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2-Type Integral Surfaces in $S^5(1)$

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Abstract. The main purpose of this paper is to classify integral surfaces of the unit sphere $S^5(1)$ which are mass-symmetric and of 2-type. If we consider $S^5(1)$ as a Sasakian manifold, then we prove that a mass-symmetric 2-type integral surface of $S^5(1)$ lies fully in $S^5(1)$ and is the product of a plane circle and a helix of order 4 or the product of two circles.

1. Introduction.

Let M^n be a (connected) n-dimensional submanifold of Euclidean space E^{m+1}. Let x, H and Δ respectively be the position vector field, the mean curvature vector field and the Laplace operator of the induced metric on M^n. Then, the position vector x and the mean curvature vector H of M^n in E^{m+1} satisfy (see e.g. [4])

$$\Delta x = -nH . \tag{1.1}$$

This formula yields the following well-known result: M^n is a minimal submanifold in E^{m+1} if and only if all coordinate functions of E^{m+1}, restricted to M, are harmonic functions, that is $\Delta x = 0$ (i.e. they are eigenfunctions of Δ with eigenvalue 0). Moreover, in this context, T. Takahashi [9] proved that the submanifolds M^n for which

$$\Delta x = \lambda x \tag{1.2}$$

i.e. for which all coordinate functions are eigenfunctions of Δ with the same eigenvalue $\lambda \in \boldsymbol{R}$, are precisely either the minimal submanifolds of E^{m+1} ($\lambda = 0$) or the minimal submanifolds M^n of hyperspheres S^m in E^{m+1} (the case when $\lambda \neq 0$, actually $\lambda = n/r^2$ where r is the radius of S^m).

One branch of research in submanifold theory was introduced by B. Y. Chen in [4], [5], namely, the study of submanifolds of finite type. In terms of B. Y. Chen's theory of submanifolds in E^m of finite type, condition (1.2) asserts that M^n is of 1-type in E^m.

In general, a submanifold M^n of Euclidean space E^{m+1} is said to be of k-type if

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the position vector x of M^n in E^{m+1} can be decomposed as

$$x = x_0 + x_1 + \cdots + x_k$$

where $x_0 \in E^{m+1}$ is a fixed vector and x_i $(i=1, \cdots, k)$ are non-constant E^{m+1}-valued maps on M^n, such that

$$\Delta x_i = \lambda_i x_i \quad \text{for} \quad i=1, \cdots, k \quad \text{and} \quad \lambda_1 < \cdots < \lambda_k, \quad \lambda_i \in \mathbf{R}.$$

Many important submanifolds in Euclidean space turn out to be of finite type in this sense (see [4] for details).

A compact submanifold M^n of a hypersphere S^m of E^{m+1} is said to be mass-symmetric in S^m if the center of mass x_0 of M^n in E^{m+1} is exactly the center of S^m in E^{m+1}. Mass-symmetric 2-type submanifolds of a hypersphere can be regarded as the "simplest" submanifolds of E^{m+1} next to minimal submanifolds. Many important submanifolds are known to be mass-symmetric and of 2-type. In Chen's book [4], some basic results for mass-symmetric 2-type surfaces in an m-sphere S^m were established. In particular, it was proved that a compact surface in S^3 is mass-symmetric and of 2-type if and only if it is the product of two circles of different radii ([4, Theorem 4.5, p. 279]). M. Barros and O. Garay [2] showed that the same result holds without the assumption of mass-symmetric. Also stationary 2-type mass-symmetric compact surfaces of S^m were classified in [1] by M. Barros and B. Y. Chen. In particular, they showed that such surfaces are flat and lie fully either in a 5-sphere or in a 7-sphere. They showed also that there exist no mass-symmetric 2-type surfaces which lie fully in $S^4(1)$. Afterwards O. Garay [6] showed that a mass-symmetric 2-type Chen surface (i.e. the allied mean curvature vector $\alpha(H)$ vanishes identically on M) is either pseudoumbilical or flat. Furthermore, if the surface is flat, then it lies fully in a totally geodesic 3-sphere or in a totally geodesic 5-sphere or in a totally geodesic 7-sphere.

