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A Characterization of Real Quadratic Numbers by Diophantine Algorithms

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§0. Introduction.

We consider a transformation T on

$$X=\{(x, y) \mid 0 \leqq x<1, 0 \leqq y<1-x\}-\{(0, 1)\}$$

defined by

$$T(x, y)=\begin{cases} \left(\dfrac{x}{1-x}, \dfrac{y}{1-x}\right) & \text{if } (x, y) \in \Delta_1 , \\ \left(\dfrac{2x-1}{x}, \dfrac{y}{x}\right) & \text{if } (x, y) \in \Delta_2 , \\ \left(\dfrac{1-2x}{1-x}, \dfrac{1-x-y}{1-x}\right) & \text{if } (x, y) \in \Delta_3 , \end{cases}$$

with

$$\Delta_1=\{(x, y) \in X \mid 0 \leqq x<1/2,\ 0 \leqq y \leqq 1-2x\} ,$$
$$\Delta_2=\{(x, y) \in X \mid 1/2 \leqq x<1,\ 0 \leqq y<1-x\}$$

and

$$\Delta_3=\{(x, y) \in X \mid 0 \leqq x<1/2,\ 1-2x<y \leqq 1-x\} .$$

This transformation was fiirst introduced, in [1], to consider properties of the sequence

$$C(\alpha, \beta)=\{[n\alpha+\beta]-[(n-1)\alpha+\beta] \mid n=1, 2, \cdots\} ,$$

(where $[\cdot]$ denotes its integral part).

The purpose of this paper is to study algebraic properties of periodic points under

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the transformation T. We will show the following theorem:

THEOREM. *Let α be a quadratic surd and $(\alpha, \beta)\in X$. Then β is an element of the quadratic field $\mathbf{Q}(\alpha)$ if and only if (α, β) is a periodic point of the transformation T.*

Here, the periodic point means that there exist natural numbers k and n such that

$$T^{n+j}(\alpha, \beta)=T^{n+kl+j}(\alpha, \beta) \qquad \text{for} \quad 1\leqq j\leqq k \quad \text{and} \quad l\in \mathbf{N}.$$

The fundamental idea used to obtain this theorem comes from the following notion:

DEFINITION. Let α be a quadratic surd and β is in $\mathbf{Q}(\alpha)$. Then β is said to be α-reduced if $(\alpha, \beta, \bar{\alpha}, \bar{\beta})\in X\times X^*$, where

$$X^*=\{(z, w)\,;\, z\leqq 0, z<w\leqq 0\}\cup\{(z, w)\,;\, z>1, 0\leqq w<z\}.$$

Here and henceforth, $\bar{x}$ denotes its algebraic conjugate when x is a quadratic surd, or x itself when x is a rational number.

With this notion, it is possible to show that if β is α-reduced, then the cardinality of the set $\{T^n(\alpha, \beta)\,;\, n=1, 2, \cdots\}$ is finite, and so the point (α, β) is periodic.

A similar approach goes back to Lagrange–Galois' work: a real number α is quadratic if and only if the partial quotients of the continued fraction expansion of α is periodic. Furthermore, it has been taken up later by A. L. Schmidt in his work on quadratic numbers over Gaussian integers [5] and by K. Schmidt on Pisot-Salem numbers [6].

§1. Periodicity under some transformations.

In this section, we introduce two one-dimensional transformations S and U, which correspond to the algorithms related to the mediant convergents (see [1] and [2]). Theorems, obtained in this section, pave the way for discussions in the subsequent section.

Let S be a transformation of $X_1=[0, 1)$ onto itself defined by

$$S(x)=\begin{cases} S_1(x)=\dfrac{x}{1-x} & \text{if} \quad 0\leqq x<1/2\,, \quad \text{and} \\[2ex] S_2(x)=\dfrac{2x-1}{x} & \text{if} \quad 1/2\leqq x<1 \qquad \text{(Fig. 1)}. \end{cases}$$

We put $\alpha_{S,n}=S^n(\alpha)$ $(n\geqq 0)$ and

$$i_{S,n+1}=\begin{cases} 1 & \text{if} \quad 0\leqq \alpha_{S,n}<1/2 \\ 2 & \text{if} \quad 1/2\leqq \alpha_{S,n}<1 \end{cases} \qquad (n\geqq 0).$$

So we get a sequence $(i_{S,1}, i_{S,2}, \cdots)\in\{1, 2\}^{\mathbf{N}}$ for every $\alpha\in X_1$, which we call the name of α with respect to S. We prove "Lagrange type" theorem, i.e. α is quadratic if and only if the name of α with respect to S is periodic.

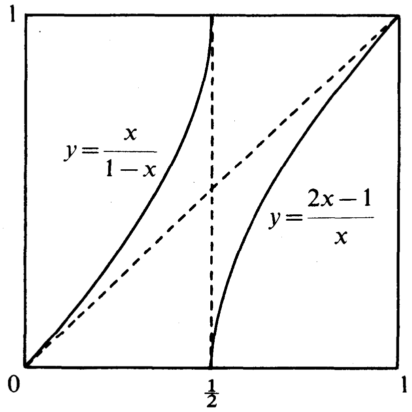

FIGURE 1

DEFINITION 1. Let $\alpha \in X_1$ be a quadratic surd. Then α is said to be S-reduced if $\bar{\alpha} \leqq 0$ or $\bar{\alpha} > 1$, where $\bar{x}$ denotes the algebraic conjugate of x.

We show two lemmas concerning S-reduced numbers.

LEMMA 1. *For an S-reduced number α, we have*
(A) *$\alpha_{S,1}$ is also S-reduced.*
(B) *There exists a unique element $\alpha_{S,-1}$ in $S^{-1}(\alpha)$, which is S-reduced.*

PROOF. We put $\bar{X}_1 = [0, 1) \times ((-\infty, 0] \cup (1, \infty))$ and define two partitions of $\bar{X}_1$ by

$$\begin{cases} D_{S,1} = [0, 1/2) \times (1, \infty) \\ D_{S,2} = [0, 1/2) \times (-\infty, 0] \\ D_{S,3} = [1/2, 1) \times (1, \infty) \\ D_{S,4} = [1/2, 1) \times (-\infty, 0], \end{cases}$$

and

$$\begin{cases} E_{S,1} = [0, 1) \times (-\infty, -1) \\ E_{S,2} = [0, 1) \times [-1, 0] \\ E_{S,3} = [0, 1) \times (1, 2) \\ E_{S,4} = [0, 1) \times [2, \infty). \end{cases}$$

