset D_i=\{y\mid N(y,\,S(P_i))=1\}$, we have $Q_j(C)=0$. D_i is therefore an $\underline{F}$-support of Q_i.

THEOREM 2. *Suppose that* $\underline{F}>\underline{E}$ (w.B). *If there exists* $N\in Weak((\underline{Y},\,\underline{B},\,\underline{Q}),\,(\underline{X},\,\underline{A}))$ *such that* N *is weakly sufficient for* $\underline{F}$ *and* $N(Q_i)=P_i$ *for all* $i\in I$. *Then the following two assertions are equivalent.*

(1) $\underline{E}$ *is a majorized experiment.*

(2) $\underline{F}$ *is a majorized experiment.*

PROOF. By our assumptions, N is weakly Blackwell sufficient ([8]). Hence there exists $N'\in Weak((\underline{X},\,\underline{A},\,N(\underline{Q})),\,(\underline{Y},\,\underline{B}))=Weak((\underline{X},\,\underline{A},\,\underline{P}),\,(\underline{Y},\,\underline{B}))$ such that $N'(N(Q_i))=Q_i$ for all $i\in I$. Thus, $N'(P_i)=Q_i$ and $\underline{E}\sim\underline{F}$ (w.B). Accordingly, Theorem 1 yields the assertion of this theorem.

The example in the following lemma is given in ([3]). For completeness of our article we shall state it here.

LEMMA 1. *There are an experiment* $\underline{E}=(\underline{X},\,\underline{A},\,\underline{P})$ *and a pairwise sufficient sub-σ-field* $\underline{B}$ *of* $\underline{A}$ *such that* $\underline{E}$ *is majorized but* $\underline{F}=(\underline{X},\,\underline{B},\,\underline{P}|\underline{B})$ *is not majorized.*

PROOF. Let $\underline{X}$ be an uncountable set, $\underline{A}$ the σ-field consisting of all countable and co-countable subsets of $\underline{X}$ and $\underline{P}=\{P_x\mid x\in\underline{X}\}$, where P_x is a probability measure on $\underline{A}$ satisfying $P_x(\{x\})=1$. $\underline{E}=(\underline{X},\,\underline{A},\,\underline{P})$ is clearly a majorized experiment. For each pair P_x, P_y in $\underline{P}$ and $0\leqq a<1$, we put $A(P_x,\,P_y;\,a)=\{x'\mid 0<dP_x/d(P_x+P_y)\cdot I(S(P_x)\cup S(P_y))\leqq a\}$. Let $\underline{S}$ be the σ-ring generated by all $A(P_x,\,P_y;\,a)$. For a fixed $z\in\underline{X}$, put $\underline{C}_z=\{A\in\underline{A}\mid A$ is countable and $z\in A^c\}\cup\{A\in\underline{A}\mid A$ is co-countable and $z\in A\}$. $\underline{C}_z$ is a sub-σ-field of $\underline{A}$. Suppose that $\underline{F}=(\underline{X},\,\underline{C}_z,\,\underline{P}|\underline{C}_z)$ is majorized. Then an $\underline{F}$-support $S(P_z|\underline{C}_z)$ ($\in\underline{C}_z$) must exist and since it must contain z, it is a co-countable set containing z. Hence there exists $y\in S(P_z|\underline{C}_z)$ such that $y\neq z$. We have $\{y\}\in\underline{C}_z$ and $P_z(\{y\})=0$ but $P_y(\{y\})=1$, which contradicts our definition of supports. Accordingly $\underline{F}$ is not majorized. Consider $A,\,B\in\underline{S}$ satisfying $A\cap B=\varnothing$. $A,\,B$ are countable sets since $\underline{S}$ is the σ-ring consisting of all countable sets. If $z\in A$ ($z\in B$), we have $B\in\underline{C}_z$ ($A\in\underline{C}_z$) and $B(A)$ separates A and

B. If $z \in (A \cup B)^c$; we have $A^c \in \underline{C}_z$ and it separates A and B. Hence $\underline{C}_z$ separates $\underline{S}$ and $\underline{C}_z$ is pairwise sufficient for $\underline{P}$ ([2]).