Finally, Y. Miyata in [7] studied mass-symmetric 2-type surfaces of constant curvature in S^m and obtained, among others, the following results:

i) If $f: M \to S^m$ is a mass-symmetric 2-type immersion of a surface M of positive constant curvature into S^m, then f is a diagonal sum of two different standard minimal immersions of M into spheres.

ii) There are no mass-symmetric 2-type surfaces of constant negative curvature in a sphere.

iii) Let M be a flat surface and f a full mass-symmetric 2-type Chen immersion of M into S^m. If $m \geq 9$, then f is a diagonal sum of two different minimal immersions into spheres. If $m=7$, there exists a full mass-symmetric 2-type Chen immersion which is not a diagonal sum of minimal immersions.

In [1] and [7] one can find many results for 2-type surfaces in S^m.

In this paper we shall classify mass-symmetric 2-type integral surfaces of the Sasakian manifold $S^5(1) \subset E^6$. In particular, we will prove that, if we consider the unit sphere $S^5(1)$ as a Sasakian manifold then a mass-symmetric 2-type integral surface M of $S^5(1)$ lies fully in $S^5(1)$ and is the product of a plane circle and a helix of order 4 or the product of two circles. Furthermore, M belongs to a 1-parameter family of such surfaces.

2. Preliminaries.

We consider the space $\boldsymbol{C}^{m+1}$ of $m+1$ complex variables and let J denote its usual almost complex structure, namely by identifying $z \in \boldsymbol{C}^{m+1}$ with $(x_1, \cdots, x_{m+1}, y_1, \cdots, y_{m+1}) \in E^{2m+2}$ we consider $Jz = (-y_1, \cdots, -y_{m+1}, x_1, \cdots, x_{m+1})$.

$$S^{2m+1} = \{z \in \boldsymbol{C}^{m+1} : |z| = 1\} .$$

We give S^{2m+1} its usual contact structure. Define a tangent vector field ξ, a 1-form η and a (1, 1) tensor field φ on S^{2m+1} as follows:

Let $\langle\,,\rangle$ denote the induced metric from $\boldsymbol{C}^{m+1}$ on S^{2m+1} (so S^{2m+1} has constant sectional curvature 1),

$$\xi = -Jz\,, \quad \eta(X) = \langle X, \xi\rangle \quad \text{and} \quad \varphi = s \circ J$$

where s denotes the orthogonal projection from $T_z\boldsymbol{C}^{m+1}$ on T_zS^{2m+1}. Using these definitions, we obtain for all tangent vector fields X and Y on S^{2m+1} that

$$\begin{aligned} &\varphi^2 X = -X + \eta(X)\xi\,, \\ &\eta(\xi) = 1\,, \quad \eta(X) = \langle X, \xi\rangle\,, \\ &d\eta(X, Y) = \langle X, \varphi Y\rangle\,, \\ &N = -2d\eta \otimes \xi\,, \end{aligned} \tag{2.1}$$

where N is defined by $N(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$. It is well-known [3] that these formulas imply that $(\varphi, \xi, \eta, \langle\,,\rangle)$ determines a Sasakian structure on S^{2m+1}. Therefore, we also have

$$\nabla'_X \xi = -\varphi X\,, \qquad (\nabla'_X \varphi)Y = \langle X, Y\rangle\xi - n(Y)X \tag{2.2}$$

where ∇' denotes the Levi-Civita connection of $\langle\,,\rangle$. For more details see [3].

A Riemannian manifold M^n, isometrically immersed in S^{2m+1}, is called an *integral submanifold* if and only if η restricted to M^n vanishes.

In this paper we consider the unit hypersphere $S^5(1) \subset \boldsymbol{C}^3 \cong E^6$ centered at the origin and with the Sasakian structure $(\varphi, \xi, \eta, \langle\,,\rangle)$. Assume that

$$x : M \to S^5(1) \tag{2.3}$$

is a mass-symmetric 2-type immersion of an integral surface M into $S^5(1)$. Denote by $\overline{\nabla}$ the usual Levi-Civita connection of E^6 and by ∇, ∇' the induced connections on M and $S^5(1)$, respectively. Let H, h, A and D denote the mean curvature vector, the second fundamental form, the Weingarten maps and the normal connection of M in E^6, respectively. Finally denote by H', h', A' and D' the corresponding quantities for M in $S^5(1)$. Then we have $H=H'-x$ and, for any vector n normal to M in $S^5(1)$, $A_n=A'_n$.

Let Δ be the Laplacian of M associated with the induced metric. This Laplacian can be extended in a natural way to E^6-valued smooth maps u of M as follows:

$$\Delta u=\sum_{i=1}^{2}(\bar{\nabla}_{\nabla_{X_i}X_i}u-\bar{\nabla}_{X_i}\bar{\nabla}_{X_i}u) \tag{2.4}$$

where $\{X_1, X_2\}$ is a local orthonormal frame field on M.