We consider a transformation $\bar{S}$ on $\bar{X}_1$ defined by

$$\bar{S}(x, y) = \begin{cases} (S_1(x), S_1(y)) = \left(\dfrac{x}{1-x}, \dfrac{y}{1-y}\right) & \text{if } 0 \leqq x < 1/2 \\[2ex] (S_2(x), S_2(y)) = \left(\dfrac{2x-1}{x}, \dfrac{2y-1}{y}\right) & \text{if } 1/2 \leqq x < 1. \end{cases}$$

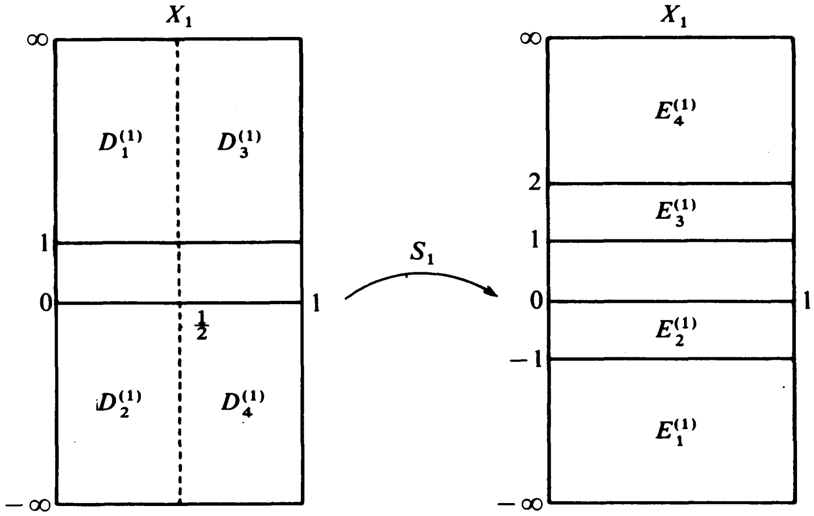

FIGURE 2

We call $\bar{S}$ a natural extension of S.

(A) For an S-reduced number α, we have $(\alpha, \bar{\alpha}) \in \bar{X}_1$. So there exists some i ($i=1, 2, 3$ or 4) such that $(\alpha, \bar{\alpha}) \in D_{S,i}$, therefore

$$(\alpha_{S,1}, \overline{\alpha_{S,1}}) = \bar{S}(\alpha, \bar{\alpha}) \in E_{S,i} \qquad \text{(Fig. 2)}.$$

(B) For every $(\alpha, \bar{\alpha}) \in \bar{X}_1$, there exists i such that $(\alpha, \bar{\alpha}) \in E_{S,i}$. If we put

$$\alpha_{S,-1} = \begin{cases} \dfrac{\alpha}{1+\alpha} & \text{if } (\alpha, \bar{\alpha}) \in E_{S,1} \cup E_{S,2} \\ \dfrac{1}{2-\alpha} & \text{if } (\alpha, \bar{\alpha}) \in E_{S,3} \cup E_{S,4}, \end{cases}$$

$(\alpha_{S,-1}, \overline{\alpha_{S,-1}}) \in D_{S,i}$ and $\alpha_{S,-1}$ is S-reduced.

LEMMA 2. *There are only finite number of S-reduced quadratic surds with a given discriminant $d>0$.*

PROOF. Let α be a root of

$$ax^2 - 2bx + c = 0 \qquad (a, b, c \in \boldsymbol{Z},\ a>0,\ d=b^2-ac>0)$$

and S-reduced.

(1) If $0 \leqq \alpha < 1$ and $\bar{\alpha} > 1$, then, from $0 \leqq (b-\sqrt{d})/a$ and $(b+\sqrt{d})/a > 1$, we get $0<a<d$ and $\sqrt{d} < b < d+\sqrt{d}$.

(2) If $0 \leqq \alpha < 1$ and $\bar{\alpha} \leqq 0$, then we get $0<a<d$ and $-\sqrt{d} < b < \sqrt{d}$.

From (1) and (2), the number of such α's is finite.

PROPOSITION 1. *If $\alpha \in X_1$ is a quadratic surd and S-reduced, the name of α with respect to S is purely periodic, that is, there exists a natural number k such that $i_{S,j} = i_{S,kl+j}$*

for $1 \leqq j \leqq k$ *and* $l \in N$.

NOTICE. In the sequel, we denote such periodicity by

$$(i_{S,1}, i_{S,2}, \cdots) = (\overline{i_{S,1}, \cdots, i_{S,k}}) .$$

PROOF. From Lemma 1 (A), if α is S-reduced, then $\alpha_{S,1}$ is S-reduced, and therefore $\alpha_{S,2}, \alpha_{S,3}, \cdots$ are also S-reduced. Moreover it is clear from the definition of S that if α is a quadratic surd with a discriminant d, then $\alpha_{S,1}, \alpha_{S,2}, \cdots$ are quadratic with the same discriminant. From Lemma 2, there exist i and k such that $\alpha_{S,i} = \alpha_{S,i+k}$. From Lemma 1 (B), we get $\alpha_{S,i-1} = \alpha_{S,i+k-1}$ and in the same way

$$\begin{aligned} &\alpha_{S,i-2} = \alpha_{S,i+k-2} , \\ &\cdots\cdots\cdots\cdots\cdots\cdots , \\ &\alpha = \alpha_{S,k} . \end{aligned}$$

Therefore the name of α with respect to S is a purely periodic sequence with the period k.

