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On the Fractal Curves Induced from Endomorphisms on a Free Group of Rank 2

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0. Introduction.

Dekking showed in [3] and [4] that some endomorphisms θ on a free group of rank 2 provide us with fractal curves which induce several space tilings on $\boldsymbol{R}^2$.

In fact, let $G\langle a, b\rangle$ be a free group with generators a and b, a map $f: G\langle a, b\rangle \to \boldsymbol{Z}^2 \subset \boldsymbol{R}^2$ be a homomorphism, and L_θ be a linear representation of the endomorphism θ, that is, f and L_θ satisfies the commutative relation:

$$\begin{array}{ccc} G\langle a, b\rangle & \xrightarrow{\theta} & G\langle a, b\rangle \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\ \boldsymbol{R}^2 & \xrightarrow{L_\theta} & \boldsymbol{R}^2 . \end{array}$$

Let $K: G\langle a, b\rangle \to \boldsymbol{R}^2$ be a map which assigns to each element of $G\langle a, b\rangle$ a polygonal curve in the plane as follows: for $W = w_1 w_2 \cdots w_k \in G\langle a, b\rangle$, $K[W]$ is a polygon joining the points $f(w_1) + \cdots + f(w_j)$ $(1 \le j \le k)$ in order (exact definitions will be found in §1). In this situation, the following result is obtained.

THEOREM ([3], [4]). *Let θ be an endomorphism of $G\langle a, b\rangle$ satisfying the following conditions*:

(1) *θ has short range cancellations, that is, for any reduced word stu $(s, t, u \in \{a^{\pm 1}, b^{\pm 1}\})$, cancellation does not erase all letters of any of the subwords $\theta(s)$, $\theta(t)$ and $\theta(u)$ in $\theta(stu)$,*

(2) *L_θ is expansive, that is, the absolute values of both eigenvalues of L_θ are greater than* 1,

(3) *$K[\theta(aba^{-1}b^{-1})]$ is double point free.*

Then there exists a limit set K_θ of $L_\theta^{-n}K[\theta^n(aba^{-1}b^{-1})]$ as a curve and the set F_θ enclosed by K_θ is a space tiling set of $\boldsymbol{R}^2$:

$$\bigcup_{\alpha \in \boldsymbol{Z}^2} (F_\theta + \alpha) = \boldsymbol{R}^2 ,$$

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$$\mathrm{int}(F_\theta+\alpha)\cap\mathrm{int}(F_\theta+\alpha')=\varnothing \qquad \textit{if } \alpha\neq\alpha' \textit{ and } \alpha,\alpha'\in\boldsymbol{Z}^2$$

and satisfies self-similarity: that is,

$$L_\theta F_\theta=\bigcup_{j=1}^{|\det L_\theta|}(F_\theta+\alpha_j),$$

where $\alpha_1,\cdots,\alpha_j\in\boldsymbol{Z}^2$ *are chosen as follows: Let* U *be the unit square,* $F[\theta(aba^{-1}b^{-1})]$ *be the closed set enclosed by* $K[\theta(aba^{-1}b^{-1})]$. *Then there exist* $\alpha_j\in\boldsymbol{Z}^2$ $(1\leq j\leq|\det L_\theta|)$ *such that* $F[\theta(aba^{-1}b^{-1})]=\bigcup_{j=1}^{|\det L_\theta|}(U+\alpha_j)$.

This is the assertion by Dekking. Our purpose of this paper is to reconstruct the result more precisely, because the treatment of the case in which the endomorphism has cancellations is not clear in his works. But we have many interesting examples which have short range cancellations. Essential idea in this paper is to construct an endomorphism $\hat{\Theta}$ of free group of rank 3 which has no cancellations, starting from an endomorphism of $G\langle a,b\rangle$ which has cancellations. Using this idea, we prove Dekking's assertion precisely, and if L_θ is isomorphic to a rotation followed by a scalar multiplication λ_θ, we are able to calculate the Hausdorff dimension of K_θ as

$$\dim_H K_\theta=\frac{\log\lambda_{\hat{\Theta}}}{\log|\lambda_\theta|},$$

where $\lambda_{\hat{\Theta}}$ is a maximum solution of a cubic equation (see §6).

1. Definitions and notations.

Let $G=G\langle a,b\rangle$ be a free group with two generators a and b. We consider G as the quotient set of the free semigroup S^* generated by $S:=\{a,b,a^{-1},b^{-1}\}$ by the equivalence relation $\sim$, where we denote $W\sim V$ if W and V determine the same element after cancellation and we call elements of G *reduced words*.

Let $f: G\to\boldsymbol{Z}^2\subset\boldsymbol{R}^2$ be the canonical homomorphism, i.e., f is determined by

$$f(a)=\begin{pmatrix}1\\0\end{pmatrix},\qquad f(b)=\begin{pmatrix}0\\1\end{pmatrix}$$

and the relations

$$f(W^{-1})=-f(W),\quad f(VW)=f(V)+f(W)\qquad \text{for } V,W\in G,$$

where VW means the reduced word of the concatenation of V and W.

We define a map K, which assigns polygonal curves in the plane to reduced words, by

$$K[s]=\{\alpha f(s): 0\leq\alpha\leq 1\}\qquad \text{for}\quad s\in S,$$

and for any reduced word $W=s_1\cdots s_m$ $(s_i\in S)$, by

$$K[W]=\bigcup_{i=1}^{m}(K[s_i]+f(s_1\cdots s_{i-1})),$$

where we denote $A+z=\{a+z : a\in A\}$ for $A\subset \boldsymbol{R}^2$ and $z\in\boldsymbol{R}^2$. We call a reduced word W *closed* if $f(W)=0$. In this case, we modify the definition of $K[W]$ slightly: for any closed word W,

$$K[W]:=f(A)+K[W'],$$

where A is the longest word satisfying the decomposition $W=AW'A^{-1}$ (reduced).

We further consider an endomorphism θ of G, i.e., θ is determined by $\theta(a)$, $\theta(b)$ and the relations

$$\theta(W^{-1})=(\theta(W))^{-1}, \quad \theta(VW)=\theta(V)\theta(W) \qquad \text{for} \quad V, W\in G.$$

Let L_θ be a linear representation of θ on $\boldsymbol{R}^2$, i.e., L_θ is a linear mapping which satisfies the commutative relation:

$$\begin{array}{ccc} G & \xrightarrow{\theta} & G \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ \boldsymbol{R}^2 & \xrightarrow{L_\theta} & \boldsymbol{R}^2. \end{array}$$

Let M_θ be a matrix representation of L_θ, then $M_\theta=(m_{ij})$ is given by

$$\begin{pmatrix} m_{11} \\ m_{21}\end{pmatrix}=f(\theta(a)) \quad\text{and}\quad \begin{pmatrix} m_{12} \\ m_{22}\end{pmatrix}=f(\theta(b)).$$

We say that an endomorphism θ has *short range cancellations* if for any reduced word stu $(s, t, u\in S)$, cancellation does not erase all letters of any of the subwords $\theta(s)$, $\theta(t)$ and $\theta(u)$ in $\theta(stu)\in S^*$.

We mention here the assumption made for the endomorphisms in this paper.

ASSUMPTION 1. We assume the endomorphism θ of G satisfies the following conditions:

(1) the polygonal curve $K[\theta(aba^{-1}b^{-1})]$ is double point free,

(2) the linear representation map L_θ is expansive, i.e., the absolute values of the eigenvalues λ_i, $i=1, 2$, of M_θ are greater than 1,

(3) θ has short range cancellations.

We say *Assumption* 1′ *is satisfied* if L_θ is isomorphic to a rotation followed by a scalar multiplication λ_θ instead of Assumption 1 (2). Under Assumption 1, for each endomorphism θ we will see the existence of a "fractal" curve K_θ as a limit set of $L_\theta^{-n}K[\theta^n(aba^{-1}b^{-1})]$ and that the set F_θ which is enclosed by K_θ has a space tiling property.

PROPOSITION 1.1 (Retiling Principle). *Let F_n be the closed bounded set enclosed by $K[\theta^n(aba^{-1}b^{-1})]$. Then $K[\theta^n(aba^{-1}b^{-1})]$ and F_n have the following properties: for each n,*

(1) *the curve $K[\theta^n(aba^{-1}b^{-1})]$ is double point free,*

(2) *(n-step space tiling)*

$$\bigcup_{\alpha\in L_\theta^n(\mathbf{Z}^2)} (F_n+\alpha)=\mathbf{R}^2,$$

$$\operatorname{int}(F_n+\alpha)\cap\operatorname{int}(F_n+\alpha')=\varnothing, \qquad \text{if} \quad \alpha\neq\alpha'\in L_\theta^n(\mathbf{Z}^2),$$

(3) *(n-step self-similarity)*

$$F_{n+1}=\bigcup_{\alpha_i\in\mathscr{D}} (F_n+L_\theta^n\alpha_i)$$

where a subset $\mathscr{D}$ of the lattice $\mathbf{Z}^2$ is chosen so that $F_1=\bigcup_{\alpha\in\mathscr{D}}(F_0+\alpha)$ and therefore, the cardinality of $\mathscr{D}$ is equal to $|\det L_\theta|$.