Since M is 2-type and mass-symmetric, the position vector x of M with respect to the origin of E^6 can be written as follows:

$$x=x_p+x_q\,, \quad \Delta x_p=\lambda_p x_p\,, \quad \Delta x_q=\lambda_q x_q \tag{2.5}$$

where x_p, x_q are non-constant E^6-valued maps on M.

Furthermore, since M is an integral submanifold of the Sasakian manifold $S^5(1)$, we can choose a local field of orthonormal frames $X_1, X_2, \xi_1=\varphi X_1, \xi_2=\varphi X_2, \xi$ in $S^5(1)$ such that X_1, X_2 are tangent to M and ξ_1 is parallel to the mean curvature vector H' of M in $S^5(1)$. From the definition of an integral submanifold and (2.1) we have that the unit vector ξ is normal to M and to ξ_1, ξ_2. So the vectors ξ_1, ξ_2, ξ, x form a basis of the normal space of M in E^6. If, for convenience, we put $(e_1, \cdots, e_6)=(X_1, X_2, \xi_1, \xi_2, \xi, x)$, then we denote by $\{\omega_i\}$, $i=1, \cdots, 6$, the dual frame of the frame $\{e_i\}$ and by $\{\omega_i^j\}$, $i, j=1, \cdots, 6$, the corresponding connection forms. Thus we have

$$\bar{\nabla}e_i=\sum_{j=1}^{6}\omega_i^j e_j\,. \tag{2.6}$$

We have

$$H=H'-x=\frac{\operatorname{tr} A_1}{2}\xi_1-x \tag{2.7}$$

where A_1 is the Weingarten map A_{ξ_1} of M associated with ξ_1. We note also that $A_x=-I$, where I is the identity map.

Applying (2.4) to H we have, by direct computation, the well known formula (see [4, p. 273])

$$\Delta H=\Delta^{D'}H'+\alpha'(H')+\operatorname{tr}\bar{\nabla}A_H+(\operatorname{tr}A_1^2+2)H'-2|H|^2x \tag{2.8}$$

where

$$\alpha'(H')=\sum_{j=4}^{5}\operatorname{tr}(A_{H'}A_{e_j})e_j \tag{2.9}$$

is the allied mean curvature vector of M in $S^5(1)$ and

$$\operatorname{tr}\bar{\nabla}A_H=\sum_{i=1}^{2}((\nabla_{X_i}A_H)X_i+A_{D_{X_i}H}X_i)\,. \tag{2.10}$$

Moreover, since $Dx=0$, we have that DH' is perpendicular to x. So $\langle \Delta^{D'}H', x\rangle=0$.

On the other hand, since $\Delta x=-2H$, by using (2.5) we find

$$\Delta H=\frac{\operatorname{tr} A_1}{2}(\lambda_p+\lambda_q)\xi_1-\left(\lambda_p+\lambda_q-\frac{\lambda_p\lambda_q}{2}\right)x\,. \tag{2.11}$$

Combining (2.8) with (2.11) we obtain $\operatorname{tr} A_1=\text{const.}$ When $\operatorname{tr} A_1=0$ M is a minimal surface of $S^5(1)$ and so is of 1-type by Takahashi's theorem. Thus we may assume that $\operatorname{tr} A_1=\text{const.}\neq 0$.

Since M is an integral surface we have $\omega_6^t=0$, $t=3, 4, 5, 6$ and from (2.2) we have $\omega_5^j=0$ if $j=1, 2, 5, 6$ and $\omega_5^3(X_i)=-\langle \xi_i, \xi_1\rangle$, $\omega_5^4(X_i)=-\langle \xi_i, \xi_2\rangle$, $i=1, 2$.