If we assume $\underline{E} \sim \underline{F}$ (L) instead of $\underline{E} \sim \underline{F}$ (w.B), Theorems 1 and 2 do not hold. By Lemma 1, there are an experiment $\underline{E}=(\underline{X}, \underline{A}, \underline{P})$ and a pairwise sufficient sub-σ-field $\underline{B}$ of $\underline{A}$ such that $\underline{E}$ is majorized but $\underline{F}=(\underline{X}, \underline{B}, \underline{P}|\underline{B})$ is not majorized. $\underline{E} > \underline{F}$ (L) is obvious. Since $\underline{B}$ is pairwise sufficient, we have $d(\underline{F}, \underline{E})=0$ and therefore $\underline{F} > \underline{E}$ (L) ([6]). Hence $\underline{E} \sim \underline{F}$ (L) and $\underline{E}$ is majorized but $\underline{F}$ is not majorized.

LEMMA 2. *Suppose that $\underline{B}$ is a pairwise sufficient sub-σ-field of $\underline{A}$ and $\underline{F}=(\underline{X}, \underline{B}, \underline{P}|\underline{B})$ is majorized. Then $\underline{E}=(\underline{X}, \underline{A}, \underline{P})$ is also majorized.*

PROOF. Let S_i be an $\underline{F}$-support of $P_i|\underline{B}$. We shall show that S_i is also an $\underline{E}$-support of P_i. Since $\underline{B}$ is pairwise sufficient, we have $P_i(A)=\int E_{i,j}(I(A)|\underline{B})dP_i$ and $P_j(A)=\int E_{i,j}(I(A)|\underline{B})dP_j$ for all $A \in \underline{A}$, where $E_{i,j}(I(A)|\underline{B})$ is the conditional expectation of $I(A)$ common to P_i and P_j. If $A \subset S_i$, we may assume $\{E_{i,j}(I(A)|\underline{B})>0\} \subset S_i$ since $S_i \in \underline{B}$. Hence we have, for $A \in \underline{A}$ with $A \subset S_i$, $P_i(A)=0 \Rightarrow P_i(\{E_{i,j}(I(A)|\underline{B})>0\})=0 \Rightarrow P_j(\{E_{i,j}(I(A)|\underline{B})>0\})=0 \Rightarrow P_j(A)=0$. Consequently S_i is an $\underline{E}$-support of P_i. $\underline{E}$ is therefore majorized.

THEOREM 3. *Let $\underline{B}$ be a sub-σ-field of $\underline{A}$. If $\underline{F}=(\underline{X}, \underline{B}, \underline{P}|\underline{B})$ is LeCam equivalent to $\underline{E}=(\underline{X}, \underline{A}, \underline{P})$ i.e. $\underline{F} \sim \underline{E}$ (L) and if $\underline{F}$ is majorized, then $\underline{E}$ is also majorized. (It is clear $\underline{E} > \underline{F}$ (L) always holds since $\underline{B}$ is a sub-σ-field of $\underline{A}$).*

PROOF. We have $d(\underline{F}, \underline{E})=0$ and therefore $\underline{B}$ is pairwise sufficient ([6]). Hence, by Lemma 2, $\underline{E}$ is majorized.

At last in this section, we remark that weak domination is not a class property. We shall give an example. Let $\underline{X}$ be the set of real numbers, $\underline{A}$ the σ-field consisting of all subsets of $\underline{X}$ and $\underline{P}=\{P_x \mid x \in [0, \infty)\}$, where $P_0(\{0\})=1$ and $P_x(\{x\})=P_x(\{-x\})=1/2$ for all $x>0$. $\underline{E}=(\underline{X}, \underline{A}, \underline{P})$ is obviously a weakly dominated experiment. Let $\underline{B}=\{A \mid A \in \underline{A}, A=-A\}$. If f is a bounded $\underline{A}$-measurable function, then $E(f|\underline{B})=(f(x)+f(-x))/2$ is a $\underline{B}$-measurable function satisfying $\int_B f dP=\int_B E(f|\underline{B})dP$ $(B \in \underline{B}, P \in \underline{P})$. $\underline{B}$ is sufficient. Let $\underline{B}^*=\{A \mid$ there exists $A' \in \underline{B}$ such that $A \triangle A'$ is a countable set$\}$. $\underline{B}^*$ is obviously a σ-field containing $\underline{B}$. If we put $N(\cdot, A)=E(I(A)|\underline{B})$, we have $N \in Weak((\underline{X}, \underline{B}^*, \underline{P}|\underline{B}^*), (\underline{X}, \underline{A}))$ and $P=N(P|\underline{B}^*)$ for all $P \in \underline{P}$. So $\underline{E} \sim \underline{F}$ $(:=(\underline{X}, \underline{B}^*, \underline{P}|\underline{B}^*))$ (w.B). On the other hand, it is easily shown that $\underline{B}^*$ is not closed under the formation of unions of arbitrary many number of sets in it. Hence $\underline{F}$ is not weakly dominated ([5]) though $\underline{E}$ is weakly dominated and $\underline{E} \sim \underline{F}$ (w.B).