PROOF. From the fact that F_0 is the unit square we know $\bigcup_{\alpha\in\mathbf{Z}^2}(F_0+\alpha)=\mathbf{R}^2$ and $\operatorname{int}(F_0+\alpha)\cap\operatorname{int}(F_0+\alpha')=\varnothing$ $(\alpha\neq\alpha'\in\mathbf{Z}^2)$. Therefore, we know also that the set $L_\theta(F_0)$ has a space tiling property, that is, $\bigcup_{\alpha\in L_\theta(\mathbf{Z}^2)}(L_\theta(F_0)+\alpha)=\mathbf{R}^2$ and $\operatorname{int}(L_\theta(F_0)+\alpha)\cap\operatorname{int}(L_\theta(F_0)+\alpha')=\varnothing$ $(\alpha\neq\alpha'\in L_\theta(\mathbf{Z}^2))$. This space tiling is divided by the net

$$\bigcup_{\alpha\in L_\theta(\mathbf{Z}^2)} L_\theta(K[aba^{-1}b^{-1}]+\alpha),$$

we call it *a tiling net*, and this is constructed by lines $L_\theta K[s]$, $s\in\{a,b\}$. Now replace the lines $L_\theta K[s]$ with the polygonal curves $K[\theta(s)]$, $s\in\{a,b\}$. Then, by Assumption 1 (1), we have

$$\bigcup_{\alpha\in L_\theta(\mathbf{Z}^2)} (F_1+\alpha)=\mathbf{R}^2,$$

$$\operatorname{int}(F_1+\alpha)\cap\operatorname{int}(F_1+\alpha')=\varnothing \qquad (\alpha\neq\alpha'\in L_\theta(\mathbf{Z}^2))$$

and

$$F_1=\bigcup_{\alpha\in\mathscr{D}} (F_0+\alpha),$$

that is, F_1 has first step space tiling and self-similar properties in Proposition 1.1 and its tiling net is given by

$$\bigcup_{\alpha\in L_\theta(\mathbf{Z}^2)} (K[\theta(aba^{-1}b^{-1})]+\alpha)$$

which is constructed from $K[s]$, $s\in\{a,b\}$. Let us consider again a space tiling given by $\bigcup_{\alpha\in L_\theta^2(\mathbf{Z}^2)}(L_\theta(F_1)+\alpha)$. Then the tiling net is constructed by the lines $L_\theta K[s]$, $s\in\{a,b\}$. Therefore, replace the lines $l_\theta K[s]$ with the polygonal curves $K[\theta(s)]$, $s\in\{a,b\}$. Then we have again a second step space tiling whose net is constructed by $K[\theta^2(aba^{-1}b^{-1})]$, and also we see second step self-similarity:

$$F_2=\bigcup_{\alpha\in\mathscr{D}}(F_1+L_\theta\alpha)\,.$$

We are able to continue this procedure and obtain the results.

We call this procedure *Retiling Principle.*

2. First reduction.

Let θ be an endomorphism of G satisfying Assumption 1. For any $W\in G$ we define the *adjoint* θ_W with respect to W of the endomorphism θ by

$$\theta_W(V)=W\theta(V)W^{-1}\qquad\text{for any}\quad V\in G\,.$$

Then from the definition of L_θ we see easily the following lemma.

LEMMA 2.1. *For any $W\in G$ the linear representation L_{θ_W} of the adjoint θ_W coincides with L_θ.*

LEMMA 2.2. *For any $W\in G$, the closed curve $K[\theta_W^n(aba^{-1}b^{-1})]$ is congruent to $K[\theta^n(aba^{-1}b^{-1})]$; more explicitly,*

$$K[\theta_W^n(aba^{-1}b^{-1})]=\sum_{k=0}^{n-1}f(\theta^kW)+K[\theta^n(aba^{-1}b^{-1})]\qquad\text{for all}\quad n\in\boldsymbol{N}\,.$$

PROOF. From the definition of θ_W, we see $\theta_W(aba^{-1}b^{-1})=W\theta(aba^{-1}b^{-1})W^{-1}$. Therefore, for any n,

$$\theta_W^n(aba^{-1}b^{-1})=W_n\theta^n(aba^{-1}b^{-1})W_n^{-1}\,,$$

where $W_n=W\theta(W)\cdots\theta^{n-1}(W)$. Thus, from the definition of the map K, we see

$$K[\theta_W^n(aba^{-1}b^{-1})]=\sum_{k=0}^{n-1}f(\theta^kW)+K[\theta^n(aba^{-1}b^{-1})]\,.$$

We are interested in the shape of the limit set of $L_\theta^{-n}K[\theta^n(aba^{-1}b^{-1})]$. By Lemmas 2.1 and 2.2, we know $L_\theta^{-n}K[\theta_W^n(aba^{-1}b^{-1})]$ given by the adjoint θ_W is congruent to $L_\theta^{-n}K[\theta^n(aba^{-1}b^{-1})]$ for any $W\in G$. Therefore, for the investigation of the shape of the limit set, it is enough to discuss the shape given by θ_W.

LEMMA 2.3. *Supppose we represent $\theta(a)$ and $\theta(b)$ as follows:*

$$\theta(a)=vAx\,,\quad\theta(b)=yBz\qquad(x,y,z,v\in S)\,,$$

then we see that

$\theta(b^{-1})\theta(a)$ *has cancellations iff* $v=y$,

$\theta(a)\theta(b)$ *has cancellations iff* $x=y^{-1}$,

$\theta(b)\theta(a^{-1})$ *has cancellations iff* $z=x$, *and*

$\theta(a^{-1})\theta(b^{-1})$ *has cancellations iff* $z=v^{-1}$.

The proof is easy.

Let us consider the set of four pairs P_θ:

$$P_\theta=\{(\theta(s)\theta(t)) : (s, t)=(b^{-1}, a), (a, b), (b, a^{-1}), (a^{-1}, b^{-1})\} ,$$

the pairs of which constitute the pieces of the closed word $\theta(aba^{-1}b^{-1})$. By Lemma 2.3, we have easily

LEMMA 2.4. *If any three pairs of* P_θ *have cancellations, then all four pairs of* P_θ *have cancellations.*

LEMMA 2.5. *If all pairs of* P_θ *have cancellations, then there exists a word* $W \in G$ *such that the adjoint* θ_W *does not have cancellations for at least two pairs of* P_{θ_W}.

PROOF. If all pairs of P_θ have cancellations, then by Lemma 2.3, there exists an $x \in S$ such that

$$\theta(a)=x^{-1}Ax , \qquad \theta(b)=x^{-1}Bx .$$

Let W be the *longest* word satisfying the decomposition

$$\theta(a)=W^{-1}AW , \qquad \theta(b)=W^{-1}BW ,$$

and let us consider the adjoint θ_W with respect to W of the endomorphism θ. Then the adjoint θ_W is given by

$$\theta_W(a)=A , \qquad \theta_W(b)=B .$$

If the adjoint θ_W has cancellations for some three pairs in P_{θ_W}, then by Lemma 2.4 the adjoint θ_W has cancellations for all pairs. This contradicts the choice of W.

LEMMA 2.6. *If an endomorphism* θ *has cancellations for at most* 2 *pairs of* P_θ, *then there exists a word* W *such that the adjoint* θ_W *has cancellations for only one pair of* P_{θ_W}.

PROOF. We will give a proof only for the following case: (1) $\theta(b^{-1})\theta(a)$ and $\theta(a)\theta(b)$ does not have cancellations and (2) $\theta(b)\theta(a^{-1})$ and $\theta(b)\theta(a)$ have cancellations. The conclusions for other cases are obtained in the same manner. By (1), θ is denoted as follows: for some $x, y, z \in S$,

$$\theta(a)=xAz , \qquad \theta(b)=yB$$

and

$$x \neq y\,, \qquad z \neq y^{-1}\,.$$

On the other hand, from (2) we have: for some $x \in S$,

$$\theta(a) = xAx^{-1}\,, \qquad \theta(b) = B'x^{-1}\,.$$

Therefore, the endomorphism is denoted as follows: for some x, y ($x \neq y \in S$)

$$\theta(a) = xAx^{-1}\,, \qquad \theta(b) = yBx^{-1}\,.$$

Let W be the longest word satisfying the form: $\theta(a) = WA''W^{-1}$ and $\theta(b) = yB''W^{-1}$, and let us consider the adjoint θ_W of θ with respect to W. Then θ_W is denoted by

$$\theta_W(a) = A''\,, \qquad \theta_W(b) = W^{-1}yB''\,.$$

The adjoint θ_W has no cancellation at $\theta_W(a)\theta_W(b)$ nor at $\theta_W(b^{-1})\theta_W(a)$ and also cannot have cancellations at both $\theta_W(b)\theta_W(a^{-1})$ and $\theta_W(b)\theta_W(a)$ simultaneously. In fact, if both of $\theta_W(b)\theta_W(a^{-1})$ and $\theta_W(b)\theta_W(a)$ have cancellations, then this contradicts the choice of W.

LEMMA 2.7. (1) *If an endomorphism θ has cancellations only at $\theta(b^{-1})\theta(a)$, then there exists a word $W \in G$ such that the adjoint θ_W has cancellations only at $\theta_W(b)\theta_W(a^{-1})$.*

(2) *If an endomorphism θ has cancellations only at $\theta(a)\theta(b)$, then there exists a word $W \in G$ such that the adjoint θ_W has a cancellation only at $\theta_W(a^{-1})\theta_W(b^{-1})$.*

PROOF. From the assumption of (1), there exists $x, y, z \in S$ such that

$$\theta(a) = yAx\,, \qquad \theta(b) = yBz$$

and

$$x \neq z\,, \quad x \neq y^{-1}\,, \quad z \neq y^{-1}\,.$$

Let W be the longest word satisfying

$$\theta(a) = W^{-1}Ax\,, \qquad \theta(b) = W^{-1}Bz\,.$$

Take the adjoint θ_W with respect to W given by

$$\theta_W(a) = AxW^{-1}\,, \qquad \theta_W(b) = BzW^{-1}\,.$$

Then we see that θ_W has cancellations only at $\theta_W(b)\theta_W(a^{-1})$. Statement (2) is obtained in the same manner.

THEOREM 2.1 (First Reduction Theorem). *Let θ be an endomorphism satisfying Assumption 1, and θ has cancellations. Then we can choose a word W such that the adjoint θ_W satisfies one of the following conditions*:

(1) *θ_W has a cancellation only at $\theta_W(b)\theta_W(a^{-1})$ in four pairs*

$$\{\theta_W(s)\theta_W(t) : (s, t) = (a, b), (b, a^{-1}), (a^{-1}, b^{-1}), (b^{-1}, a)\}\,,$$

(2) *θ_W has a cancellation only at $\theta_W(a^{-1})\theta_W(b^{-1})$ in four pairs*

$$\{\theta_W(s)\theta_W(t) : (s, t)=(a, b), (b, a^{-1}), (a^{-1}, b^{-1}), (b^{-1}, a)\},$$

(3) *θ_W has no cancellations.*

Let η be an automorphism on G defined by

$$\eta : \begin{cases} a \to b^{-1} \\ b \to a . \end{cases}$$

Then the inverse η^{-1} and the matrix representation L_η are given by

$$\eta^{-1} : \begin{cases} a \to b \\ b \to a^{-1} \end{cases} \quad \text{and} \quad L_\eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

LEMMA 2.8. *If the endomorphism θ has a cancellation only at $\theta(a^{-1})\theta(b^{-1})$, then the endomorphism $\theta' := \eta\theta\eta^{-1}$ has a cancellation only at $\theta'(b)\theta'(a^{-1})$.*

PROOF. We see from the definition of η that $\theta(a^{-1})\theta(b^{-1})$ has a cancellation iff $\theta(\eta^{-1}(b))\theta(\eta^{-1}(a^{-1}))$ has a cancellation and iff $\eta\theta\eta^{-1}(b)\eta\theta\eta^{-1}(a^{-1})$ has a cancellation. Therefore, the case that θ has a cancellation at $\theta(a^{-1})\theta(b^{-1})$ can be reduced to the case that θ' has a cancellation at $\theta'(b)\theta'(a^{-1})$.

