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# Invariant Differential Operators and Spherical Sections on a Homogeneous Vector Bundle

Katsuhiro MINEMURA

*Japan Women's University*
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## Introduction.

In this paper we study the structure of the algebra of invariant differential operators on a homogeneous vector bundle over a reductive coset space, define the eigensections and spherical sections, and determine the dimension of the space of the spherical sections in case of a vector bundle over a Riemannian symmetric space.

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and $K$ be a closed subgroup with Lie algebra $\mathfrak{k}$. We assume that $G/K$ is reductive, that is, there exists an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{p}$ in $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ (direct sum). Let $\mathfrak{g}_c$, $\mathfrak{k}_c$ and $\mathfrak{p}_c$ be the complexifications of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Let $U(\mathfrak{g}_c)$ and $U(\mathfrak{k}_c)$ be the universal enveloping algebra of $\mathfrak{g}_c$ and $\mathfrak{k}_c$ respectively, $S(\mathfrak{p}_c)$ the symmetric algebra of $\mathfrak{p}_c$, and $\mathsf{T}$ the canonical anti-automorphism of $U(\mathfrak{g}_c)$ defined by $1^{\mathsf{T}}=1$, $x^{\mathsf{T}}=-x$, $(xy)^{\mathsf{T}}=y^{\mathsf{T}}x^{\mathsf{T}}$ ($x$, $y\in\mathfrak{g}_c$). Let $\tau$ be a representation of $K$ on a complex vector space $V$ of finite dimension, $d\tau$ be its differential representation of $U(\mathfrak{k}_c)$ and let $\mathscr{I}$ be the kernel of $d\tau$ in $U(\mathfrak{k}_c)$. $\mathscr{I}^{\mathsf{T}}$ denotes the image of $\mathscr{I}$ under $\mathsf{T}$. Let $E_\tau$ be the homogeneous vector bundle over $G/K$ associated to $\tau$ and $D(E_\tau)$ be the algebra of $G$-invariant differential operators on $E_\tau$. In §1 we establish a linear isomorphism between $D(E_\tau)$ and $K$-invariants in $S(\mathfrak{p}_c)\otimes\mathrm{End}\,V$ (Proposition 1.2). Then under a certain condition, we have an algebra isomorphism $D(E_\tau)\cong U(\mathfrak{g}_c)^K/U(\mathfrak{g}_c)^K\cap U(\mathfrak{g}_c)\mathscr{I}^{\mathsf{T}}$ (Theorem 1.3).

Let $\chi$ be a finite-dimensional representation of the algebra $D(E_\tau)$. In §2, we give a definition of an eigensection of $E_\tau$ of type $\chi$ (Definition 2.1) and give a definition of a spherical section (Definition 2.4) which is a generalization of the so-called (zonal) spherical functions ([5], [7]). In particular when $K$ is compact, we can give an upper bound of the dimension of the space of spherical sections (Theorem 2.5).

In §3 we determine the dimension of the space of spherical sections by using the Poisson transform (Definition 3.2) when $G/K$ is a Riemannian symmetric space (Theorem 3.5).

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We shall use $C$ for the field of complex numbers. If $M$ is an analytic manifold, $D$ a differential operator and $f$ a differentiable function, the value of $Df$ at $x$ in $M$ is often denoted by $f(x; D)$. If $E$ is an analytic vector bundle over $M$, we denote by $\mathscr{B}(E)$ the space of hyperfunctions of type $E$ over $M$ (for details, see [13]). In this paper we shall usually call a hyperfunction of type $E$ over $M$ a hyperfunctional section (or simply a hyperfunction) of $E$ over $M$. For a group $G$ and $G$-modules $V$ and $W$, let $V^G$ denote the set of $G$-invariants in $V$ and let $\mathrm{Hom}_G(V, W)$ denote the set of homomorphisms which commute with the action of $G$. For a vector space $H$, let $H^*$ denote the dual space of $H$ and let $\langle\ ,\ \rangle$ denote the canonical bilinear form on $H^* \times H$.

## 1. The algebra of invariant differential operators.

In this section we assume that $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$ and $K$ is a closed subgroup of $G$ with Lie algebra $\mathfrak{k}$ such that $G/K$ is reductive, that is, there exists an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{p}$ complementary to $\mathfrak{k}$ in $\mathfrak{g}$. We fix such $\mathfrak{p}$ once and for all. We keep to the notation in the introduction.