THEOREM 1. *A real number* $\alpha \in X_1$ *is quadratic if and only if the name of* α *with respect to* S *is periodic.*

PROOF. Let

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

which correspond to $S_1{}^{-1}(x) = x/(1+x)$ and $S_2{}^{-1}(x) = 1/(2-x)$, respectively. For every name $\{i_{S,1}, i_{S,2}, \cdots\}$ of α with respect to S, we put

$$\begin{pmatrix} r_{S,n} & s_{S,n} \\ t_{S,n} & u_{S,n} \end{pmatrix} = A_{i_{S,1}} A_{i_{S,2}} \cdots A_{i_{S,n}} \qquad (n \geqq 1) .$$

Then it is easy to show that

$$\alpha = \frac{t_{S,n} + u_{S,n} \cdot \alpha_{S,n}}{r_{S,n} + s_{S,n} \cdot \alpha_{S,n}} , \tag{1}$$

and that $t_{S,n}/r_{S,n}$ and $u_{S,n}/s_{S,n}$ are the convergents of α (see [1]). We denote $\bar{S}^n(\alpha, 0)$ by $(\alpha_{S,n}, \alpha^*_{S,n})$, and we see that $(\alpha_{S,n}, \alpha^*_{S,n}) \in \bar{X}_1$ for all $n \geqq 0$. We have $(\alpha_{S,n}, \overline{\alpha_{S,n}}) \in \bar{X}_1$ for large n in the following way. Noting that $\det A_1 = \det A_2 = 1$ and so $r_{S,n} u_{S,n} - t_{S,n} s_{S,n} = 1$, we see that

$$\begin{aligned} |\overline{\alpha_{S,n}} - \alpha^*_{S,n}| &= \left| \frac{t_{S,n} - r_{S,n}\bar{\alpha}}{-u_{S,n} + s_{S,n}\bar{\alpha}} - \frac{t_{S,n}}{-u_{S,n}} \right| \\ &= \left| \frac{\bar{\alpha}}{-u_{S,n}(-u_{S,n} + s_{S,n}\bar{\alpha})} \right| \end{aligned}$$

$$=\left|\frac{\bar{\alpha}}{-u_{S,n}s_{S,n}(\bar{\alpha}-u_{S,n}/s_{S,n})}\right| \to 0 \quad \text{as} \quad n\to\infty \tag{2}$$

(see [1]). Let

$$\overline{X_1(\varepsilon)}=[0,1)\times(-\varepsilon,0]\cup[0,1)\times(1,1+\varepsilon)\ ,$$

where ε is positive but small enough. We consider the following two cases.
(1) If for every N, there exists $n\geqq N$ such that $(\alpha_{S,n}, \alpha^*_{S,n})\in\bar{X}_1-\overline{X_1(\varepsilon)}$, then $(\alpha_{S,n}, \overline{\alpha_{S,n}})\in\bar{X}_1$ by (2).
(2) If there exists N such that $(\alpha_{S,n}, \alpha^*_{S,n})\in\overline{X_1(\varepsilon)}$ for every $n\geqq N$, then $\{i_{S,1}, i_{S,2}, \cdots\}=\{1, 1, \cdots\}$ or $\{2, 2, \cdots\}$.
From (1) and (2), it follows that the name of α with respect to S is periodic. Conversely, if $\{i_{S,1}, i_{S,2}, \cdots\}$ is periodic, that is, there exist N and k with $\alpha_{S,N}=\alpha_{S,N+k}$, then from (1),

$$\begin{aligned}\alpha&=\frac{t_{S,N}+u_{S,N}\cdot\alpha_{S,N}}{r_{S,N}+s_{S,N}\cdot\alpha_{S,N}}\\ &=\frac{t_{S,N+k}+u_{S,N+k}\cdot\alpha_{S,N+k}}{r_{S,N+k}+s_{S,N+k}\cdot\alpha_{S,N+k}}\\ &=\frac{t_{S,N+k}+u_{S,N+k}\cdot\alpha_{S,N}}{r_{S,N+k}+s_{S,N+k}\cdot\alpha_{S,N}}\,.\end{aligned}$$

So $\alpha_{S,N}$ is quadratic, therefore α is quadratic.

REMARK. The converse of Proposition 1 also holds from Theorem 1. In fact if, the name of α with respect to S is a purely periodic sequence with a period k, then from (1), $\alpha=(t_{S,k}+u_{S,k}\cdot\alpha)/(r_{S,k}+s_{S,k}\cdot\alpha)$, and so α is quadratic.

From Theorem 1, there exists N such that $\alpha_{S,N}$ is S-reduced. Since $\alpha_{S,n}$'s, $n\geqq N$, are S-reduced and $\alpha=\alpha_{S,kl}$ for every $l\geqq 0$, we see that α is S-reduced from Lemma 1 (A).

Now we define the second transformation U from [0, 1) to [0, 1) as follows. We see properties analogous to those of S. We only state the definitions and the properties without any proofs.

We define the transformation U on $X_2=[0,1)$ by

$$U(x)=\begin{cases} U_1(x)=\dfrac{x}{1-x} & \text{if} \quad 0\leqq x<1/2\\[2ex] U_2(x)=\dfrac{1-x}{x} & \text{if} \quad 1/2\leqq x<1 \qquad \text{(Fig. 3)}\,. \end{cases}$$

The name of α with respect to U is given by $\alpha_{U,n}=U^n(\alpha)$ $(n\geqq 0)$ and

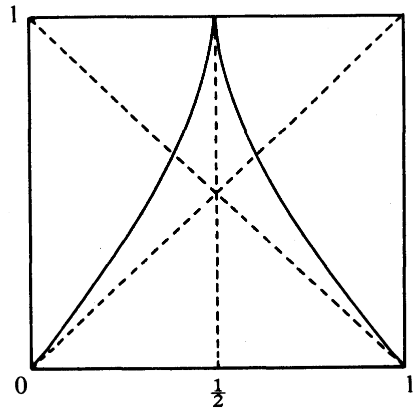

FIGURE 3

$$i_{U,n+1}=\begin{cases}1 & \text{if } 0\leqq\alpha_{U,n}<1/2\\ 3 & \text{if } 1/2\leqq\alpha_{U,n}<1\end{cases}\qquad (n\geqq 0)\,.$$

DEFINITION 1*. A quadratic number $\alpha\in X_2$ is called U-reduced if

$$(\alpha, \bar{\alpha})\in\bar{X}_2=[0, 1)\times(-\infty, 0]\,.$$

The associated matrices with respect to U are given by

$$A_1=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\qquad A_3=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix},$$

which give the following relation;

$$\alpha=\frac{t_{U,n}+u_{U,n}\cdot\alpha_{U,n}}{r_{U,n}+s_{U,n}\cdot\alpha_{U,n}} \tag{3}$$

where

$$\begin{pmatrix}r_{U,n} & s_{U,n}\\ t_{U,n} & u_{U,n}\end{pmatrix}=A_{i_{U,1}}A_{i_{U,2}}\cdots A_{i_{U,n}}\qquad (n\geqq 1)\,.$$

Then we have

LEMMA 1*. *For an U-reduced number* α, *we have*

(A) $\alpha_{U,1}$ *is also U-reduced.*

(B) *There exists a unique element* $\alpha_{U,-1}$ *in* $U^{-1}(\alpha)$, *which is U-reduced.*

The proof is obtained by introducing a natural extension $\bar{U}$ (Fig. 4):