Concerning the reduction of Lemma 2.8, we see in the following way that the shape of $K[\theta^n(aba^{-1}b^{-1})]$ and $K[\theta'^n(aba^{-1}b^{-1})]$ are essentially the same.

LEMMA 2.9. *Let $\theta' = \eta\theta\eta^{-1}$, then $L_\theta^{-n}K[\theta^n(aba^{-1}b^{-1})]$ is congruent to $L_{\theta'}^{-n}K[\theta'^n(aba^{-1}b^{-1})]$. More explicitly, we have*

$$L_\theta^{-n}K[\theta^n(aba^{-1}b^{-1})] = L_\eta^{-1}(L_{\theta'}^{-n}K[\theta'^n(aba^{-1}b^{-1})] + f(b^{-1})) .$$

PROOF. From the definition of η and the map K, we have

$$\begin{aligned} K[\theta^n(aba^{-1}b^{-1})] &= K[\eta^{-1}\theta'^n(b^{-1}aba^{-1})] \\ &= K[\eta^{-1}\theta'^n(b^{-1}aba^{-1}b^{-1}b)] \\ &= K[\eta^{-1}\theta'^n(aba^{-1}b^{-1})] + f(\eta^{-1}\theta'^n(b^{-1})) . \end{aligned}$$

Therefore, from $L_\theta^{-1} = L_\eta^{-1}L_{\theta'}^{-1}L_\eta$, we get

$$L_\theta^{-n}K[\theta^n(aba^{-1}b^{-1})] = L_\eta^{-1}L_{\theta'}^{-n}L_\eta K[\eta^{-1}\theta'^n(aba^{-1}b^{-1})] + L_\eta^{-1}L_{\theta'}^{-n}f(\theta'^n(b^{-1})) .$$

From the relation

$$L_\eta K[\eta^{-1}(W)] = K[W] \qquad \text{for} \quad W \in G ,$$

we have

$$L_\theta^{-n}K[\theta^n(aba^{-1}b^{-1})] = L_\eta^{-1}L_{\theta'}^{-n}K[\theta'^n(aba^{-1}b^{-1})] + L_\eta^{-1}f(b^{-1}) .$$

THEOREM 2.2 (First Reduction Theorem). *Let θ be an endomorphism satisfying Assumption* 1. *Then considering an adjoint θ_W and taking $\eta\theta_W\eta^{-1}$, if necessary, we can reduce θ to an endomorphism θ' satisfying one of the following*:

(1) *θ' has a cancellation only at $\theta(b)\theta(a^{-1})$, or*

(2) *θ' has no cancellations.*

3. Lifting endomorphisms.

In this section, we induce an endomorphism, which is called a lifting endomorphism, on a free group of rank 3. By First Reduction Theorem and Lemmas 2.8 and 2.9, we assume that the endomorphism θ has a cancellation only at $\theta(b)\theta(a^{-1})$ from now on. Then $\theta(a)$ and $\theta(b)$ are uniquely decomposed by some words $A, B, C \in G$ as follows:

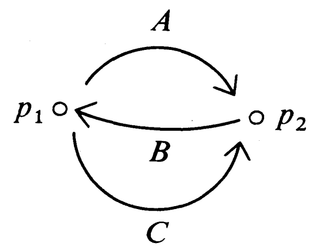
$$\theta(a)=AB\,, \qquad \theta(b)=CB \tag{3-1}$$

where

not only AB, CB but also BC, BA, $C^{-1}A$ and CA^{-1} have no cancellations. (3-2)

This is possible because $\theta(a)\theta(b)$, $\theta(a^{-1})\theta(b^{-1})$ and $\theta(b^{-1})\theta(a)$ have no cancellations. For any $W=s_1s_2\cdots s_k \in G$, $\theta(W)$ is represented as a product of A, B, C, A^{-1}, B^{-1} and C^{-1} according to (3-1). Moreover from the property (3-2) we have the following proposition.

PROPOSITION 3.1. *Let $\mathscr{G}$ be a directed graph which is constructed from* 2 *terminals, named p_1 and p_2, and* 3 *arcs, named A, B and C, as follows*:



and for any $w \in G$ let us write $\theta(w)$ as a sequence of $A^{\pm1}$, $B^{\pm1}$, $C^{\pm1}$ according to (3-1) *as follows*:

$$\theta(w)=A_1A_2\cdots A_k\,.$$

Then the sequence $A_1A_2\cdots A_k$ is $\mathscr{G}$-admissible, that is, the sequence constitutes a path of the directed graph $\mathscr{G}$.

We call $\theta(w)=A_1A_2\cdots A_k$ the *block representation* of $\theta(w)$. We see that cancellations of $\theta(s)\theta(t)$ causes *block* cancellations. We introduce a free group $\tilde{G}=\tilde{G}\langle A, B, C\rangle$ of rank 3 where the words A, B and C are regarded as generators. Let $i\colon \tilde{G}\to G$ be a homomorphism sending the generators A, B, C of $\tilde{G}$ to the words A, B, C of G, and define an endomorphism Θ of $\tilde{G}$, which is called a *lifting endomorphism* of θ, as follows:

for $W \in \tilde{G}\langle A, B, C\rangle$

$$\Theta(W) := \text{the block representation of } \theta(i(W)). \tag{3-3}$$

Then we have the following proposition by using the relation

$$i(ABCA^{-1}B^{-1}C^{-1}) = \theta(aba^{-1}b^{-1}).$$

PROPOSITION 3.2. *The following diagram commutes*:

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\Theta} & \tilde{G} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ G & \xrightarrow{\theta} & G, \end{array}$$

and in particular,

$$i(\Theta^{n-1}(ABCA^{-1}B^{-1}C^{-1})) = \theta^n(aba^{-1}b^{-1}).$$

4. Second reduction.

Starting from an endomorphism θ which has short range cancellations, we have obtained a reduced endomorphism θ' and its lifting Θ on $\tilde{G}$, and we noted that the investigation of $\theta'^n(aba^{-1}b^{-1})$ and $\theta^n(aba^{-1}b^{-1})$ is equivalent to that of $\Theta^n(ABCA^{-1}B^{-1}C^{-1})$ in §3. In this section, we construct a second reduced endomorphism $\hat{\Theta}$ which has no cancellations. By Theorem 2.1, Lemma 2.8 and Lemma 2.9, we can assume that the endomorphism θ has a cancellation only at $\theta(b)\theta(a^{-1})$. Then $\theta(a)$ and $\theta(b)$ are decomposed as in (3-1) and the words A, B, C satisfy (3-2). Let us decompose A, B and C as follows:

$$A = vA't, \quad B = yB'x, \quad C = wC'u \qquad (v, t, y, x, w, u \in S,\ A', B', C' \in G). \tag{4-1}$$

Then from (3-2) we have the following relations:

$$\begin{aligned} &y \neq t^{-1}, u^{-1} \\ &t \neq u \\ &x \neq v^{-1}, w^{-1} \\ &v \neq w. \end{aligned} \tag{4-2}$$

LEMMA 4.1. *Under the assumption that θ has a cancellation only at $\theta(b)\theta(a^{-1})$, we see that*

(1) *$\theta(A)\theta(B)$ has cancellations iff*

$$(t, u, y) \in \{(a, a^{-1}, b^{-1}), (a, b^{-1}, b^{-1}), (b, b^{-1}, a^{-1}), (b, a^{-1}, a^{-1})\},$$

(2) *$\theta(C)\theta(A^{-1})$ has cancellations iff*

$$(t, u, y) \in \{(b, a, a), (b, a, b), (a, b, a), (a, b, b)\}\ ,$$

(3) *$\theta(B^{-1})\theta(C^{-1})$ has cancellations iff*

$$(t, u, y) \in \{(a^{-1}, a, b^{-1}), (b^{-1}, a, b^{-1}), (a^{-1}, b, a^{-1}), (b^{-1}, b, a^{-1})\}\ ,$$

(4) *$\theta(B)\theta(C)$ has cancellations iff*

$$(x, v, w) \in \{(a, a, b^{-1}), (a, b, b^{-1}), (b, a, a^{-1}), (b, b, a^{-1})\}\ ,$$

(5) *$\theta(A^{-1})\theta(B^{-1})$ has cancellations iff*

$$(x, v, w) \in \{(a, b^{-1}, a), (a, b^{-1}, b), (b, a^{-1}, a), (b, a^{-1}, b)\}\ ,$$

(6) *$\theta(C^{-1})\theta(A)$ has cancellations iff*

$$(x, v, w) \in \{(a^{-1}, a^{-1}, b^{-1}), (b^{-1}, a^{-1}, b^{-1}), (a^{-1}, b^{-1}, a^{-1}), (b^{-1}, b^{-1}, a^{-1})\}\ .$$

PROOF. In case (1), $\theta(A)\theta(B)$ has a cancellation iff $\theta(t)\theta(y)$ has a cancellation, that is, $(t, y) = (a, b^{-1})$ or (b, a^{-1}). On the other hand, from (4-2), we know that $u \neq t, y^{-1}$ and $y \neq t^{-1}, u^{-1}$. Therefore

$$(t, u, y) \in \{(a, a^{-1}, b^{-1}), (a, b^{-1}, b^{-1}), (b, b^{-1}, a^{-1}), (b, a^{-1}, a^{-1})\}\ .$$

Other cases are obtained in the same manner.