Let $\tau$ be a representation of $K$ on a complex vector space $V_\tau$ of finite dimension and let $E_\tau$ be the vector bundle over $G/K$ associated to $\tau$. Let $C^\infty(G, V_\tau)$ be the space of $C^\infty$ functions on $G$ with values in $V_\tau$ and let $C^\infty(G, \tau)$ be the space of the elements $f$ in $C^\infty(G, V_\tau)$ such that $f(gk)=\tau(k^{-1})f(g)$ for all $g$ in $G$ and all $k$ in $K$. We identify the fiber of $E_\tau$ at $eK$ with $V_\tau$ and define $u^\sim(g)=g^{-1}u(gK)$ for $g$ in $G$ and $u$ in $C^\infty(E_\tau)$. Then $u^\sim$ lies in $C^\infty(G, \tau)$ and $C^\infty(E_\tau)$ is isomorphic to $C^\infty(G, \tau)$ by the correspondence $u \mapsto u^\sim$. Hereafter we identify $C^\infty(E_\tau)$ with $C^\infty(G, \tau)$.

Let $\mathfrak{g}$ act on $C^\infty(G, V_\tau)$ by

$$Xf(g)=\left(\frac{d}{dt} f(g \exp tX)\right)_{t=0} \qquad (X \in \mathfrak{g}).$$

Then this action induces a representation of $U(\mathfrak{g}_c)$ on $C^\infty(G, V_\tau)$. Let $\sigma$ be another finite-dimensional representation of $K$ on $V_\sigma$ and let $L(V_\tau, V_\sigma)$ denote the space of all linear maps from $V_\tau$ to $V_\sigma$. For $X$ in $U(\mathfrak{g}_c)$ and $T$ in $L(V_\tau, V_\sigma)$, set $(X \otimes T)f = T \cdot Xf$ ($f \in C^\infty(G, V_\tau)$). This assigns a differential operater from $G \times V_\tau$ to $G \times V_\sigma$ to each element in $U(\mathfrak{g}_c) \otimes L(V_\tau, V_\sigma)$ ([15, 5.4.5]). Let $D(E_\tau, E_\sigma)$ denote the space of all homogeneous differential operators from $E_\tau$ to $E_\sigma$ (for the definition see [15, 5.4.1]). If $\tau$ equals $\sigma$, we denote $D(E_\tau, E_\sigma)$ simply by $D(E_\tau)$ and call its elements invariant differential operators of $E_\tau$.

Since $\mathfrak{g}$ and $\mathfrak{p}$ are $K$-modules, $K$ acts on $U(\mathfrak{g}_c)$ and $S(\mathfrak{p}_c)$ canonically. Let $K$ act on $L(V_\tau, V_\sigma)$ by

$$kT=\sigma(k) \cdot T \cdot \tau(k^{-1}) \qquad (T \in L(V_\tau, V_\sigma))$$

and let $K$ act on $U(\mathfrak{g}_c) \otimes L(V_\tau, V_\sigma)$ and $S(\mathfrak{p}_c) \otimes L(V_\tau, V_\sigma)$ by tensor product. By [15, Proposition 5.4.11], for any element $D \in (U(\mathfrak{g}_c) \otimes L(V_\tau, V_\sigma))^K$

$$D(C^{\infty}(G, \tau)) \subset C^{\infty}(G, \sigma).$$

Therefore $D$ defines an element $\Delta \in D(E_\tau, E_\sigma)$ through the isomorphism $C^{\infty}(E_\tau) \cong C^{\infty}(G, \tau)$ and $C^{\infty}(E_\sigma) \cong C^{\infty}(G, \sigma)$. We define a map $\mu_0$ from $(U(\mathfrak{g}_c) \otimes L(V_\tau, V_\sigma))^K$ to $D(E_\tau, E_\sigma)$ by sending $D$ to the above $\Delta$. Thus, for $D \in (U(\mathfrak{g}_c) \otimes L(V_\tau, V_\sigma))^K$ and $u \in C^{\infty}(E_\tau)$, putting $\Delta = \mu_0(D)$ we have

$$(\Delta u)^{\sim} = D(u^{\sim}).$$

Let $\Lambda$ be the symmetrization from $S(\mathfrak{g}_c)$ to $U(\mathfrak{g}_c)$ and let $\nu$ be the linear map from $S(\mathfrak{g}_c) \otimes L(V_\tau, V_\sigma)$ to $U(\mathfrak{g}_c) \otimes L(V_\tau, V_\sigma)$ defined by $\nu(p \otimes T) = \Lambda(p) \otimes T$ ($p \in S(\mathfrak{g}_c)$, $T \in L(V_\tau, V_\sigma)$). We regard $S(\mathfrak{p}_c)$ as the subalgebra of $S(\mathfrak{g}_c)$ generated by $\mathfrak{p}$ and 1. Let $\nu_K$ be the restriction of $\nu$ to $(S(\mathfrak{p}_c) \otimes L(V_\tau, V_\sigma))^K$. Since $\nu$ is a $K$-isomorphism, $\nu_K$ is an isomorphism from $(S(\mathfrak{p}_c) \otimes L(V_\tau, V_\sigma))^K$ onto $(\Lambda(S(\mathfrak{p}_c)) \otimes L(V_\tau, V_\sigma))^K$. We define $\zeta_K$ to be the map $\mu_0 \cdot \nu_K$ from $(S(\mathfrak{p}_c) \otimes L(V_\tau, V_\sigma))^K$ into $D(E_\tau, E_\sigma)$.