By direct computation, we get

$$\Delta^{D'}H'=\sum_{i=1}^{2}(D'_{\nabla_{X_i}X_i}H'-D'_{X_i}D'_{X_i}H')=\frac{\operatorname{tr} A_1}{2}\Delta^D\xi_1 \tag{2.12}$$

$$=\frac{\operatorname{tr} A_1}{2}[-(\operatorname{tr}\nabla\omega_3^4)\xi_2+|D\xi_1|^2\xi_1-(\omega_3^4(X_2)+\omega_1^2(X_2))\xi]$$

where we have put

$$|D\xi_1|^2=\sum_{i=1}^{2}|D_{X_i}\xi_1|^2=\sum_{i=1}^{2}(\omega_3^4(X_i))^2+1\,, \tag{2.13}$$

$$\operatorname{tr}\nabla\omega_3^4=\sum_{i=1}^{2}(\nabla_{X_i}\omega_3^4)(X_i)=\sum_{i=1}^{2}(X_i\omega_3^4(X_i)-\omega_3^4(\nabla_{X_i}X_i))\,. \tag{2.14}$$

From [3, Lemma 1, p. 102] we have $A_\xi=0$. Thus from (2.9) and (2.10) we get

$$\alpha'(H')=\frac{\operatorname{tr} A_1}{2}\operatorname{tr}(A_1A_2)\xi_2\,, \tag{2.15}$$

$$\operatorname{tr}\bar{\nabla}A_H=\frac{\operatorname{tr} A_1}{2}\sum_{i=1}^{2}((\nabla_{X_i}A_1)X_i+\omega_3^4(X_i)A_2X_i)\,. \tag{2.16}$$

Now, from (2.8), (2.11), (2.12), (2.15) and (2.16) we obtain the following useful equations

$$\begin{array}{ll}\text{(i)} & \sum_{i=1}^{2}((\nabla_{X_i}A_1)X_i+\omega_3^4(X_i)A_2X_i)=0\,,\\ \text{(ii)} & |D\xi_1|^2+\operatorname{tr} A_1^2=\lambda_p+\lambda_q-2\,,\\ \text{(iii)} & \operatorname{tr}\nabla\omega_3^4-\operatorname{tr} A_1A_2=0\,,\\ \text{(iv)} & \omega_3^4(X_2)+\omega_1^2(X_2)=0\,.\end{array} \tag{2.17}$$

We continue with some further calculations. Using the Codazzi equation

$$(\nabla_X A_1)Y-A_{D_X\xi_1}Y-(\nabla_Y A_1)X+A_{D_Y\xi_1}X=0$$

and $\operatorname{tr} A_2=0$, we compute

$$0=\operatorname{grad}\operatorname{tr}A_1=\sum_{i=1}^{2}(\operatorname{tr}\nabla_{X_i}A_1)X_i=\sum_{i=1}^{2}((\nabla_{X_i}A_1)X_i-\omega_3^4(X_i)A_2X_i)\,.$$

Combining this with (2.17 (i)) we obtain

$$\sum_{i=1}^{2}(\nabla_{X_i}A_1)X_i=0 \tag{2.18}$$

and

$$\sum_{i=1}^{2}\omega_3^4(X_i)A_2X_i=0\,. \tag{2.19}$$

From [3, Lemma 2, p. 103] we have

$$A_1X_2=A_2X_1\,. \tag{2.20}$$

So,

$$\text{if}\quad A_1=\begin{bmatrix} a & b \\ b & c \end{bmatrix},\qquad \text{then}\quad A_2=\begin{bmatrix} b & c \\ c & -b \end{bmatrix}.$$

We have $\det A_2\neq 0$, because if we assume $\det A_2=0$, from (2.18) we conclude $\omega_1^2=0$ and from (2.17 (iv)) $\omega_3^4(X_2)=0$. Thus from (2.17 (ii)) and (2.13) we obtain $\omega_3^4(X_1)(X_2\omega_3^4(X_1))=0$. On the other hand, since $\langle R^{\perp}(X_1, X_2)\xi_1, \xi_2\rangle=1-X_2\omega_3^4(X_1)$, the equation of Ricci implies $X_2\omega_3^4(X_1)=1$. This is a contradiction. Therefore, $\det A_2\neq 0$ and (2.19) gives $\omega_3^4=0$. Then applying (2.13) and (2.14) to (2.17 (ii)) and (2.17 (iii)) respectively, we find $\operatorname{tr}A_1^2=\text{const.}$ and $\operatorname{tr}A_1A_2=0$. Thus, we get $b=0$, $a=$ const. and $c=$ const.

We are now ready to state and prove the main results.

3. Main results.

The following lemma shows that M is flat.