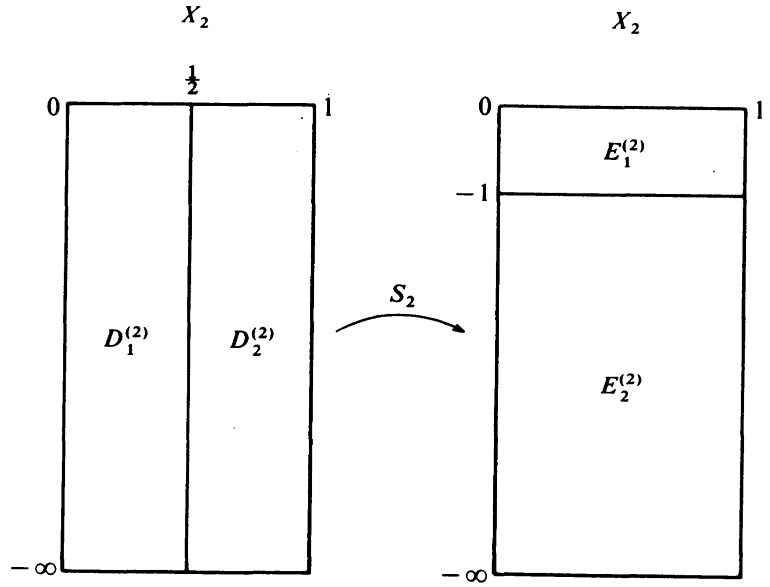

FIGURE 4

$$\bar{U}(x, y) = \begin{cases} (U_1(x), U_1(y)) = \left(\dfrac{x}{1-x}, \dfrac{y}{1-y}\right) & \text{if } 0 \leqq x < 1/2 \\ (U_2(x), U_2(y)) = \left(\dfrac{1-x}{x}, \dfrac{1-y}{y}\right) & \text{if } 1/2 \leqq x < 1 . \end{cases}$$

LEMMA 2*. *There are only finite number of U-reduced quadratic numbers with a given discriminant $d > 0$.*

The proof is obtained by the analogy with the proof of Lemma 2. We will also have the following:

PROPOSITION 1*. *If $\alpha \in X_2$ is a quadratic surd and U-reduced, the name of α with respect to U is purely periodic.*

THEOREM 1*. *The number $\alpha \in X_2$ is quadratic if and only if the name of α with respect to U is periodic.*

§ 2. Periodicity under the transformation T.

We consider the transformation T defined in §0. We note that $\{\Delta_1, \Delta_2, \Delta_3\}$ is a partition of X (Fig. 5) and that

$$T(\Delta_1) = X, \quad T(\Delta_2) = X, \quad \text{and} \quad T(\Delta_3) = X - \{(x, y) \in X \mid y = 1 - x\} .$$

Putting $\bar{X} = X \times X^*$, we define a transformation $\bar{T}$ from $\bar{X}$ to $\bar{X}$ as follows:

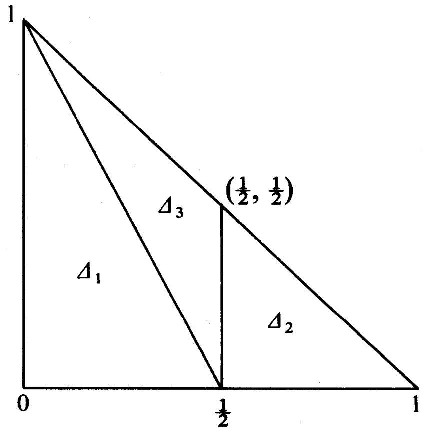

FIGURE 5

$$\bar{T}(x, y, z, w) = \begin{cases} \left(\dfrac{x}{1-x}, \dfrac{y}{1-x}, \dfrac{z}{1-z}, \dfrac{w}{1-z}\right) & \text{if } (x, y) \in \Delta_1 \\ \left(\dfrac{2x-1}{x}, \dfrac{y}{x}, \dfrac{2z-1}{z}, \dfrac{w}{z}\right) & \text{if } (x, y) \in \Delta_2 \\ \left(\dfrac{1-2x}{1-x}, \dfrac{1-x-y}{1-x}, \dfrac{1-2z}{1-z}, \dfrac{1-z-w}{1-z}\right) & \text{if } (x, y) \in \Delta_3 . \end{cases}$$

The transformation $\bar{T}$ is a natural extension of T. Putting $(\alpha_n, \beta_n) = T^n(\alpha, \beta)$ $(n \geqq 0)$ and $i_{n+1} = i$ if $(\alpha_n, \beta_n) \in \Delta_i$, we get a sequence $(i_1, i_2, i_3, \cdots)$, which we call the name of (α, β).

LEMMA 3. *Let α be a quadratic surd and $\beta \in \mathbf{Q}(\alpha)$ be α-reduced. Then we have*
(A) *β_1 is α_1-reduced.*
(B) *There exists a unique point $(\alpha_{-1}, \beta_{-1})$ in $T^{-1}(\alpha, \beta)$ so that β_{-1} is α_{-1}-reduced.*

PROOF. We define a partition $\{\Delta_1^*, \Delta_2^*, \Delta_3^*\}$ of X^* by

$$\Delta_1^* = \{(z, w) \in X^* \mid z \leqq 0, z < w \leqq 0\}$$
$$\Delta_2^* = \{(z, w) \in X^* \mid z > 1, 0 \leqq w < 1\}$$
$$\Delta_3^* = \{(z, w) \in X^* \mid z > 1, 1 \leqq w < z\} .$$