For simplicity we denote the differential representation of $\tau$ also by the same symbol $\tau$.

LEMMA 1.1. *Let $\tau$ (resp. $\sigma$) be a finite-dimensional representation of $K$ on $V_\tau$ (resp. $V_\sigma$), and let $E_\tau$ (resp. $E_\sigma$) be the associated bundle to $\tau$ (resp. $\sigma$) over $G/K$. Then the map $\zeta_K$ from $(S(\mathfrak{p}_c) \otimes L(V_\tau, V_\sigma))^K$ to $D(E_\tau, E_\sigma)$ is an onto isomorphism.*

PROOF. Let $\Delta$ be an arbitrary element in $D(E_\tau, E_\sigma)$. Then there exists an element $D$ in $U(\mathfrak{g}_c) \otimes L(V_\tau, V_\sigma)$ whose restriction to $C^{\infty}(G, \tau)$ induces $\Delta$ by the identification $C^{\infty}(E_\tau) \cong C^{\infty}(G, \tau)$ ([15, Proposition 5.4.11]). Let $X_1, X_2, \cdots, X_m$ (resp. $Y_1, Y_2, \cdots, Y_n$) be a basis of $\mathfrak{p}$ (resp. $\mathfrak{k}$). By the decomposition $U(\mathfrak{g}_c) = \Lambda(S(\mathfrak{p}_c))U(\mathfrak{k}_c)$ ([3, Proposition 2.4.15]), with multi-indices $\alpha$ and $\beta$, $D$ is written as

$$D = \sum_{\alpha, \beta} \Lambda(X^{\alpha}) Y^{\beta} \otimes A_{\alpha, \beta} \qquad (A_{\alpha, \beta} \in L(V_\tau, V_\sigma)).$$

Set $D_{\mathfrak{p}}$ to be the element $\sum_{\alpha, \beta} \Lambda(X^{\alpha}) \otimes A_{\alpha, \beta}\, \tau((Y^{\beta})^{\mathrm{T}})$ and rewrite it as $\sum_{\alpha} \Lambda(X^{\alpha}) \otimes B_{\alpha}$. Then for any $f \in C^{\infty}(G, \tau)$ and $g \in G$,

$$\begin{aligned}(Df)(g) &= \sum_{\alpha, \beta} A_{\alpha, \beta}(\Lambda(X^{\alpha}) Y^{\beta} f)(g) \\ &= \sum_{\alpha, \beta} A_{\alpha, \beta} \tau((Y^{\beta})^{\mathrm{T}})(\Lambda(X^{\alpha}) f)(g) \\ &= (D_{\mathfrak{p}} f)(g).\end{aligned}$$

Since $\Delta$ belongs to $D(E_\tau, E_\sigma)$,

$$(Df)(gk) = \sigma(k^{-1})(Df)(g) \qquad (g \in G, k \in K).$$

Therefore,

$$(Df)(gk) = \sigma(k^{-1})(Df)(g)$$
$$= \sigma(k^{-1})(D_{\mathfrak{p}}f)(g).$$

On the other hand, since $f$ belongs to $C^\infty(G, \tau)$,

$$(Df)(gk) = (D_{\mathfrak{p}}f)(gk)$$
$$= \sum_\alpha B_\alpha \tau(k^{-1})(\mathrm{Ad}(k)\Lambda(X^\alpha)f)(g).$$

Therefore, we have

$$(D_{\mathfrak{p}}f)(g) = ((\sum_\alpha \sigma(k)B_\alpha \tau(k^{-1})(\mathrm{Ad}(k)\Lambda(X^\alpha)))f)(g)$$
$$= ((kD_{\mathfrak{p}})f)(g).$$

Since $G/K$ is a reductive coset space, $C^\infty(G, \tau)$ has sufficiently many elements in order to conclude that $D_{\mathfrak{p}} = kD_{\mathfrak{p}}$ for all $k \in K$. Therefore $D_{\mathfrak{p}}$ belongs to $(\Lambda(S(\mathfrak{p}_c)) \otimes L(V_\tau, V_\sigma))^K$ and $\mu_0(D_{\mathfrak{p}}) = \Delta$. Hence $\zeta_K$ is surjective. Injectivity of $\zeta_K$ is proved by the same arguments. This finishes the proof.