LEMMA 3.1. *Let M be a mass-symmetric 2-type integral surface in $S^5(1)$ in E^6. Then M is flat.*

PROOF. Note that the ambient space $S^5(1)$ is a Sasakian manifold. So from (2.2) and the fact that M is an integral surface we have

$$\begin{aligned}\overline{\nabla}_{X_j}\xi_i&=\nabla'_{X_j}\xi_i=(\nabla'_{X_j}\varphi)X_i+\varphi(\nabla'_{X_j}X_i)\\ &=\delta_{ij}\xi+\varphi(\nabla_{X_j}X_i+h'(X_i, X_j))\,,\qquad i,j=1,2\,.\end{aligned}$$

On the other hand

$$\overline{\nabla}_{X_j}\xi_i=-A_iX_j+D_{X_j}\xi_i \tag{3.1}$$

and moreover using (2.20) again

$$\varphi(h'(X_i, X_j)) = \varphi(\langle A_1 X_i, X_j\rangle \xi_1 + \langle A_2 X_i, X_j\rangle \xi_2)$$

$$= -(\langle A_i X_1, X_j\rangle X_1 + \langle A_i X_2, X_j\rangle X_2) = -A_i X_j\,, \qquad i, j = 1, 2\,.$$

Thus, we conclude that $\varphi(\nabla_{X_j} X_i) = 0$ and from (2.1) that $\nabla_{X_j} X_i$ is parallel to ξ. But $\nabla_{X_j} X_i$ is tangent to M. So $\nabla_{X_j} X_i = 0$ and the lemma follows.

From the equation of Gauss we get $1 + ac - c^2 = 0$. So $c \neq 0$ and $a = (c^2 - 1)/c$.

We need the following definition (see [8, p. 20]).

DEFINITION 3.2. If $\gamma(s)$ is a curve in a Riemannian manifold N, parametrized by arc length s, we say that γ is a *Frenet curve of osculating order* r when there exist orthonormal vector fields $E_1, E_2, \cdots, E_r$, along γ, such that:

$$\dot\gamma = E_1\,, \quad \nabla_{\dot\gamma} E_1 = \kappa_1 E_2\,, \quad \nabla_{\dot\gamma} E_2 = -\kappa_1 E_1 + \kappa_2 E_3\,, \quad \cdots\,,$$

$$\nabla_{\dot\gamma} E_{r-1} = -\kappa_{r-2} E_{r-2} + \kappa_{r-1} E_r\,, \quad \nabla_{\dot\gamma} E_r = -\kappa_{r-1} E_{r-1}$$

where $\kappa_1, \kappa_2, \cdots, \kappa_{r-1}$ are positive C^∞ functions of s. κ_j is called the j-th *curvature* of γ.

So, for example, a geodesic is a Frenet curve of osculating order 1; a circle is a Frenet curve of osculating order 2 with κ_1 a constant; a helix of order r is a Frenet curve of osculating order r, such that $\kappa_1, \kappa_2, \cdots, \kappa_{r-1}$ are constants.

THEOREM 3.3. *Let M be a mass-symmetric 2-type integral surface in $S^5(1)$ in E^6. Then M is locally the Riemannian product of a circle and a helix of order 4 or the product of two circles.*

PROOF. We shall prove that the X_1-curve is a helix of order 4 or a circle and the X_2-curve is a circle. Next we obtain that, under the hypothesis of Theorem 3.3, M lies fully in $S^5(1)$.

First of all we observe that for the second fundamental form h of M in E^6 we have

$$h(X_1, X_1) = a\xi_1 - x\,, \quad h(X_1, X_2) = c\xi_2\,, \quad h(X_2, X_2) = c\xi_1 - x\,. \tag{3.2}$$

From this and (3.1) we get

$$\begin{aligned} &\overline\nabla_{X_1} X_1 = a\xi_1 - x\,, \quad \overline\nabla_{X_1}\xi_1 = -aX_1 + \xi\,, \quad \overline\nabla_{X_1}\xi_2 = -cX_2\,, \\ &\overline\nabla_{X_1} x = X_1\,, \quad \overline\nabla_{X_1}\xi = = -\xi_1\,. \end{aligned} \tag{3.3}$$

Also we get

$$\begin{aligned} &\overline\nabla_{X_2} X_2 = c\xi_1 - x\,, \quad \overline\nabla_{X_2}\xi_1 = -cX_2\,, \quad \overline\nabla_{X_2}\xi_2 = -cX_1 + \xi\,, \\ &\overline\nabla_{X_2} x = X_2\,, \quad \overline\nabla_{X_2}\xi = -\xi_2\,, \quad \overline\nabla_{X_2} X_1 = c\xi_2\,. \end{aligned} \tag{3.4}$$