LEMMA 4.2. *Under the same assumpiton as in Lemma* 4.1, *we have*

(1) (i) *The possibilities for $\theta(A)\theta(B)$, $\theta(C)\theta(A^{-1})$ or $\theta(B^{-1})\theta(C^{-1})$ to have cancellations are mutually exclusive.*

(ii) *Let us denote*

$$\begin{aligned} \theta(A) &= A_1 A_2 \cdots A_s \\ \theta(B) &= B_1 B_2 \cdots B_t \\ \theta(C) &= C_1 C_2 \cdots C_u\ , \end{aligned} \tag{4-3}$$

where $A_i, B_j, C_k \in \{A, B, C, A^{-1}, B^{-1}, C^{-1}\}$. Then we have

$$(A_s, B_1, C_u) = \begin{cases} (B, B^{-1}, \text{not } B) & \text{if } \theta(A)\theta(B) \text{ has cancellations.} \\ (B, \text{not } B^{-1}, B) & \text{if } \theta(C)\theta(A^{-1}) \text{ has cancellations.} \\ (\text{not } B, B^{-1}, B) & \text{if } \theta(B^{-1})\theta(C^{-1}) \text{ has cancellations}\ . \end{cases}$$

(2) (i) *The possibilities for $\theta(B)\theta(C)$, $\theta(A^{-1})\theta(B^{-1})$ or $\theta(C^{-1})\theta(A)$ to have cancellations are mutually exclusive.*

(ii) *Let $\theta(A)$, $\theta(B)$ and $\theta(C)$ be as in* (4-3). *Then we have*

$$(A_1, B_t, C_1) = \begin{cases} (\text{not } B^{-1}, B, B^{-1}) & \text{if } \theta(B)\theta(C) \text{ has cancellations.} \\ (B^{-1}, B, \text{not } B) & \text{if } \theta(A^{-1})\theta(B^{-1}) \text{ has cancellations.} \\ (B^{-1}, \text{not } B, B^{-1}) & \text{if } \theta(C^{-1})\theta(A) \text{ has cancellations.} \end{cases}$$

PROOF. The statements (1-i) and (2-i) are obtained from Lemma 4.1. The statements (1-ii) and (2-ii) are obtained in the following manner: Assume that $\theta(A)\theta(B)$ has cancellations, then $\theta(t)\theta(y)$ has a cancellation and $\theta(B^{-1})\theta(C^{-1})$ has no cancellations by (1-i). Therefore we know from (3-1), (4-1) and the fact that $(t, y)=(a, b^{-1})$ or (b, a^{-1}) that

$$A_s=B \quad \text{and} \quad B_1=B^{-1}$$

and from the fact that $\theta(B^{-1})\theta(C^{-1})$ has no cancellations, $B_1^{-1}C_u^{-1}$ has no cancellations, that is,

$$C_u=\text{not } B .$$

Statement (2-ii) is obtained in the same manner.

We now define the second reduced endomorphism $\hat{\Theta}$ of Θ using Lemma 4.2:

Case (1): if $\theta(A)\theta(B)$, $\theta(C)\theta(A^{-1})$ or $\theta(B^{-1})\theta(C^{-1})$ has cancellations, and $\theta(B)\theta(C)$, $\theta(A^{-1})\theta(B^{-1})$ and $\theta(C^{-1})\theta(A)$ have no cancellations, then set

$$\begin{aligned} \hat{\Theta}(A)&=\Theta(A)B^{-1} \\ \hat{\Theta}(B)&=B\Theta(B) \\ \hat{\Theta}(C)&=\Theta(C)B^{-1} . \end{aligned} \tag{4-4}$$

Case (2): if $\theta(B)\theta(C)$, $\theta(A^{-1})\theta(B^{-1})$ or $\theta(C^{-1})\theta(A)$ has cancellations and $\theta(A)\theta(B)$, $\theta(C)\theta(A^{-1})$ and $\theta(B^{-1})\theta(C^{-1})$ have no cancellations, then set

$$\begin{aligned} \hat{\Theta}(A)&=B\Theta(A) \\ \hat{\Theta}(B)&=\Theta(B)B^{-1} \\ \hat{\Theta}(C)&=B\Theta(C) . \end{aligned} \tag{4-5}$$

Case (3): if $\theta(A)\theta(B)$, $\theta(C)\theta(A^{-1})$ or $\theta(B^{-1})\theta(C^{-1})$ has cancellations, and $\theta(B)\theta(C)$, $\theta(A^{-1})\theta(B^{-1})$ or $\theta(C^{-1})\theta(A)$ have no cancellations, then set

$$\begin{aligned} \hat{\Theta}(A)&=B\Theta(A)B^{-1} \\ \hat{\Theta}(B)&=B\Theta(B)B^{-1} \\ \hat{\Theta}(C)&=B\Theta(C)B^{-1} . \end{aligned} \tag{4-6}$$

Then we see

THEOREM 4.1 (Second Reduction Theorem). *The endomorphism $\hat{\Theta}$ of $\tilde{G}$ has no cancellations on any $\mathscr{G}$-admissible words.*

PROOF. We give a proof for the case (1) above. Let $\mathscr{A}$ be the set of all $\mathscr{G}$-admissible words of length 2, that is, $\mathscr{A}^+ :=\{AB, BC, CA^{-1}, A^{-1}B^{-1}, B^{-1}C^{-1}, C^{-1}A\}$ and $\mathscr{A}=\mathscr{A}^+ \cup \mathscr{A}^-$, where $\mathscr{A}^-=\{T^{-1}S^{-1} \mid ST\in\mathscr{A}^+\}$. Note that $\theta(S)\theta(T)$ has cancellations iff $\Theta(S)\Theta(T)$ has cancellations. In (1) we assume that the endomorphism Θ has cancellations only at $\Theta(A)\Theta(B)$, $\Theta(C)\Theta(A^{-1})$ or $\Theta(B^{-1})\Theta(C^{-1})$ and has no cancellations at $\Theta(B)\Theta(C)$, $\Theta(A^{-1})\Theta(B^{-1})$ and $\Theta(C^{-1})\Theta(A)$. Let us assume that $\Theta(A)\Theta(B)$ has a cancellation. Then we know by Lemma 4.2 with notation (4-3) that $(A_s, B_1, C_u)=(B, B^{-1}$, not $B)$ and B_tA_1, $C_1^{-1}A_1$ and B_tC_1 have no cancellations. Therefore, the reduced endomorphism $\hat{\Theta}$:

$$\hat{\Theta}(A)=A_1A_2\cdots A_{s-1}$$

$$\hat{\Theta}(B)=B_2\cdots B_t$$

$$\hat{\Theta}(C)=C_1\cdots C_uB$$

has no cancellations at $\hat{\Theta}(S)\hat{\Theta}(T)$ for $ST\in\mathscr{A}$, that is, $\hat{\Theta}$ has no cancellations on any $\mathscr{G}$-admissible words. Other cases also are proved in the same manner.

At the end of this section, we discuss the relation between the endomorphisms Θ and $\hat{\Theta}$. Let $\hat{G}$ be the subgroup of $\tilde{G}$ of all $\mathscr{G}$-admissible words with even length. We remark that elements of $\hat{G}$ correspond exactly to closed paths of the graph $\mathscr{G}$. Let

$$\hat{\hat{G}}=\{W\in\hat{G} \mid f(i(W))=0\}.$$

We call words of $\hat{\hat{G}}$ *circle words*. We define two circle words $V=A_1\cdots A_{2k}$ and $W=B_1\cdots B_{2k}$ to be equivalent if there exists some j such that

$$B_jB_{j+1}\cdots B_{2k}B_1\cdots B_{j-1}=A_1\cdots A_{2k}$$

and denote it $V\equiv W\,(circle)$. Then we have the following proposition.

PROPOSITION 4.1. *Let Θ be the lifting of θ and $\hat{\Theta}$ be the second reduction of Θ. Then we have that*

(1) *the subgroup $\hat{G}\subset\tilde{G}$ is invariant under both Θ and $\hat{\Theta}$,*

(2) *for any $W\in\hat{\hat{G}}$,*

$$\Theta(W)\equiv\hat{\Theta}(W)\ (circle).$$

In particular,

(3) $\hat{\Theta}^n(ABCA^{-1}B^{-1}C^{-1})\equiv\Theta^n(ABCA^{-1}B^{-1}C^{-1})$ (*circle*).

PROOF. The statement (1) is trivial from the definitions of Θ, $\hat{\Theta}$ and (3-1). Let us assume $\hat{\Theta}$ is reduced as in case (1). For any $W=A_1A_2\cdots A_{2k}\in\hat{G}$, if the starting terminal of A_1 is p_1, then

$$\begin{aligned}\hat{\Theta}(A_{2j-1}A_{2j})&=\hat{\Theta}(A_{2j-1})\hat{\Theta}(A_{2j})\\&=\Theta(A_{2j-1})B^{-1}B\Theta(A_{2j})\\&=\Theta(A_{2j-1})\Theta(A_{2j})\\&=\Theta(A_{2j-1}A_{2j})\end{aligned}$$

and so

$$\hat{\Theta}(A_1 \cdots A_{2k}) = \Theta(A_1 \cdots A_{2k}),$$

and if the starting terminal of A_1 is p_2, then

$$\hat{\Theta}(A_{2j-1}A_{2j}) = B\Theta(A_{2j-1}A_{2j})B^{-1}$$

and so

$$\hat{\Theta}(A_1 \cdots A_{2k}) = B\Theta(A_1 \cdots A_{2k})B^{-1}.$$

This is the conclusion (2). Other cases are obtained in the same way.

5. Polygonal curve associated with the reduced endomorphism $\hat{\Theta}$.

In this section, we discuss the relation between polygonal curves $L_\theta^{-n} K[\theta^n(aba^{-1}b^{-1})]$ and $L_{\hat{\Theta}}^{-n+1} K[\hat{\Theta}^n(ABCA^{-1}B^{-1}C^{-1})]$.

Let a map $\tilde{f}$: $\tilde{G} = \tilde{G}\langle A, B, C\rangle \to \boldsymbol{Z}^3 \subset \boldsymbol{R}^3$ be a canonical homomorphism, i.e. $\tilde{f}$ is given by $\tilde{f}(A) = {}^t(1, 0, 0)$, $\tilde{f}(B) = {}^t(0, 1, 0)$ and $\tilde{f}(C) = {}^t(0, 0, 1)$ and L_Θ be the linear representation of the endomorphism Θ. Then the following diagram commutes:

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\Theta} & \tilde{G} \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ \boldsymbol{R}^3 & \xrightarrow{L_\Theta} & \boldsymbol{R}^3 . \end{array} \tag{5-1}$$

The matrix representation M_Θ of L_Θ is given as follows: put

$$\begin{pmatrix} m_a \\ m_b \end{pmatrix} := f(A), \quad \begin{pmatrix} n_a \\ n_b \end{pmatrix} := f(B) \quad \text{and} \quad \begin{pmatrix} o_a \\ o_b \end{pmatrix} := f(C).$$

Then from (3-1) M_θ and M_Θ are given by

$$M_\theta = \begin{pmatrix} m_a + n_a & o_a + n_a \\ m_b + n_b & o_b + n_b \end{pmatrix} \tag{5-2}$$

and

$$M_\Theta = \begin{pmatrix} m_a & n_a & o_a \\ m_a + m_b & n_a + n_b & o_a + o_b \\ m_b & n_b & o_b \end{pmatrix}. \tag{5-3}$$

And $M_{\hat{\Theta}}$ is given by

$$M_{\hat{\Theta}} = M_\Theta + T_N, \qquad N = 1, 2 \text{ or } 3, \tag{5-4}$$

where

$$T_1=\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{in case (1), that is, } \hat{\Theta} \text{ is reduced by (4-4)}.$$

$$T_2=\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{in case (2), that is, } \hat{\Theta} \text{ is reduced by (4-5)}.$$

$$T_3=T_1+T_2=0 \qquad \text{in case (3), that is, } \hat{\Theta} \text{ is reduced by (4-6)}.$$

LEMMA 5.1. *Let us denote the characteristic polynomials of the matrices M_θ, M_Θ and $M_{\hat{\Theta}}$ by Φ_θ, Φ_Θ and $\Phi_{\hat{\Theta}}$, respectively. Then we have*

$$\Phi_{\hat{\Theta}}(\lambda)=\begin{cases} (\lambda-1)\Phi_\theta(\lambda) & \textit{in case } (1) \\ (\lambda+1)\Phi_\theta(\lambda) & \textit{in case } (2) \\ \lambda\Phi_\theta(\lambda)\,(=\Phi_\Theta(\lambda)) & \textit{in case } (3)\,. \end{cases}$$

The proof is easy.