(A) For an α-reduced number β, there exists i $(i = 1, 2 \text{ or } 3)$ with

$$(\alpha, \beta, \bar{\alpha}, \bar{\beta}) \in \Delta_i \times X^* .$$

Then we get $(\alpha_1, \beta_1, \bar{\alpha}_1, \bar{\beta}_1) = \bar{T}(\alpha, \beta, \bar{\alpha}, \bar{\beta}) \in X \times \Delta_i^*$ (Fig. 6). Therefore β_1 is α_1-reduced.
(B) For an α-reduced number β, there exists i $(i = 1, 2 \text{ or } 3)$ with

$$(\alpha, \beta, \bar{\alpha}, \bar{\beta}) \in X \times \Delta_i^* .$$

If we put

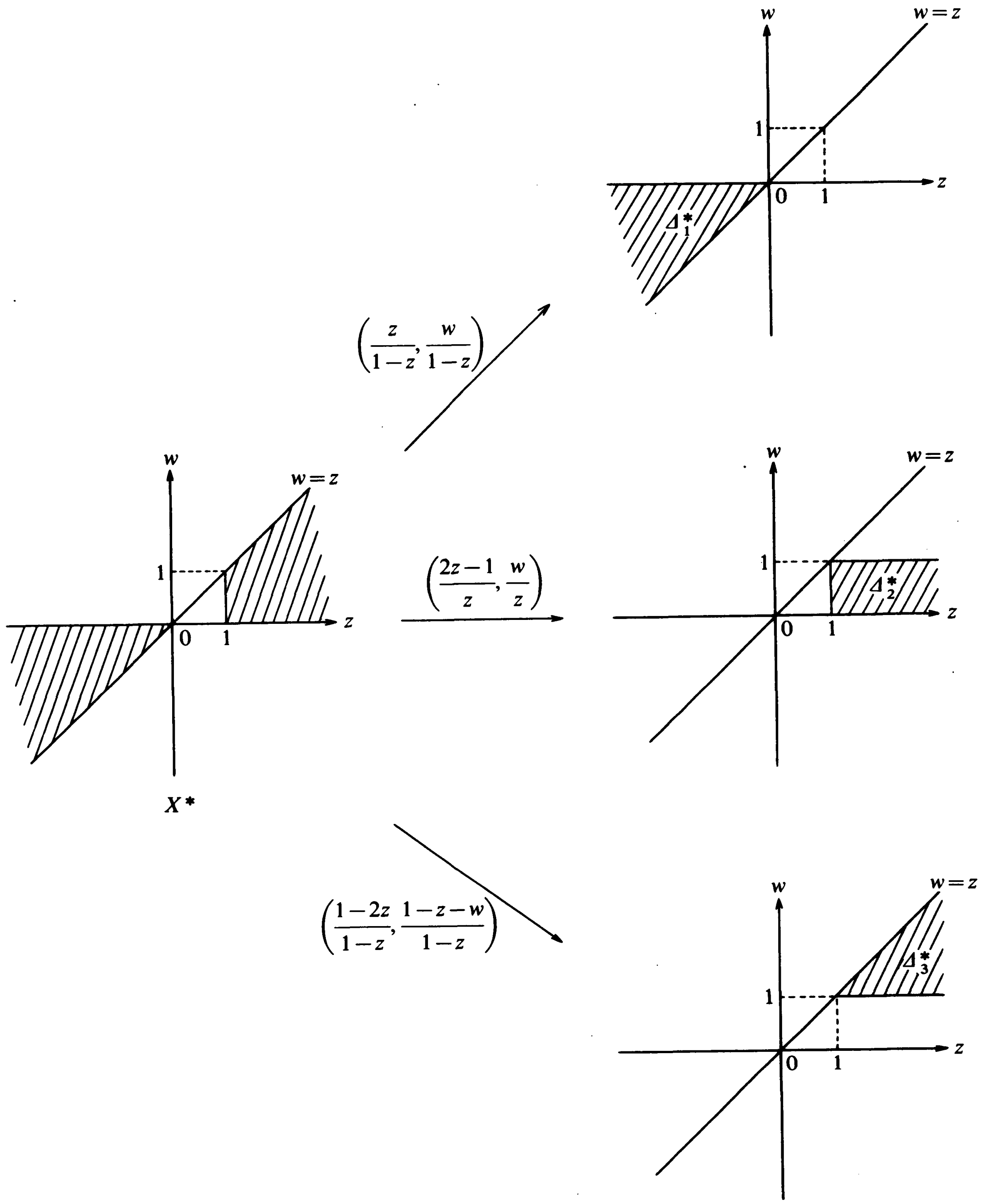
w
w=z
1
0
1
z
Δ_1^*
$\left(\frac{z}{1-z}, \frac{w}{1-z}\right)$
w
w=z
1
0
1
z
X^*
$\left(\frac{2z-1}{z}, \frac{w}{z}\right)$
w
w=z
1
0
1
z
Δ_2^*
$\left(\frac{1-2z}{1-z}, \frac{1-z-w}{1-z}\right)$
w
w=z
Δ_3^*
1
0
1
z

FIGURE 6

$$(\alpha_{-1}, \beta_{-1}) = \begin{cases} \left(\dfrac{\alpha}{1+\alpha}, \dfrac{\beta}{1+\alpha}\right) & \text{if} \quad (\bar{\alpha}, \bar{\beta}) \in \Delta_1^* \\ \left(\dfrac{1}{2-\alpha}, \dfrac{\beta}{2-\alpha}\right) & \text{if} \quad (\bar{\alpha}, \bar{\beta}) \in \Delta_2^* \\ \left(\dfrac{1-\alpha}{2-\alpha}, \dfrac{1-\beta}{2-\alpha}\right) & \text{if} \quad (\bar{\alpha}, \bar{\beta}) \in \Delta_3^* , \end{cases}$$

we have $(\alpha_{-1}, \beta_{-1}) \in \Delta_i \times X^*$. Therefore β_{-1} is α_{-1}-reduced.

LEMMA 4. *Let $\alpha \in X$ be quadratic and $\beta \in \boldsymbol{Q}(\alpha)$ be α-reduced. Then the cardinality of the set $\{\alpha_0, \alpha_1, \alpha_2, \cdots\}$ is finite.*

PROOF. By Proposition 1, the sequence of α with respect to S is purely periodic. We denote its period by k. We denote the sequence $(\alpha_{S,0}, \alpha_{S,1}, \cdots)$ by $(\alpha_0, \alpha_1, \cdots)$. For every quadratic surd α and α-reduced $\beta \in \boldsymbol{Q}(\alpha)$, we have $\alpha \in \{\alpha_{S,0}, \cdots, \alpha_{S,k-1}\}$ because α is also S-reduced. For any $i \geqq 0$, if $(\alpha_i, \beta_i) \in \Delta_1 \cup \Delta_2$ and there exists l $(0 \leqq l \leqq k-1)$ such that $\alpha_i = \alpha_{S,l}$, then $\alpha_{i+1} = \alpha_{S,l+1}$ $(0 \leqq l \leqq k-2)$ or $\alpha_{i+1} = \alpha_{S,0}$ $(l = k-1)$. When $(\alpha_i, \beta_i) \in \Delta_3$ and there exists l $(0 \leqq l \leqq k-1)$ such that $\alpha_i = \alpha_{S,l}$, we put

$$\alpha'_{i+1} = \frac{1-2\alpha_{S,l}}{1-\alpha_{S,l}} = 1 - \frac{\alpha_{S,l}}{1-\alpha_{S,l}} = 1 - \alpha_{S,l+1}$$

and

$$\beta'_{i+1} = 1 - \frac{\beta_i}{1-\alpha_i} .$$

There are two possibilities:

(1) $0 \leqq \alpha'_{i+1} < 1/2$ (equivalently $1/2 < \alpha_{S,l+1} < 1$). If $(\alpha'_{i+1}, \beta'_{i+1}) \in \Delta_1$, then we have

$$1 - \frac{\alpha'_{i+1}}{1-\alpha'_{i+1}} = 1 - \frac{1-\alpha_{S,l+1}}{1-(1-\alpha_{S,l+1})} = \frac{2\alpha_{S,l+1}-1}{\alpha_{S,l+1}} = \alpha_{S,l+2} .$$

If $(\alpha'_{i+1}, \beta'_{i+1}) \in \Delta_3$, then we have

$$\frac{1-2\alpha'_{i+1}}{1-\alpha'_{i+1}} = \frac{1-2(1-\alpha_{S,l+1})}{1-(1-\alpha_{S,l+1})} = \frac{2\alpha_{S,l+1}-1}{\alpha_{S,l+1}} = \alpha_{S,l+2} .$$

(2) $1/2 \leqq \alpha'_{i+1} < 1$ (equivalently $0 \leqq \alpha_{S,l+1} < 1/2$). If $(\alpha_{S,l+1}, \beta_{S,l+1}) \in \Delta_1$, we have

$$1 - \frac{2\alpha'_{i+1}-1}{\alpha'_{i+1}} = 1 - \frac{2(1-\alpha_{S,l+1})-1}{1-\alpha_{S,l+1}} = \frac{\alpha_{S,l+1}}{1-\alpha_{S,l+1}} = \alpha_{S,l+2} .$$

If $(\alpha_{S,l+1}, \beta_{S,l+1}) \in \Delta_3$, we have

$$\frac{2\alpha'_{i+1}-1}{\alpha'_{i+1}}=\frac{2(1-\alpha_{S,l+1})-1}{1-\alpha_{S,l+1}}=\frac{1-2\alpha_{S,l+1}}{1-\alpha_{S,l+1}}=\alpha_{S,l+2}\,.$$

In both cases, every element of the sequence $(\alpha_0, \alpha_1, \cdots)$ is included in

$$\mathfrak{A}(\alpha)=\{\alpha_{S,0}, \cdots, \alpha_{S,k-1}, \alpha'_0, \cdots, \alpha'_{k-1}, 1-\alpha'_0, \cdots, 1-\alpha'_{k-1}\}\,.$$

Since the cardinality of the set $\mathfrak{A}(\alpha)$ is finite, the cardinality of the set $\{\alpha_0, \alpha_1, \cdots\}$ is finite.

Now, we define the matrices associated with T as follows;

$$B_1=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\quad B_2=\begin{pmatrix}2&-1&0\\1&0&0\\0&0&1\end{pmatrix},\quad B_3=\begin{pmatrix}2&-1&0\\1&-1&0\\1&0&-1\end{pmatrix}.$$

We define $p_n^{(i)}$, $q_n^{(i)}$, $r_n^{(i)}$, and ε_n by

$$\begin{pmatrix}q_n^{(1)}&q_n^{(2)}&0\\p_n^{(1)}&p_n^{(2)}&0\\r_n^{(1)}&r_n^{(2)}&\varepsilon_n\end{pmatrix}=B_{i_1}B_{i_2}\cdots B_{i_n}\qquad(n\geqq 0)\,.$$

From [1], we get

$$\alpha=\frac{p_n^{(1)}+p_n^{(2)}\alpha_n}{q_n^{(1)}+q_n^{(2)}\alpha_n},\tag{4}$$

$$\beta=\frac{r_n^{(1)}+r_n^{(2)}\alpha_n+\varepsilon_n\beta_n}{q_n^{(1)}+q_n^{(2)}\alpha_n}.\tag{5}$$

LEMMA 5. *Let α be a quadratic surd and $\beta\in\boldsymbol{Q}(\alpha)$ be α-reduced. Then the cardinality of the set $\{\beta_0, \beta_1, \beta_2, \cdots\}$ is finite.*

PROOF. We put

$$\alpha_j=\frac{n_j+l_j\sqrt{d}}{m_j},\quad \beta_j=\frac{u_j+v_j\sqrt{d}}{t_j}\quad(m_j, n_j, l_j, t_j, u_j, v_j\in\boldsymbol{Z},\ j\geqq 0)$$

From Lemma 4, it follows that there exist at most finite number of such m_j's. From (5) and $\varepsilon_n=\pm 1$, we get

$$\beta_n=\varepsilon_n\{(q_n^{(1)}+q_n^{(2)}\alpha_n)\beta-r_n^{(1)}-r_n^{(2)}\alpha_n\}\,.$$

Therefore we have $t_j=m_j\times t_0$ and there are finitely many such t_j's. Since β is α-reduced, we see that β_n is α_n-reduced for any $n\geqq 1$ by Lemma 3 (A). Noting that

$$0<\beta_n<1-\alpha_n\,,\qquad |\overline{\beta_n}|<|\overline{\alpha_n}|\,,$$

we conclude from Lemma 4 that there are only finitely many such β_n's.

Therefore we obtain the following proposition.

PROPOSITION 2. *Let α be a quadratic surd and $\beta\in \boldsymbol{Q}(\alpha)$ be α-reduced. Then the name of (α, β) is purely periodic.*

PROOF. From Lemma 3 (A), β_n is α_n-reduced for any $n>0$. From Lemma 5, there exist i and N such that $\beta_i=\beta_{i+N}$. From Lemma 3 (B), we have $\alpha_{i-1}=\alpha_{i+N-1}$ and $\beta_{i-1}=\beta_{i+N-1}$. In the same way, we have

$$\begin{array}{ccc} \alpha_{i-2}=\alpha_{i+N-2} & \text{and} & \beta_{i-2}=\beta_{i+N-2}\,, \\ \cdots\cdots & \cdots & \cdots\cdots\,, \\ \alpha_0=\alpha_N & \text{and} & \beta_0=\beta_N\,. \end{array}$$

Finally we show the results on more general case.