Let $X_1 = E_1$. From (3.3) we obtain

$$\bar{\nabla}_{E_1} E_1 = a\xi_1 - x = \kappa_1 E_2\,, \qquad \text{where } E_2 = \frac{a\xi_1 - x}{\sqrt{a^2+1}},\ \kappa_1 = \sqrt{a^2+1}\,.$$

$$\bar{\nabla}_{E_1} E_2 = -\sqrt{a^2+1}\,E_1 + \frac{a}{\sqrt{a^2+1}}\xi = -\kappa_1 E_1 + \kappa_2 E_3$$

where

$$E_3 = \xi,\ \kappa_2 = \frac{a}{\sqrt{a^2+1}} \quad \text{if } a>0\,, \quad \text{or } E_3 = -\xi,\ \kappa_2 = \frac{-a}{\sqrt{a^2+1}} \quad \text{if } a<0\,.$$

$$\bar{\nabla}_{E_1} E_3 = -\xi_1 = -\kappa_2 E_2 + \kappa_3 E_4\,,$$

where

$$E_4 = -\frac{\xi_1 + ax}{\sqrt{a^2+1}} \quad \text{if } a>0\,, \quad \text{or } E_4 = \frac{\xi_1 + ax}{\sqrt{a^2+1}} \quad \text{if } a<0\,, \quad \kappa_3 = \frac{1}{\sqrt{a^2+1}}\,.$$

$$\bar{\nabla}_{E_1} E_4 = -\frac{1}{\sqrt{a^2+1}}\xi = -\kappa_3 E_3 \qquad \text{if } a>0\,, \quad \text{or}$$

$$\bar{\nabla}_{E_1} E_4 = \frac{1}{\sqrt{a^2+1}}\xi = -\kappa_3 E_3 \qquad \text{if } a<0\,.$$

Thus $\kappa_4 = 0$ and the X_1-curve is a helix of order 4. The case $a=0$ corresponds to $\kappa_2 = 0$ and hence the X_1-curve is a circle.

Now we put $X_2 = v_1$. From (3.4) we obtain

$$\bar{\nabla}_{v_1} v_1 = c\xi_1 - x = \kappa_1 v_2\,, \qquad \text{where } v_2 = \frac{c\xi_1 - x}{\sqrt{c^2+1}},\ \kappa_1 = \sqrt{c^2+1}\,,$$

$$\bar{\nabla}_{v_1} v_2 = -\sqrt{c^2+1}\,v_1\,.$$

So $\kappa_2 = 0$ and the X_2-curve is a circle. This completes the proof of the theorem.

Now, on M we may choose local coordinates such that the immersion (2.3) is $x = x(u, v)$ with $x_u = X_1$ and $x_v = X_2$. Thus, from equations (3.3) and (3.4), by direct computation we find

$$(3.5)\qquad \begin{aligned} &\text{(i)} && x_{uuuu} + \frac{c^4+1}{c^2} x_{uu} + x = 0\,,\\ &\text{(ii)} && x_{vvv} + (c^2+1)x_v = 0\,,\\ &\text{(iii)} && c^2 x_{uu} - (c^2-1)x_{vv} + x = 0\,. \end{aligned}$$

We want to find the general solution of the system (3.5). We need the following lemma.

LEMMA 3.4. *Suppose $c^2 \neq 1$. Then the general solution of the ordinary differential equation*

$$f^{(iv)} + \frac{c^4+1}{c^2} f'' + f = 0 \tag{3.6}$$

is

$$f(t) = c_1 \cos ct + c_2 \sin ct + c^3 \cos \frac{t}{c} + c_4 \sin \frac{t}{c}, \tag{3.7}$$

$$c_i = \text{const.}, \qquad i = 1, 2, 3, 4 .$$

The functions $\cos ct$, $\sin ct$, $\cos t/c$, $\sin t/c$ *are linearly independent and the function* $f(t)$ *is periodic with period* $T = 2\pi\sqrt{lm}$ *if and only if* c^2 *is the rational number* $c^2 = l/m$, l, m *integers.*

PROOF. The differential equation (3.6) is of 4-th order, linear and homogeneous. So the general solution of this is given by (3.7). Let $A \cos ct + B \sin ct + C \cos t/c + D \sin t/c = 0$. If we take $t = 0$, πc, $2\pi c$, π/c, $2\pi/c$, we see that $A = B = C = D = 0$ unless $c^2 = 1$. So the functions $\cos ct$, $\sin ct$, $\cos t/c$, $\sin t/c$ are linearly independent.