LEMMA 5.2. *Put $v_1=\vec{f}(A)+\vec{f}(B)={}^t(1,1,0)$, $v_2=\vec{f}(C)+\vec{f}(B)={}^t(0,1,1)$ and denote by $\mathscr{L}(v_1, v_2)$ the plane spanned by v_1 and v_2. Then we have*

(1) *$\mathscr{L}(v_1, v_2)$ is invariant under both L_Θ and $L_{\hat{\Theta}}$.*

(2) *The matrix representations of $L_\Theta|_{\mathscr{L}(v_1,v_2)}$ and $L_{\hat{\Theta}}|_{\mathscr{L}(v_1,v_2)}$ with respect to the base (v_1, v_2) coincide with M_θ.*

The proof is easy.

LEMMA 5.3. *Let W be a word of $\hat{G}$. Then we have*

$$\vec{f}(W)\in\mathscr{L}(v_1, v_2)\,.$$

PROOF. It is easy to see that for any $ST\in\mathscr{A}$ as in the proof of Theorem 4.1,

$$\vec{f}(ST)\in\mathscr{L}(v_1, v_2)\,.$$

Therefore $\vec{f}(W)$ belongs to $\mathscr{L}(v_1, v_2)$ for any $W\in\hat{G}$.

Let a map π be a homomorphism from $\mathscr{L}(v_1, v_2)$ to $\boldsymbol{R}^2$, i.e., a map $\pi: \mathscr{L}(v_1, v_2)\to\boldsymbol{R}^2$ satisfies

$$\pi(v_1)=\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \pi(v_2)=\begin{pmatrix}0\\1\end{pmatrix}.$$

PROPOSITION 5.1. *The following commutative relations hold*:

(1)
$$\begin{array}{ccc} \hat{G} & \xrightarrow{\Theta} & \hat{G} \\ {\scriptstyle \tilde{f}}\downarrow & & {\scriptstyle \tilde{f}}\downarrow \\ \mathscr{L}(v_1, v_2) & \xrightarrow{L_\Theta} & \mathscr{L}(v_1, v_2) \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi}\downarrow \\ \boldsymbol{R}^2 & \xrightarrow{L_\theta} & \boldsymbol{R}^2 , \end{array}$$

(2)
$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\hat{\Theta}} & \tilde{G} \\ {\scriptstyle \tilde{f}}\downarrow & & {\scriptstyle \tilde{f}}\downarrow \\ \mathscr{L}(v_1, v_2) & \xrightarrow{L_{\hat{\Theta}}} & \mathscr{L}(v_1, v_2) \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi}\downarrow \\ \boldsymbol{R}^2 & \xrightarrow{L_\theta} & \boldsymbol{R}^2 . \end{array}$$

PROOF. We already knew that the commutative relation (5-1) holds. We know also by Lemma 5.2 that $\mathscr{L}(v_1, v_2)$ is L_Θ and $L_{\hat{\Theta}}$-invariant, and we know by Proposition 4.1 (1) that $\hat{G}$ is Θ and $\hat{\Theta}$-invariant. We know also by Lemma 5.3 that

$$\tilde{f}(W) \in \mathscr{L}(v_1, v_2) \qquad \text{for any } W \in \hat{G} .$$

Therefore, the following diagrams commute:

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\Theta} & \hat{G} \\ {\scriptstyle \tilde{f}}\downarrow & & {\scriptstyle \tilde{f}}\downarrow \\ \mathscr{L}(v_1, v_2) & \xrightarrow{L_\Theta} & \mathscr{L}(v_1, v_2) \end{array} \quad \text{and} \quad \begin{array}{ccc} \hat{G} & \xrightarrow{\hat{\Theta}} & \hat{G} \\ {\scriptstyle \tilde{f}}\downarrow & & {\scriptstyle \tilde{f}}\downarrow \\ \mathscr{L}(v_1, v_2) & \xrightarrow{L_{\hat{\Theta}}} & \mathscr{L}(v_1, v_2) . \end{array}$$

Using the relation of Lemma 5.2 (2), we obtain the conclusion.

LEMMA 5.4. *For any* $W \in \hat{G}$

$$\pi(\tilde{f}(W)) = L_\theta^{-1}(f(i(W))) .$$

PROOF. It is sufficient to see that for any $ST \in \mathscr{A}$

$$\pi(\tilde{f}(ST)) = L_\theta^{-1}(f(i(ST))) .$$

We know that

$$i(AB) = \theta(a) , \quad i(AC^{-1}) = \theta(ab^{-1}) ,$$
$$i(CB) = \theta(b) , \quad (i(CA^{-1}) = \theta(ba^{-1}))$$

and

$$\tilde{f}(AB) = \tilde{f}(BA), \quad \tilde{f}(CB) = \tilde{f}(BC) .$$

Therefore

$$L_\theta^{-1} f(i(AB)) = L_\theta^{-1} f(\theta(a)) = f(a) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \pi(\tilde{f}(AB)) .$$

Similarly we have

$$L_\theta^{-1} f(i(CB)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \pi(\tilde{f}(CB))$$

and

$$L_\theta^{-1} f(i(AC^{-1})) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \pi(\tilde{f}(AC^{-1}))\,.$$

Combining Lemma 5.4 and Proposition 5.1, we have the proposition:

PROPOSITION 5.2. *The following diagrams commute*:

(1)

$$\begin{tikzcd}
 & \hat{G} \arrow[rr, "\hat{f}"] \arrow[dd, "i"] & & \mathscr{L}(v_1, v_2) \arrow[dd, "\pi"] \\
\hat{G} \arrow[ur, "\Theta"] \arrow[rr, "\hat{f}"] \arrow[dd, "i"'] & & \mathscr{L}(v_1, v_2) \arrow[ur, "L_\Theta"'] \arrow[dd, "\pi"] & \\
 & G \arrow[rr, "L_\theta^{-1} f"] & & \boldsymbol{R}^2 \\
G \arrow[ur, "\theta"] \arrow[rr, "L_\theta^{-1} f"'] & & \boldsymbol{R}^2 \arrow[ur, "L_\theta"'] &
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & \hat{G} \arrow[rr, "\hat{f}"] \arrow[dd, "i"] & & \mathscr{L}(v_1, v_2) \arrow[dd, "\pi"] \\
\hat{G} \arrow[ur, "\hat{\Theta}"] \arrow[rr, "\hat{f}"] \arrow[dd, "i"'] & & \mathscr{L}(v_1, v_2) \arrow[ur, "L_{\hat{\Theta}}"'] \arrow[dd, "\pi"] & \\
 & G \arrow[rr, "L_\theta^{-1} f"] & & \boldsymbol{R}^2 \\
G \arrow[ur, "\theta"] \arrow[rr, "L_\theta^{-1} f"'] & & \boldsymbol{R}^2 \arrow[ur, "L_\theta"'] &
\end{tikzcd}$$

Let us define a map $\tilde{K}\colon \tilde{G} \to \boldsymbol{R}^2$ as follows:

$$\tilde{K}[W] := L_\theta^{-1} K[i(W)] \qquad \text{for} \quad W \in \tilde{G}\,.$$

LEMMA 5.5. *For any $W \in \hat{G}$,*

(1) *end points of the curve $\tilde{K}[W]$ coincide with end points of the polygon $L_\theta^{-1}\tilde{K}[\Theta(W)]$,*

(2) *end points of the curve $\tilde{K}[W]$ coincide with end points of the polygon $L_\theta^{-1}\tilde{K}[\hat{\Theta}(W)]$.*

PROOF. It is sufficient to see that the statements are valid for any $W\in\mathscr{A}$. From the definition of $\tilde{K}$, the end points of $\tilde{K}[W]$ are given by $\begin{pmatrix}0\\0\end{pmatrix}$ and $L_\theta^{-1}f(i(W))$. On the other hand the end points of $L_\theta^{-1}\tilde{K}[\Theta(W)]$ are given by $\begin{pmatrix}0\\0\end{pmatrix}$ and $L_\theta^{-1}L_\theta^{-1}f(i(\Theta(W)))$. From Proposition 5.2 (1), we see

$$L_\theta^{-1}L_\theta^{-1}f(i(\Theta(W)))=L_\theta^{-1}L_\theta L_\theta^{-1}f(i(W))=L_\theta^{-1}f(i(W))\,.$$

This is nothing but the statement (1). The statement (2) is obtained in the same manner.

PROPOSITION 5.3. *Let $\mathscr{K}$ be the set of compact subsets of $\boldsymbol{R}^2$ and d be the Hausdorff metric on $\mathscr{K}$. Then there exists a limit set $K_{\hat{\Theta}}$ such that*

$$K_{\hat{\Theta}}=\lim_{n\to\infty}L_\theta^{-n}\tilde{K}[\hat{\Theta}^n(ABCA^{-1}B^{-1}C^{-1})]$$

where the limit is taken in the Hausdorff metric.