THEOREM 2. *Let $(\alpha, \beta)\in X$ and α be a quadratic surd. The number β belongs to $\boldsymbol{Q}(\alpha)$ if and only if the name of (α, β) is periodic.*

PROOF. We consider the transformation $\bar{T}$ on the extended domain $X\times \boldsymbol{R}^2$. Such an extension is possible because $\bar{T}$ is defined by two linear fractional transformations. To prove the necessity, it is enough to show that there exists $N\in \boldsymbol{N}$ such that $(\alpha_{N-1}, \beta_{N-1}, \bar{\alpha}_{N-1}, \bar{\beta}_{N-1})\in \bar{X}$. From the definition of β, we have

$$\beta_n=\delta_{n-1}+\frac{\gamma_{n-1}\beta_{n-1}}{\varepsilon_{n-1}-\alpha_{n-1}}\,, \tag{6}$$

where

$$(\gamma_i, \delta_i, \varepsilon_i)=\begin{cases} (1, 0, 1) & \text{if} \quad (\alpha_i, \beta_i)\in \Delta_1 \\ (1, 0, 0) & \text{if} \quad (\alpha_i, \beta_i)\in \Delta_2 \\ (-1, 1, 1) & \text{if} \quad (\alpha_i, \beta_i)\in \Delta_3\,. \end{cases}$$

From (6), we get

$$\begin{aligned} \beta_n=&\left(\delta_{n-1}+\frac{\gamma_{n-2}\delta_{n-2}}{\varepsilon_{n-1}-\alpha_{n-1}}+\cdots+\frac{\gamma_{n-2}\cdots\gamma_1\gamma_0}{(\varepsilon_{n-1}-\alpha_{n-1})\cdots(\varepsilon_1-\alpha_1)}\right) \\ &+\frac{\gamma_{n-2}\cdots\gamma_1\gamma_0\cdot\beta_0}{(\varepsilon_{n-1}-\alpha_{n-1})\cdots(\varepsilon_1-\alpha_1)(\varepsilon_0-\alpha_0)}\,. \end{aligned} \tag{7}$$

Now we define

$$(\alpha_n, \beta_n, \bar{\alpha}_n, \beta_n^*)=\bar{T}^n(\alpha_0, \beta_0, \bar{\alpha}_0, 0)\,.$$

From $(\alpha_0, \beta_0, \bar{\alpha}_0, 0)\in \bar{X}$, we have $(\alpha_n, \beta_n, \bar{\alpha}_n, \beta_n^*)\in \bar{X}$ for every $n\geqq 0$ by Lemma 3 (A). Noting $\gamma_i=\pm 1$, we have

$$|\bar{\beta}_n-\beta_n^*|=\left|\frac{\beta_0}{(\varepsilon_{n-1}-\overline{\alpha_{n-1}})\cdots(\varepsilon_1-\overline{\alpha_1})(\varepsilon_0-\overline{\alpha_0})}\right|$$

by (7). We show $|\bar{\beta}_n-\beta_n^*|\to 0$ as $n\to\infty$.
(1) if $(\alpha_i, \beta_i)\in\Delta_1$ and $|\varepsilon_i-\bar{\alpha}_i|<1$, then $1<\bar{\alpha}_i<2$ by $\varepsilon_i=1$. For

$$\left|(\varepsilon_i-\bar{\alpha}_i)\left(\varepsilon_{i+1}-\frac{\bar{\alpha}_i}{1-\bar{\alpha}_i}\right)\right|=\begin{cases}|\bar{\alpha}_i| & \text{if} \quad \varepsilon_{i+1}=0\\ |1-\bar{\alpha}_i| & \text{if} \quad \varepsilon_{i+1}=1\,,\end{cases}$$

the left hand side of the above formula is larger than 1.
(2) If $(\alpha_i, \beta_i)\in\Delta_2$ and $|\varepsilon_i-\bar{\alpha}_i|<1$, then $-1<\bar{\alpha}_i<0$. Hence

$$\left|(\varepsilon_i-\bar{\alpha}_i)\left(\varepsilon_{i+1}-\frac{2\bar{\alpha}_i-1}{\bar{\alpha}_i}\right)\right|>1\,,$$

which follows in the same way as (1).
(3) If $(\alpha_i, \beta_i)\in\Delta_3$ and $|\varepsilon_i-\bar{\alpha}_i|<1$, then $1<\bar{\alpha}_i<2$. Hence

$$\left|(\varepsilon_i-\bar{\alpha}_i)\left(\varepsilon_{i+1}-\frac{1-2\bar{\alpha}_i}{1-\bar{\alpha}_i}\right)\right|>1\,,$$

which follows in the same way as (1).
From (1), (2) and (3), the sequence $(\alpha_0, \alpha_1, \alpha_2, \cdots)$ is periodic and we see $|\bar{\beta}_n-\beta_n^*|\to 0$ (as $n\to\infty$).

Now we consider the case that $(\bar{\alpha}_n, \beta_n^*)$ belongs to the boundary of X^* for any n larger than some N. We put

$$X^*(\varepsilon)=\{(z, w)\in X^* \mid |w|<\varepsilon \text{ or } |z-w|<\varepsilon\}\,.$$

To analyze the behavior of $(\Delta_i\times X^*)$ by $\bar{T}$, we consider the following two cases.
(1) There exists N such that for any $n\geqq N$,

$$(\bar{\alpha}_n, \beta_n^*)\in\{(z, w)\in X^*(\varepsilon) \mid |w|<\varepsilon\}\,.$$

(2) There exists N such that for any $n\geqq N$,

$$(\bar{\alpha}_n, \beta_n^*)\in\{(z, w)\in X^*(\varepsilon) \mid |z-w|<\varepsilon\}\,.$$

In the case (1), we get $(i_1, i_2, \cdots)\in\{1, 2\}^N$ and such a point (α, β) belongs to

$$I_0=\{(\alpha, \beta)\in X \mid 0\leqq\alpha<1,\ \beta=0\}\,.$$

From $I_0\subset\Delta_1\cup\Delta_2$, for every $(\alpha, \beta)\in I_0$, we have

$$(\alpha_n, \beta_n)=(S^n(\alpha), 0)\,.$$

In the case (2), we get $(i_1, i_2, \cdots)\in\{1, 3\}^N$ and such a pair (α, β) belongs to

$$J_0=\{(\alpha, \beta)\in X \mid \beta=\alpha\}\,.$$

On $J_0\subset\Delta_1\cup\Delta_3$, we define a transformation T' by

$$T'(x, y) = \begin{cases} \left(\dfrac{x}{1-x}, \dfrac{y}{1-x}\right) & \text{if } (x, y) \in \Delta_1 \\ \left(\dfrac{1-2x}{1-x}, \dfrac{1-x-y}{1-x}\right) & \text{if } (x, y) \in \Delta_3 . \end{cases}$$

We see that T' and U are isomorphic by the mapping U_1. The sequence $(i_{U,1}, i_{U,2}, i_{U,3}, \cdots)$ is periodic as we have seen in §1. Hence $(i_1, i_2, \cdots)$ is periodic because the periodicity is preserved by the isomorphism.