If the function $f(t)$ is periodic with period T then

$$(c_1(\cos cT - 1) + c_2 \sin cT) \cos ct + (-c_1 \sin cT + c_2(\cos cT - 1)) \sin ct$$
$$+ \left(c_3\left(\cos \frac{T}{c} - 1 \right) + c_4 \sin \frac{T}{c} \right) \cos \frac{t}{c} + \left(-c_3 \sin \frac{T}{c} + c_4\left(\cos \frac{T}{c} - 1 \right) \right) \sin \frac{t}{c} = 0 .$$

Since the functions $\cos ct$, $\sin ct$, $\cos t/c$ and $\sin t/c$ are linearly independent we conclude that $cT = 2\pi l$ and $T/c = 2\pi m$ where l, m are integers. Thus the function $f(t)$ is periodic if and only if $c^2 = l/m$.

THEOREM 3.5. *Let $x : M \to S^5(1) \subset E^6$ be a mass-symmetric 2-type immersion of an integral surface M into $S^5(1)$. Then M lies fully in E^6 and the position vector $x = x(u, v)$ of M in E^6 is given by*

$$\begin{aligned} x = \frac{1}{\sqrt{c^2+1}} \Bigg[&\left(c \cos \frac{u}{c} \right) e_1 + (\sin cu \sin \sqrt{c^2+1}\, v) e_2 \\ &- (\sin cu \cos \sqrt{c^2+1}\, v) e_3 + \left(c \sin \frac{u}{c} \right) e_4 \\ &+ (\cos cu \sin \sqrt{c^2+1}\, v) e_5 - (\cos cu \cos \sqrt{c^2+1}\, v) e_6 \Bigg] \end{aligned} \tag{3.8}$$

where $c = \text{const.} \neq 0$ and $\{e_i\}$, $i = 1, \cdots, 6$, is an orthonormal basis of E^6.

PROOF. If $c^2 \neq 1$, according to Lemma 3.4, the general solution of the differential equation (3.5 (i)) is

$$x = A^1(v)\cos\frac{u}{c} + A^2(v)\sin cu + A^3(v)\sin\frac{u}{c} + A^4(v)\cos cu$$

where $A^i(v)$, $i=1, \cdots, 4$, are E^6-valued smooth functions of the variable v. Since the functions $\cos u/c$, $\sin cu$, $\sin u/c$, $\cos cu$ are linearly independent, every function $A^i(v)$ must be a solution of the equation (3.5 (ii)). So

$$A^i(v) = \frac{1}{\sqrt{c^2+1}}[(\sin\sqrt{c^2+1}v)A_1^i - (\cos\sqrt{c^2+1}v)A_2^i + cA_3^i], \qquad i=1, 2, 3, 4$$

where A_j^i, $i=1, \cdots, 4$, $j=1, 2, 3$, are constant vectors in E^6. Thus the solution of the equations (3.5) (i) and (ii) is given by

$$\begin{aligned} x = \frac{1}{\sqrt{c^2+1}}\Big[&(\sin\sqrt{c^2+1}vA_1^1 - \cos\sqrt{c^2+1}vA_2^1 + cA_3^1)\cos\frac{u}{c} \\ &+(\sin\sqrt{c^2+1}vA_1^2 - \cos\sqrt{c^2+1}vA_2^2 + cA_3^2)\sin cu \\ &+(\sin\sqrt{c^2+1}vA_1^3 - \cos\sqrt{c^2+1}vA_2^3 + cA_3^3)\sin\frac{u}{c} \\ &+(\sin\sqrt{c^2+1}vA_1^4 - \cos\sqrt{c^2+1}vA_2^4 + cA_3^4)\cos cu\Big]. \end{aligned}$$

On the other hand, from this and (3.5 (iii)) we find $(A_1^1, A_2^1, A_3^2, A_1^3, A_2^3, A_3^4) = (0, 0, 0, 0, 0, 0)$. Thus the position vector x of M is given by (3.8) where $e_1, \cdots, e_6$ are the constant vectors $A_3^1, A_1^2, A_2^2, A_3^3, A_1^4, A_2^4$, respectively.