PROOF. We already knew that $\hat{\Theta}^n[ABCA^{-1}B^{-1}C^{-1}]$ is $\mathscr{G}$-admissible and $\hat{\Theta}$ has no cancellations on $\mathscr{G}$-admissible words. Put $\hat{\Theta}^n(ABCA^{-1}B^{-1}C^{-1})=A_1^{(n)}A_2^{(n)}\cdots A_{2k}^{(n)}$. Then

$$\begin{aligned}\tilde{K}[\hat{\Theta}^n(ABCA^{-1}B^{-1}C^{-1})]=\tilde{K}[A_1^{(n)}A_2^{(n)}]\cup(L_\theta^{-1}f(i(A_1^{(n)}A_2^{(n)}))+\tilde{K}[A_3^{(n)}A_4^{(n)}])\cup\\ \cdots\cup(L_\theta^{-1}f(i(A_1^{(n)}\cdots A_{2k-2}^{(n)}))+\tilde{K}[A_{2k-1}^{(n)}A_{2k}^{(n)}])\,.\end{aligned}$$

On the other hand, from the fact that the endomorphism $\hat{\Theta}$ has no cancellations and that $L_\theta^{-1}fi(\hat{\Theta}(W))=f(i(W))$ as shown in Proposition 5.2, we have

$$\begin{aligned}&L_\theta^{-1}[\tilde{K}(\hat{\Theta}^{n+1}(ABCA^{-1}B^{-1}C^{-1}))]\\ &=L_\theta^{-1}[\tilde{K}(\hat{\Theta}(A_1^{(n)}A_2^{(n)})\cdots\hat{\Theta}(A_{2k-1}^{(n)}A_{2k}^{(n)}))]\\ &=L_\theta^{-1}\tilde{K}[\hat{\Theta}(A_1^{(n)}A_2^{(n)})]\cup(L_\theta^{-1}f(i(A_1^{(n)}A_2^{(n)}))+L_\theta^{-1}\tilde{K}[\hat{\Theta}(A_3^{(n)}A_4^{(n)})])\\ &\quad\cdots\cup(L_\theta^{-1}(f(i(A_1^{(n)}\cdots A_{2k-2}^{(n)}))+L_\theta^{-1}\tilde{K}[\hat{\Theta}(A_{2k-1}^{(n)}A_{2k}^{(n)})])\,.\end{aligned}$$

Put $d_0=\max_{W\in\mathscr{A}}d(\tilde{K}[W],L_\theta^{-1}\tilde{K}[\hat{\Theta}(W)])$. Then from Lemma 5.5 we have

$$d(\tilde{K}[\hat{\Theta}^n(ABCA^{-1}B^{-1}C^{-1})],L_\theta^{-1}\tilde{K}[\hat{\Theta}^{n+1}(ABCA^{-1}B^{-1}C^{-1})])=d_0$$

and for any n

$$d(L_\theta^{-n}\tilde{K}[\hat{\Theta}^n(ABCA^{-1}B^{-1}C^{-1})],L_\theta^{-n-1}\tilde{K}[\hat{\Theta}^{n+1}(ABCA^{-1}B^{-1}C^{-1})])=d_0|\lambda_\theta|^{-n}\,,$$

where $|\lambda_\theta|$ is the maximum of absolute values of eigenvalues λ_1, λ_2 of M_θ. Therefore, from the completeness of the metric space $(\mathscr{K},d)$, there exists a limit set $K_{\hat{\Theta}}$ for $L_\theta^{-n}\tilde{K}(\hat{\Theta}^n(ABCA^{-1}B^{-1}C^{-1}))$.

We already knew in Proposition 3.2 that

$$i(\Theta^n(ABCA^{-1}B^{-1}C^{-1}))=\theta^{n+1}(aba^{-1}b^{-1}),$$

and, from the definition of $\tilde{K}$,

$$L_\theta^{-n}\tilde{K}[\Theta^n(ABCA^{-1}B^{-1}C^{-1})]=L_\theta^{-(n+1)}K[\theta^{n+1}(aba^{-1}b^{-1}))]. \tag{5-5}$$

Therefore we have the theorem.

THEOREM 5.1. *There exists a set K_θ as a limit of $L_\theta^{-n}K[\theta^n(aba^{-1}b^{-1})]$ and K_θ is congruent to $K_{\hat{\Theta}}$.*

PROOF. By Proposition 4.1, we know

$$\hat{\Theta}^n(ABCA^{-1}B^{-1}C^{-1})\equiv\Theta^n(ABCA^{-1}B^{-1}C^{-1}) \ (circle),$$

that is, if we denote $\Theta^n(ABCA^{-1}B^{-1}C^{-1})=A_1^{(n)}\cdots A_{2k}^{(n)}$, then there exists $j=j(n)$ such that

$$\hat{\Theta}^n(ABCA^{-1}B^{-1}C^{-1})=A_j^{(n)}A_{j+1}^{(n)}\cdots A_{2k}^{(n)}A_1^{(n)}\cdots A_{j-1}^{(n)}.$$

Therefore, we have

$$\begin{aligned}\tilde{K}[\hat{\Theta}^n(ABCA^{-1}B^{-1}C^{-1})]&=L_\theta^{-1}K[i(A_j^{(n)}\cdots A_{j-1}^{(n)})]\\&=L_\theta^{-1}f(i(A_j^{(n)}\cdots A_{2k}^{(n)}))+L_\theta^{-1}K[i(A_1^{(n)}\cdots A_{2k}^{(n)})]\\&=L_\theta^{-1}f(i(A_j^{(n)}\cdots A_{2k}^{(n)}))+\tilde{K}[\Theta^n(ABCA^{-1}B^{-1}C^{-1})].\end{aligned}$$

Put $f_n=L_\theta^{-(n+1)}f(i(A_j^{(n)}\cdots A_{2k}^{(n)}))\in\boldsymbol{R}^2$. Then from the boundedness of $L_\theta^{-n}\tilde{K}[\hat{\Theta}^n(ABC A^{-1}B^{-1}C^{-1})]$, the set $\{f_n: n=1, 2, \cdots\}$ is bounded. Therefore there exists a subsequence f_{n_k} such that $f_{n_k}\to f$. Thus from (5-5) and Proposition 5.3, the limit K_θ of $L_\theta^{-(n+1)}\tilde{K}[\theta^{n+1}(aba^{-1}b^{-1})]$ also exists, and satisfies the relation:

$$K_{\hat{\Theta}}=K_\theta+f.$$

6. Fractal curves and Hausdorff dimension.

In this section we prove that the limit set K_θ is a curve when $\hat{\Theta}$ is irreducible and aperiodic.

Let X be a set of (one-sided) infinite $\mathcal{G}$-admissible sequence, i.e.,

$$X=\{A_1A_2\cdots \mid A_i\in\{A^{\pm1}, B^{\pm1}, C^{\pm1}\},\ A_1A_2\cdots A_n \text{ is } \mathcal{G}\text{-admissible for any } n\}$$

and let $N_{\hat{\Theta}}=(n_{ij})$ be its *structure matrix* with respect to the endomorphism $\hat{\Theta}$, that is, putting $A_1=A$, $A_2=B$, $A_3=C$ and $n_{ij}=$the number of occurrence of A_i or A_i^{-1} in $\hat{\Theta}(A_j)$.

In addition to Assumption 1 or 1′, we make the following assumption:

ASSUMPTION 2. $N_{\hat{\Theta}}$ is *irreducible* and *aperiodic*, that is, there exists an n such that all the elements of $N_{\hat{\Theta}}^n$ are positive.

Then by Assumption 2, there exists uniquely a maximum eigenvalue $\lambda_{\hat{\Theta}}$ (>1) of $N_{\hat{\Theta}}$ and its eigen row vector (x_A, x_B, x_C) with $x_A, x_B, x_C>0$ and $x_A+x_B+x_C=1$ (Frobenius' theorem). Let us denote

$$\hat{\Theta}(S)=A_1\cdots A_{j(S)} \qquad \text{for} \quad S\in\{A^{\pm 1}, B^{\pm 1}, C^{\pm 1}\}.$$

Then we define the partition $\{I_{A_i} : 1\le i\le j(S)\}$ of an interval J associated with $\hat{\Theta}(S)$ as follows:

(i) $J=\bigcup_{i=1}^{j(S)} I_{A_i}$ (disjoint union),

(ii) The ratios of length of intervals satisfy

$$|I_{A_1}|:|I_{A_2}|:\cdots:|I_{A_{j(S)}}|=x_{A_1}:x_{A_2}:\cdots:x_{A_{j(S)}},$$

(iii) The order of intervals of (i) is the same as the order of corresponding letters in $\hat{\Theta}(S)$.

We consider partitions ξ_m $(m\in \boldsymbol{N})$ of $I=[0,1]$ associated with $\hat{\Theta}^m$ $(ABCA^{-1}B^{-1}C^{-1})$. Put $\hat{\Theta}^m(ABCA^{-1}B^{-1}C^{-1})=A_1^{(m)}\cdots A_{j(m)}^{(m)}$. Then the partition $\xi_m=\{I_{A_i^{(m)}} : 1\le i\le j(m)\}$ of the interval $I=[0,1]$ associated with $\hat{\Theta}^m(ABCA^{-1}B^{-1}C^{-1})$ is defined as follows:

(i) For $m=0$, let

$$\xi_0=\{I_A, I_B, I_C, I_{A^{-1}}, I_{B^{-1}}, I_{C^{-1}}\} \quad\text{and}\quad |I_S|=|I_{S^{-1}}|=\tfrac{1}{2}x_S,\ S\in\{A, B, C\},$$

(ii) starting from the partition ξ_{m-1}, the partition ξ_m is constructed by gathering the partitions of the interval $I_{A_i^{(m-1)}}$ in ξ_{m-1} associated with $\hat{\Theta}(A_i^{(m-1)})$ defined above.

We denote these partitions

$$\xi_m=\{I_{A_j^{(m)}} : \hat{\Theta}^m(ABCA^{-1}B^{-1}C^{-1})=A_1^{(m)}\cdots A_{j(m)}^{(m)}\}.$$

Then from the fact that (x_A, x_B, x_C) is an eigenvector corresponding to the eigenvalue $\lambda_{\hat{\Theta}}$ (>1), we obtain the following lemma.

LEMMA 6.1. *For the partition ξ_m $(m\ge 1)$,*

(i) *ξ_m is a refinement of ξ_{m-1}.*

(ii) *The length of intervals of ξ_m is estimated uniformly by $|I_{A_j^{(m)}}|\sim 1/|\lambda_{\hat{\Theta}}|^m$.*

Let $\varphi_m : I\to \boldsymbol{R}^2$ $(m\ge 0)$ be a polygonal map given by

$$\varphi_m(I_{A_j^{(m)}})=L_\theta^{-(m+1)}\left(\sum_{k=1}^{j-1} f(i(A_k^{(m)}))+\tilde{K}[A_j^{(m)}]\right) \qquad \text{for each } I_{A_j^{(m)}}\in\xi_m.$$

Then from the proof of Proposition 5.3 and Lemma 6.1, we have

PROPOSITION 6.1. (1) $\varphi_m(I)=L_\theta^{-m}K[\hat{\Theta}^m(ABCA^{-1}B^{-1}C^{-1})]$.