§3. Pivotality.

Suppose that $\underline{E} \sim \underline{F}$ (w.B) and $\underline{E}$ is majorized. Then, by Theorem 1, $\underline{F}$ is also majorized. Let $N \in Weak((\underline{X}, \underline{A}, \underline{P}), (\underline{Y}, \underline{B}))$ be a kernel satisfying $N(P_i)=Q_i$ for all $i \in I$

and let m be a majorizing measure for $\underline{E}$. But $N(m)$ is not necessarily a majorizing measure for $\underline{F}$. It is easily seen from the following example. Let $\underline{X}=\{1, 2, \cdots\}$, $\underline{A}=2^{\underline{X}}$ and $P(\{i\})=1/2^i$. Let $\underline{E}=(\underline{X}, \underline{A}, \{P\})$ and $\underline{F}=(\underline{Y}, \underline{B}, \{Q\})$ be experiments consisting of one probability measure. If we define $N(x, B)\equiv Q(B)$ and $K(y, A)\equiv P(A)$, it is easy to verify $N(P)=Q$ and $K(Q)=P$. Hence $\underline{E}\sim\underline{F}$ (w.B). Let m be the counting measure on $\underline{X}$. Then $N(m)(B)=0$ or ∞ and $dQ/dN(m)$ does not exist. Hence $N(m)$ is not a majorizing measure for $\underline{F}$.

At last we shall show that, if m is a pivotal measure, $N(m)$ is also a pivotal measure (hence, of course, a majorizing measure). For this purpose we need some definitions. By $CVL(\underline{E})$, we denote the closed vector sublattice of $ca(\underline{A})$ generated by $\underline{P}$. An orthogonal system in $CVL(\underline{E})$ is a subset of $CVL(\underline{E})^+-\{0\}$ such that $u\wedge v=0$ for all distinct members u and v of it.

THEOREM 4. *Suppose that* $\underline{E}\sim\underline{F}$ (w.B) *and* $\underline{E}$ *is majorized. Let* N *be a weak kernel with* $N(P_i)=Q_i$ *for all* $i\in I$. *If* m *is a pivotal measure for* $\underline{E}$, *then* $N(m)$ *is a pivotal measure for* $\underline{F}$.

PROOF. According to [7], m is expressed as $m=\sum\{u_\alpha \mid \alpha\in\Lambda\}$, where $\{u_\alpha \mid \alpha\in\Lambda\}$ $(\subset CVL(\underline{E})^+-\{0\})$ is a maximal orthogonal system of $CVL(\underline{E})$ and $(\sum_\alpha u_\alpha)(A)=sup\{\sum_{\alpha\in W}u_\alpha(A) \mid W\subset\Lambda,\ W\text{: finite}\}$. N is an isometric Banach lattice isomorphism from $CVL(\underline{E})$ onto $CVL(\underline{F})$ ([9], 55.16 Theorem). Hence $\{N(u_\alpha) \mid \alpha\in\Lambda\}$ is a maximal orthogonal system of $CVL(\underline{F})$. Since $\underline{F}$ is majorized by Theorem 1, $\sum\{N(u_\alpha) \mid \alpha\in\Lambda\}$ is a pivotal measure for $\underline{F}$. As $N(m)=N(\sum\{u_\alpha \mid \alpha\in\Lambda\})=\sum\{N(u_\alpha) \mid \alpha\in\Lambda\}$, $N(m)$ is a pivotal measure for $\underline{F}$. We remark that, if N' satisfies $N'(P_i)=Q_i$ for all $i\in I$, we have $N(m)=N'(m)$. This follows from the fact that N with $N(P_i)=Q_i$ for all $i\in I$ is uniquely determined on $CVL(\underline{E})$ ([9], 55.16 Theorem) and from the fact which we stated at the beginning of this proof.

Suppose that $\underline{E}$ and $\underline{F}$ are majorized and $\underline{E}\sim\underline{F}$ (L). If T is a transition satisfying $T(P_i)=Q_i$ for all $i\in I$ and m is a pivotal measure for $\underline{E}$, then $T(m)$ is a pivotal measure for $\underline{F}$. By substituting T for N, we can prove this assertion quite similarly to the proof of the above theorem.

The author wishes to express his gratitude to Professor H. Heyer, Professor H. Morimoto and Mr. J. Fujii for their many helpful suggestions.

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