As $x = x(u, v)$ in (3.8) is the solution of the differential system (3.5), we have at the point $x(0, 0)$

$$\begin{aligned} &x = \frac{1}{\sqrt{c^2+1}}(ce_1 - e_6), \quad x_u = \frac{1}{\sqrt{c^2+1}}(-ce_3 + e_4), \quad x_v = e_5, \\ &x_{uv} = ce_2, \quad x_{vv} = \sqrt{c^2+1}e_6, \quad x_{uvv} = c\sqrt{c^2+1}e_3. \end{aligned} \tag{3.9}$$

On the other hand, from (3.3) and (3.4) we find

$$\begin{aligned} &\langle x, x\rangle = 1, \quad \langle x, x_u\rangle = 0, \quad \langle x, x_v\rangle = 0, \quad \langle x, x_{uv}\rangle = 0, \\ &\langle x, x_{vv}\rangle = -1, \quad \langle x, x_{uvv}\rangle = 0, \quad \langle x_u, x_u\rangle = 1, \quad \langle x_u, x_v\rangle = 0, \\ &\langle x_u, x_{uv}\rangle = 0, \quad \langle x_u, x_{vv}\rangle = 0, \quad \langle x_u, x_{uvv}\rangle = -c^2, \quad \langle x_v, x_v\rangle = 1, \\ &\langle x_v, x_{uv}\rangle = 0, \quad \langle x_v, x_{vv}\rangle = 0, \quad \langle x_v, x_{uvv}\rangle = 0, \quad \langle x_{uv}, x_{uv}\rangle = c^2, \end{aligned} \tag{3.10}$$

$$\langle x_{uv}, x_{vv}\rangle=0\,,\quad \langle x_{uv}, x_{uvv}\rangle=0\,,\quad \langle x_{vv}, x_{vv}\rangle=c^2+1\,,\quad \langle x_{vv}, x_{uvv}\rangle=0\,,$$
$$\langle x_{uvv}, x_{uvv}\rangle=c^2(c^2+1)\,.$$

Combining (3.9) with (3.10) we obtain $\langle e_i, e_j\rangle=\delta_{ij}$.

If we have $c^2=1$, using a similar argument to that of the case $c^2\neq 1$ we obtain

$$x=\frac{1}{\sqrt{2}}[(\cos u)e_1+(\sin u\sin\sqrt{2}\,v)e_2-(\sin u\cos\sqrt{2}\,v)e_3$$
$$+(\sin u)e_4+(\cos u\sin\sqrt{2}\,v)e_5-(\cos u\cos\sqrt{2}\,v)e_6]\,.$$

Moreover, in this case the corresponding equations (3.9) and (3.10) are valid if we put $c=1$. If $c=-1$, changing the sign of e_1, e_2, e_3 gives the same result. Thus we again conclude $\langle e_i, e_j\rangle=\delta_{ij}$.

REMARK. Let $x: M\to S^n(1)$ be an isometric immersion of a compact surface M into the sphere $S^n(1)$. The total mean curvature is defined by

$$\tau(x)=\int_M(\alpha'^2+1)dV$$

where α' is the mean curvature of the surface M. The surface M is said to be stationary if

$$\delta\left(\int_M(\alpha'^2+1)dV\right)=0$$

for any δ, where δ is a normal variation. Weiner [10] shows that M is stationary if and only if

$$\Delta^{D'}H'=-2\alpha'^2H'+\frac{1}{\alpha'^2}(\operatorname{tr}A_{H'}^2)H'+\alpha'(H')\,, \tag{3.11}$$

(see also [1]). We obtain the following.

PROPOSITION 3.6. *If M is a mass-symmetric 2-type integral surface of $S^5(1)$, then M is not stationary.*

PROOF. Assume that M is stationary. From (2.15) we have that M is a Chen surface of $S^5(1)$, i.e. $\alpha'(H')=0$. Therefore, we obtain from (3.11)

$$\Delta^{D'}H'=\frac{\operatorname{tr}A_1}{2}\left(-\frac{(\operatorname{tr}A_1)^2}{2}+\operatorname{tr}A_1^2\right)\xi_1$$

and since $\operatorname{tr}A_1=a+c=(2c^2-1)/c\neq 0$,

$$\Delta^{D'}H'=\frac{2c^2-1}{4c^3}\xi_1\,.$$

On the other hand, from (2.12) we get

$$\Delta^{D'} H' = \frac{\operatorname{tr} A_1}{2} \xi_1 = \frac{2c^2 - 1}{2c} \xi_1 .$$

Therefore we have $2c^2 = 1$, a contradiction.

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