(2) *Let $\varphi : I\to \boldsymbol{R}^2$ be a map defined by*

$$\varphi(t)=\lim_{m\to\infty}\varphi_m(I_{A_{j(t)}^{(m)}})$$

where $I_{A_{j(t)}^{(m)}}$ is chosen by the condition $t \in I_{A_j^{(m)}}$. Then φ is continuous and

$$\varphi(I) = K_{\hat{\Theta}} .$$

LEMMA 6.2. *Under Assumption 2, the following inequality holds*:

$$\lambda_{\hat{\Theta}} > |\det L_\theta| .$$

PROOF. Denote the number of segments $\tilde{K}(A)$, $\tilde{K}(B)$ and $\tilde{K}(C)$ constituting ∂F by the vector ${}^t(2, 2, 2)$, where F is defined in Proposition 1.1. Then the number of segments of ∂F_1 is represented by $N_{\hat{\Theta}}{}^t(2, 2, 2)$. On the other hand, the area of polygon F_1 is given by $|\det L_\theta|$ and the curve ∂F_1 is double point free. Therefore, we have the following inequality:

$$|\det L_\theta| \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} > N_{\hat{\Theta}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix},$$

where ${}^t(x_1, x_2, x_3) > {}^t(y_1, y_2, y_3)$ means that $x_i \geq y_i$ for all i and $x_i > y_i$ for some i. From Assumption 2, there exists an eigen row vector $\boldsymbol{x} = {}^t(x_1, x_2, x_3)$ of $N_{\hat{\Theta}}$ with respect to $\lambda_{\hat{\Theta}}$ such that $x_i > 0$ for all i. Therefore, we see

$${}^t\boldsymbol{x}(|\det L_\theta| - \lambda_{\hat{\Theta}}) \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = {}^t\boldsymbol{x}(|\det L_\theta| E_3 - N_{\hat{\Theta}}) \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} > 0$$

(E_3 denotes the identity matrix of size 3).

Now we can estimate the Hausdorff dimension of K_θ or $K_{\hat{\Theta}}$.

PROPOSITION 6.2. *An estimate for* $\dim K_\theta$ *from above is given by*

$$\dim K_\theta \leq 1 + \frac{\log \lambda_{\hat{\Theta}} - \log|\lambda_1|}{\log|\lambda_2|} < 2$$

where $1 < |\lambda_1| \leq |\lambda_2|$ are the moduli of the eigenvalues of L_θ.

PROOF. The length of $\theta^n(aba^{-1}b^{-1})$ is of the same order as $\lambda_{\hat{\Theta}}^n$ by Proposition 3.2 and Proposition 4.1 (3). Therefore, we have the first inequality (see [1], [2]). The second inequality is obtained by Lemma 6.2.

Finally we estimate the value of Hausdorff dimension of curve K_θ under Assumption 1′, that is,

L_θ is isomorphic to a rotation followed by a scalar multiplication by λ_θ (>1).

LEMMA 6.2. *Let μ be a measure on $K_{\hat{\Theta}}$ which is induced from Lebesgue measure λ on I by the map φ. Put $r_0 = \log|\lambda_{\hat{\Theta}}|/\log|\lambda_1|$. Then the measure μ satisfies the following property: there exists a constant c such that $\mu(B) \leq c|B|^{r_0}$ for any ball B, where $|B|$*

means the radius of B.

PROOF. Let B_r be a ball with radius r. Then $\mu(B_r)=\sum_{k=1}^{j(m)}\mu(\varphi(I_{A_k^{(m)}})\cap B_r)$ for all m where $\hat{\Theta}^m(ABCA^{-1}B^{-1}C^{-1})=A_1^{(m)}\cdots A_{j(m)}^{(m)}$. From Lemma 6.1, there exists a constant c such that

$$\mu(B_r)\leq c\lambda_{\hat{\Theta}}^{-m}\#\{j:\varphi(I_{A_j^{(m)}})\cap B_r\neq\varnothing\}\qquad\text{for all}\quad m\,.$$

We choose m so that $|\lambda_\theta|^{-(m+1)}<r\leq|\lambda_\theta|^{-m}$. Then we have

$$\mu(B_r)\leq c' r^{\log|\lambda_{\hat{\Theta}}|/\log|\lambda_\theta|}\#\{j:\varphi(I_{A_j^{(m)}})\cap B_r\neq\varnothing\}\,.$$

We note that the cardinality of $\{j:\varphi(I_{A_j^{(m)}})\cap B_r\neq\varnothing\}$ is smaller than the cardinality of $\{j:L_\theta^m(\varphi(I_{A_j^{(m)}}))\cap B_1\neq\varnothing\}$. We know that the mesh $\bigcup_{\alpha\in Z^2}(K_{\hat{\Theta}}+\alpha)$ of R^2 is constructed by

$$K_{\hat{\Theta}}(S)=\lim_{n\to\infty}L_\theta^{-n}\tilde{K}[\hat{\Theta}^n(S)]\qquad\text{for}\quad S\in\{A^{\pm1},B^{\pm1},C^{\pm1}\}$$

and from the definition of φ_m and φ we see

$$L_\theta^m(\varphi(I_{A_j^{(m)}}))=K_{\hat{\Theta}}(A_j^{(m)})\,.$$

Therefore the cardinality of $\{j:L_{\hat{\Theta}}^m(\varphi(I_{A_j^{(m)}}))\cap B_1\neq\varnothing\}$ is bounded.

Therefore, by Frostman's lemma [1], we have the following estimate from below:

THEOREM 6.1. (1) *If the endomorphism θ satisfies Assumption* 1 *and Assumption* 2, *then the Hausdorff dimension of K_θ satisfies*

$$\dim_H K_\theta\leq 1+\frac{\log\lambda_{\hat{\Theta}}-\log|\lambda_1|}{\log|\lambda_2|}<2\,.$$

(2) *If the endomorphism θ satisfies Assumption* 1′ *and Assumption* 2, *then the Hausdorff dimension of K_θ is given by*

$$\dim_H K_\theta=\frac{\log\lambda_{\hat{\Theta}}}{\log|\lambda_\theta|}\,.$$

Now we state our main goal:

THEOREM 6.2. *Let θ be an endomorphism satisfying the Assumption* 1 *and Assumption* 2. *Then there exists a limit set K_θ of $L_\theta^{-n}K[\theta^n(aba^{-1}b^{-1})]$ as a Jordan curve, and the curve K_θ satisfies the following properties:*

Let F_θ be the closed set enclosed by K_θ, that is, $\partial F_\theta=K_\theta$, and $\mathscr{D}$ be a subset of Z^2 defined in Proposition 1.1 (3). *Then we have*

(1) (*Space tiling*)

$$\bigcup_{\alpha\in Z^2}(F_\theta+\alpha)=R^2$$

and

$$\mathrm{int}(F_\theta+\alpha)\cap\mathrm{int}(F_\theta+\alpha')=\varnothing \qquad \text{if } \alpha\neq\alpha', \text{ and } \alpha, \alpha'\in \boldsymbol{Z}^2 .$$

(2) (*Self-similarity*)

$$L_\theta F_\theta=\bigcup_{\alpha_j\in\mathscr{D}}(F_\theta+\alpha_j) .$$

PROOF. From Property 1.1, Theorem 5.1 and Proposition 6.1, we obtain the result.

From the theorem, we obtain a corollary on ergodic theory.

COROLLARY [2]. *Let $M=(m_{ij})$ be a 2×2 nonsingular matrix with integral elements m_{ij} and expansive. Let T_M: $\boldsymbol{T}^2\to\boldsymbol{T}^2$ be a group endomorphism associated with M, that is,*

$$T_M(x)=Mx \pmod{\boldsymbol{Z}^2}$$

and μ be the Lebesgue measure on $\boldsymbol{T}^2$. Then we obtain a Bernoulli partition of the dynamical system $(\boldsymbol{T}^2, T_M, \mu)$ constructively.

PROOF. For any M satisfying the above assumption, choose an endomorphism θ satisfying Assumption 1 and $M_\theta=M$. For the endomorphism θ, we know the existence of F_θ satisfying Theorem 6.2. The set F_θ is a fundamental domain for $\boldsymbol{Z}^2$ acting on $\boldsymbol{R}^2$. Therefore F_θ is isomorphic to $\boldsymbol{T}^2$. Let $\xi=\{A_j: 1\leq j\leq|\det M|\}$ be a partition of F_θ $(=\boldsymbol{T}^2)$ such that

$$A_j=L_\theta^{-1}(F_\theta+\alpha_j) , \qquad \alpha_j\in\mathscr{D} .$$

Then ξ is a Bernoulli partition of the group endomorphism $(\boldsymbol{T}^2, T_M, \mu)$.

7. Generalization and examples.

In this paper, we have been assuming that the endomorphism has only short range cancellations. We know many examples which have "long" range cancellations. Therefore we give a weaker condition for the cancellation of θ. We say that the endomorphism θ has short range cancellation *in a wide sense* if θ satisfies the following conditions:

> there exist endomorphisms τ and θ', both of which have only short range cancellations and satisfy the relation

$$\theta\tau=\tau\theta' \quad \text{and} \quad \theta(aba^{-1}b^{-1})=\tau(aba^{-1}b^{-1}) . \tag{7-1}$$

REMARK. We also say θ has short range cancellations in a wide sense if there exists a word $W\in G$ such that the adjoint endomorphism θ_W of θ satisfies the condition above.

LEMMA 7.1. *Assume that the endomorphism θ has short range cancellations in a* *wide sense. Then the limit set of* $L_\theta^{-n}K[\theta^{n+1}(aba^{-1}b^{-1})]$ *coincides with the limit set of* $L_\tau L_{\theta'}^{-n}K[\theta'^n(aba^{-1}b^{-1})]$.

PROOF. From the assumption, we see $\theta^n\tau(aba^{-1}b^{-1})=\tau\theta'^n(aba^{-1}b^{-1})$, and $f(\theta\tau(s))=L_\tau f(\theta'(s))$ for $s\in S$. By the condition of short range cancellation for τ, we know

$$d(K[\theta^n\tau(aba^{-1}b^{-1})),\ L_\tau K[\theta'^n(aba^{-1}b^{-1})])\leq c_0$$

for all n, where $c_0=\max_{s\in S} d(K[\tau(s)],\ L_\tau K[s])$. Therefore, using the relation

$$L_\tau^{-1}L_\theta^{-1}=L_{\theta'}^{-1}L_\tau^{-1}$$

and the assumption $\theta(aba^{-1}b^{-1})=\tau(aba^{-1}b^{-1})$, we see that the limit set of $L_\theta^{-n}K[\theta^{n+1}(aba^{-1}b^{-1})]$ coincides with the limit set of $L_\tau L_{\theta'}^{-n}K[\theta'^n(aba^{-1}b^{-1})]$.