Next we show the sufficiency. Before doing this, we need the following:

COROLLARY 1 (The converse of Proposition 2). *If the name of (α, β) is purely periodic, then α is quadratic and β is α-reduced.*

PROOF. By (4) and (5), we have α is quadratic and $\beta \in \mathbf{Q}(\alpha)$. By the necessity part of Theorem 2, there exists N such that

$$(i_1, i_2, \cdots) = (i_1, i_2, \cdots, i_{N-1}, \overline{i_N, \cdots, i_{N+k-1}}) .$$

By the pure periodicity of the name of (α, β), we get $N=1$. From the proof of the necessity part of Theorem 2, we conclude $(\alpha, \beta, \bar{\alpha}, \bar{\beta}) \in \bar{X}$.

Now, we prove the sufficiency part of Theorem 2. Assume that there exist N and k with

$$(i_1, i_2, \cdots) = (i_1, \cdots, i_N, \overline{i_{N+1}, \cdots, i_{N+K}}) .$$

Since the name of (α_N, β_N) is purely periodic, α_N is quadratic and β_N is α_N-reduced by Corollary 1. Therefore α is quadratic and $\beta \in \mathbf{Q}(\alpha)$.

§3. Invariance under substitutions.

Let $C(\alpha, \beta)$ be the characteristic sequence of $(\alpha, \beta) \in X$ in §0. We define the substitutions on the set of finite sequences of $\{0, 1\}$ by

$$\delta_1 : \begin{array}{l} 0 \longrightarrow 0 \\ 1 \longrightarrow 01 \end{array}, \qquad \delta_2 : \begin{array}{l} 0 \longrightarrow 01 \\ 1 \longrightarrow 1 \end{array}, \qquad \delta_3 : \begin{array}{l} 0 \longrightarrow 01 \\ 1 \longrightarrow 0 \end{array} .$$

The following theorem is due to [1].

THEOREM. *Let $(\alpha, \beta) \in X$, and $(i_1, i_2, \cdots)$ be the name of (α, β) introduced in §2. Then the sequence $C(\alpha, \beta)$ is given by*

$$C(\alpha, \beta) = \lim_{n \to \infty} (\delta_{i_1} \circ \delta_{i_2} \circ \cdots \circ \delta_{i_n})(0) .$$

By this theorem and Theorem 2, we have the following corollary.

COROLLARY 2. *Let α be a quadratic surd and $(\alpha, \beta) \in X$ and $(i_1, i_2, \cdots)$ be the name of (α, β). The number β belongs to $\mathbf{Q}(\alpha)$ if and only if there exist N and k such that the sequence $C(\alpha_N, \beta_N)$ is $\delta_{i_{N+1}} \circ \delta_{i_{N+2}} \circ \cdots \circ \delta_{i_{N+k}}$-invariant.*

PROOF. If $\beta \in \mathbf{Q}(\alpha)$, then there exist N and k with

$$(i_1, i_2, \cdots) = (i_1, \cdots, i_N, \overline{i_{N+1}, \cdots, i_{N+k}})$$

by Theorem 2. By [1], we have

$$C(\alpha, \beta) = \lim_{n \to \infty} (\delta_{i_1} \circ \delta_{i_2} \circ \cdots \circ \delta_{i_n})(0) .$$

From

$$C(\alpha_N, \beta_N) = \lim_{m \to \infty} (\delta_{i_{N+1}} \circ \cdots \circ \delta_{i_{N+m}})(0) ,$$

we have

$$C(\alpha_N, \beta_N) = (\delta_{i_{N+1}} \circ \cdots \circ \delta_{i_{N+k}}) \cdot C(\alpha_N, \beta_N) .$$

Conversely if $C(\alpha_N, \beta_N) = (\delta_{i_{N+1}} \circ \cdots \circ \delta_{i_{N+k}}) \cdot C(\alpha_N, \beta_N)$, we get

$$C(\alpha_N, \beta_N) = (\delta_{i_{N+1}} \circ \cdots \circ \delta_{i_{N+k}})^n \cdot C(\alpha_N, \beta_N) \qquad \text{for any} \quad n .$$

Thus we have

$$\lim_{m \to \infty} (\delta_{i_{N+1}} \circ \delta_{i_{N+2}} \circ \cdots \circ \delta_{i_{N+m}})(0) = \lim_{n \to \infty} (\delta_{i_{N+1}} \circ \cdots \circ \delta_{i_{N+k}})^n(0) .$$

For a given (α, β), there is a 1-1 correspondence between the name of (α, β) and the sequence $C(\alpha, \beta)$. Then we have

$$(i_{N+1}, i_{N+2}, \cdots) = (\overline{i_{N+1}, \cdots, i_{N+k}})$$

and

$$(i_1, i_2, \cdots) = (i_1, \cdots, i_N, \overline{i_{N+1}, \cdots, i_{N+k}}) .$$

Therefore we get $\beta \in \mathbf{Q}(\alpha)$ from Theorem 2.

References

[1] SH. ITO and S. YASUTOMI, On continued fractions, substitutions and characteristic sequences $[nx+y]-[(n-1)x+y]$, Japanese J. Math., **16** (1990), 287–306.

[2] SH. ITO, Algorithms with mediant convergents and their metrical theory, Osaka J. Math., **26** (1989), 557–578.

[3] Y. HARA and SH. ITO, On real quadratic fields and periodic expansions, Tokyo J. Math., **12** (1989), 357–370.

[4] H. NAKADA, On ergodic theory of A. Schmidt's complex continued fractions over Gaussian field,

Monatshefte für Math., **105** (1988), 131–150.
[5] A. L. Schmidt, Diophantine approximation of complex numbers, Acta Math., **134** (1975), 1–85.
[6] K. Schmidt, On periodic expansions of Pisot-numbers and Salem-numbers, Bull. London Math. Soc., **12** (1980), 269–278.
[7] B. A. Venkov, *Elementary Number Theory*, Wolter-Noordhoff, 1970.

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