By Lemma 2.2, Lemma 2.7 and Theorem 6.2, we can state the following:

THEOREM 7.1. *Let θ be an endomorphism satisfying the conditions* (1) *and* (2) *in Assumption* 1, *and suppose that θ has short range cancellations in a wide sense. Then there exists a limit set K_θ of $L_\theta^{-n}K[\theta^n(aba^{-1}b^{-1})]$ which coincides with $L_\tau K_{\theta'}$.*

EXAMPLE 1 (Twindragon boundary [5], [6]). Let θ be given by

$$\theta=\begin{cases} a\to aaba^{-1}a^{-1}a^{-1} \\ b\to ab^{-1}a^{-1}a^{-1}\,. \end{cases}$$

Then we see that θ has "long" range cancellations. Take $W:=a^{-1}a^{-1}$. Then the adjoint θ_W:

$$\theta_W=\begin{cases} a\to ba^{-1} \\ b\to a^{-1}b^{-1} \end{cases}$$

has a short range cancellation only at $\theta(a^{-1})\theta(b^{-1})$. By taking η as in Lemma 2.8, and defining the endomorphism θ' as $\theta'=\eta\theta\eta^{-1}$, we see that

$$\theta'=\begin{cases} a\to ba^{-1} \\ b\to b^{-1}a^{-1} \end{cases}$$

has cancellation at $\theta'(b)\theta'(a^{-1})$. Putting $A:=b$, $B:=a^{-1}$ and $C:=b^{-1}$, we obtain by (3-3) a lifting endomorphism Θ:

$$\Theta(A)=CB$$
$$\Theta(B)=B^{-1}A^{-1}$$
$$\Theta(C)=B^{-1}C^{-1}\,.$$

This lifting Θ belongs to the case (1) in §4, (4-4). Therefore we obtain a reduced endomorphism $\hat{\Theta}$

$$\hat{\Theta}(A)=C$$
$$\hat{\Theta}(B)=A^{-1}$$
$$\hat{\Theta}(C)=B^{-1}C^{-1}B^{-1} .$$

The structure matrix $N_{\hat{\Theta}}$ is given by

$$N_{\hat{\Theta}}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

and satisfies irreducibility and aperiodicity. We know that its characteristic polynomial is $\lambda^3-\lambda^2-2$. Therefore by Theorem 6.2, we have the following result: Let K_θ be a curve given by $\lim L_\theta^n K[\theta^n(aba^{-1}b^{-1})]$ and F_θ be the closed set enclosed by K_θ. Then we know that F_θ satisfies the space tiling property and self-similarity such that

$$L_\theta F_\theta=F_\theta \cup (F_\theta+1)$$

and

$$\dim_H K_\theta=\frac{\log \lambda_{\hat{\Theta}}}{\log \lambda_\theta} ,$$

where λ_θ is $\sqrt{2}$ (L_θ is isomorphic to the scalar multiplication by $\sqrt{2}\, e^{3\pi i/4}$) and $\lambda_{\hat{\Theta}}$ is the maximum solution of $\lambda^3-\lambda^2-2=0$.

EXAMPLE 2 (Complex radix expansion). As a generalization of Example 1, let us consider the following endomorphisms:

For positive integers m, n, let

$$\theta=\theta_{m,n}: \begin{cases} a\to a^N ba^{-(N+n)} \\ b\to (a^n b^{-1})^n a^{-N} \end{cases}$$

where $N=m+n^2$, and a^m means consecutive m-times occurrences of a. Take $W:=a^N$, then the adjoint θ_W is given by

$$\theta_W: \begin{cases} a\to ba^{-n} \\ b\to a^{-(N-n)}b^{-1}(a^n b^{-1})^{n-1} , \end{cases}$$

has a "long" cancellation. These θ or their adjoint θ' have no short range cancellations. Therefore, take an isomorphism

$$\tau: \begin{cases} a\to ba^{-n} \\ b\to a^{-N+n}b^{-1} , \end{cases}$$

and consider

$$\theta'=\begin{cases} a\to ba^{-2n+1} \\ b\to a^{-N+2n-1}b^{-1} . \end{cases}$$

Then τ and θ' satisfy the condition of Theorem 7.1 for θ_W. θ' has a cancellation at $\theta'(a^{-1})\theta'(b^{-1})$, and so take η as in Lemma 2.8, and define $\theta''=\eta\theta'\eta^{-1}$. Then

$$\theta''=\begin{cases} a\to b^{N-2n+1}a^{-1} \\ b\to b^{-2n+1}a^{-1} \end{cases}$$

and it has a cancellation at $\theta''(b)\theta''(a^{-1})$. Putting $A:=b^{N-2n+1}$, $B:=a^{-1}$ and $C:=b^{-2n+1}$, we obtain by (3-3) a lifting endomorphism Θ:

$$\Theta(A)=(CB)^{N-2n+1}$$
$$\Theta(B)=B^{-1}A^{-1}$$
$$\Theta(C)=(CB)^{-2n+1} .$$

This lifting endomorphism belongs to the case (1) in §4, (4-4). Therefore we obtain a reduced endomorphism $\hat{\Theta}$:

$$\hat{\Theta}(A)=(CB)^{N-2n}C$$
$$\hat{\Theta}(B)=A^{-1}$$
$$\hat{\Theta}(C)=(CB)^{-2n+1}B^{-1} .$$

The structure matrix $N_{\hat{\Theta}}$ is given by

$$N_{\hat{\Theta}}=\begin{pmatrix} 0 & 1 & 0 \\ N-2n & 0 & 2n \\ N-2n+1 & 0 & 2n-1 \end{pmatrix}$$

and the characteristic polynomial is

$$\lambda^3-(2n-1)\lambda^2-(N-2n)\lambda-N=0 .$$

For each $m, n\geq 1$, let us consider the following integers in the complex quadratic field $\boldsymbol{Q}(\sqrt{m}\,i)$:

$$\alpha=-n\pm\sqrt{m}\,i \qquad (n=1, 2, \cdots) \qquad \text{if } -m\equiv 2, 3 \pmod 4$$

$$\alpha=\frac{-2n+1\pm\sqrt{m}\,i}{2} \qquad (n=1, 2, 3, \cdots) \qquad \text{if } -m\equiv 1 \pmod 4$$

$$\left(\alpha=\frac{-2n+1\pm\sqrt{3}\,i}{2} \qquad (n=2, 3, \cdots) \qquad \text{if } -m=3\right).$$

Instead of the canonical homomorphism $f: G\langle a, b\rangle \to \boldsymbol{Z}^2$, let us consider a canonical homomorphism $f_\alpha: G\langle a, b\rangle \to \boldsymbol{Z}(\sqrt{m}\,i) \subset \boldsymbol{C}^1$ associated with the above α's as follows:

$$f_\alpha: \begin{cases} a \to 1 \\ b \to \pm\sqrt{m}i \end{cases} \qquad \text{if } -m \equiv 2, 3 \pmod 4$$

$$f_\alpha: \begin{cases} a \to 1 \\ b \to \dfrac{1 \pm \sqrt{m}i}{2} \end{cases} \qquad \text{if } -m \equiv 1 \pmod 4).$$

Let $T_\alpha: \boldsymbol{Z}(\sqrt{m}\,i) \to \boldsymbol{Z}(\sqrt{m}\,i)$ be an endomorphism of $\boldsymbol{Z}(\sqrt{m}\,i)$ defined by the multiplication by α, that is, T_α is defined by $T_\alpha(z) = \alpha z$. Then we have the following commutative diagram: for each $m, n \geq 1$

$$\begin{array}{ccc} G\langle a, b\rangle & \xrightarrow{\theta_{m,n}} & G\langle a, b\rangle \\ {\scriptstyle f_\alpha}\downarrow & & {\scriptstyle f_\alpha}\downarrow \\ \boldsymbol{Z}(\sqrt{m}i) & \xrightarrow{T_\alpha} & \boldsymbol{Z}(\sqrt{m}i). \end{array}$$

Therefore, we see that $\theta_{m,n}$ satisfies Assumption 1'. Thus we obtain the following theorem by a slight modification of previous arguments ([4], [5] and [6]):

THEOREM 7.2. *For each of the above α a curve K_α $(=K_{\theta_{m,n}})$ satisfying the following property is constructed on the complex plane: let X_α be the bounded closed domain enclosed by K_α. Then*

(1) (*space tiling*)

$$\bigcup_{z \in \boldsymbol{Z}(\sqrt{m}\,i)} (X_\alpha + z) = \boldsymbol{C}^1$$

and

$$\operatorname{int}(X_\alpha + z) \cap \operatorname{int}(X_\alpha + z') = \varnothing \qquad \textit{if } z \neq z' \in \boldsymbol{Z}(\sqrt{m}i),$$

(2) (*self-similarity*)

$$\alpha X_\alpha = \bigcup_{j=0}^{N-1} (X_\alpha + j),$$

(3) (*complex radix expansion*)

$$X_\alpha = \left\{ \sum_{k=1}^{\infty} a_k \alpha^{-k} \,\middle|\, a_k \in \{0, 1, 2, \cdots, N-1\} \right\},$$

(4) *The Hausdorff dimension of the curves K_α is equal to $2 \log \lambda_{m,n} / \log N$, where $\lambda_{m,n}$ is the maximum solution of*

$$\lambda^3-(2n-1)\lambda^2-(N-2n)\lambda-N=0 .$$

References

[1] T. BEDFORD, Crinkly curves, Markov partition and dimension, Ph.D. Thesis (Warwick Univ.), 1984.
[2] T. BEDFORD, Generating special Markov partitions for hyperbolic toral automorphisms using fractals, Ergod. Th. Dynam. Sys., **6** (1986), 325–333.
[3] F. M. DEKKING, Recurrent sets, Adv. in Math., **44** (1982), 78–104.
[4] F. M. DEKKING, Replicating super figures and endomorphisms of free groups, J. Combin. Theory Ser. A, **32** (1982), 315–320.
[5] W. J. GILBERT, The fractal dimension of sets derived from complex basis, Canad. Math. Bull., **29** (1986), 495–500.
[6] B. B. MANDELBROT, *The Fractal Geometry of Nature*, Freeman, New York, 1982.

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