From now on, we assume moreover that $\sigma$ equals $\tau$ and the differential of $\tau$, which is denoted also by $\tau$, is irreducible. We ragard $U(\mathfrak{g}_c)^K$ as a subset of $(U(\mathfrak{g}_c) \otimes \mathrm{End}\, V_\tau)^K$ by sending $q \mapsto q \otimes \mathrm{Id}$ and let $\mu$ be the restriction of $\mu_0$ to $U(\mathfrak{g}_c)^K$. Extending $\tau$ to the representation of $U(\mathfrak{k}_c)$ canonically, let $\mathscr{I}$ be the kernel of $\tau$ in $U(\mathfrak{k}_c)$. Since $\mathrm{Ad}(k)$ $(k \in K)$ commutes with T and $\mathscr{I}$ is $K$-invariant, $\mathscr{I}^{\mathrm{T}}$ is also $K$-invariant and $S(\mathfrak{p}_c) \otimes U(\mathfrak{k}_c)/\mathscr{I}^{\mathrm{T}}$ is regarded as a $K$-module by tensor product. Then the map $\xi$ given by

$$\xi(q \otimes z) = q \otimes (z + \mathscr{I}^{\mathrm{T}}) \qquad (q \in S(\mathfrak{p}_c),\ z \in U(\mathfrak{k}_c))$$

is clearly a $K$-homomorphism from $S(\mathfrak{p}_c) \otimes U(\mathfrak{k}_c)$ onto $S(\mathfrak{p}_c) \otimes U(\mathfrak{k}_c)/\mathscr{I}^{\mathrm{T}}$. Therefore, the restriction of $\xi$ to $(S(\mathfrak{p}_c) \otimes U(\mathfrak{k}_c))^K$ gives a homomorphism $\xi_K$ from $(S(\mathfrak{p}_c) \otimes U(\mathfrak{k}_c))^K$ into $(S(\mathfrak{p}_c) \otimes U(\mathfrak{k}_c)/\mathscr{I}^{\mathrm{T}})^K$.

Now since $\tau$ is irreducible, the map $\eta$ given by

$$\eta(q \otimes (z + \mathscr{I}^{\mathrm{T}})) = q \otimes \tau(z^{\mathrm{T}}) \qquad (q \in S(\mathfrak{p}_c),\ z \in U(\mathfrak{k}_c))$$

is a well-defined $K$-isomorphism from $S(\mathfrak{p}_c) \otimes U(\mathfrak{k}_c)/\mathscr{I}^{\mathrm{T}}$ onto $S(\mathfrak{p}_c) \otimes \mathrm{End}\, V_\tau$ and the restriction of $\eta$ to $(S(\mathfrak{p}_c) \otimes U(\mathfrak{k}_c)/\mathscr{I}^{\mathrm{T}})^K$ gives an isomorphism $\eta_K$ from $(S(\mathfrak{p}_c) \otimes U(\mathfrak{k}_c)/\mathscr{I}^{\mathrm{T}})^K$ onto $(S(\mathfrak{p}_c) \otimes \mathrm{End}\, V_\tau)^K$.

Let $\psi$ be the map given by

$$\psi(q \otimes z) = \Lambda(q)z \qquad (q \in S(\mathfrak{p}_c),\ z \in U(\mathfrak{k}_c)).$$

Then $\psi$ is a $K$-isomorphism from $S(\mathfrak{p}_c) \otimes U(\mathfrak{k}_c)$ onto $U(\mathfrak{g}_c)$ by [3, Proposition 2.4.15]. Therefore, restricted to $(S(\mathfrak{p}_c) \otimes U(\mathfrak{k}_c))^K$, $\psi$ gives an isomorphism $\psi_K$ of $(S(\mathfrak{p}_c) \otimes U(\mathfrak{k}_c))^K$

onto $U(\mathfrak{g}_c)^K$.

PROPOSITION 1.2. *Let $\tau$ be a finite-dimensional representation of $K$ on $V_\tau$ whose differential is irreducible. Let $\mu$, $\zeta_K$, $\eta_K$, $\xi_K$ and $\psi_K$ be as above. Then*

(i) *$\mu$ is an algebra homomorphism of $U(\mathfrak{g}_c)^K$ into $D(E_\tau)$ and the following diagram is commutative*:

$$\begin{tikzcd}
(S(\mathfrak{p}_c)\otimes U(\mathfrak{k}_c))^K \arrow[r, "\xi_K"] \arrow[d, "\psi_K"', "\cong"] & (S(\mathfrak{p}_c)\otimes U(\mathfrak{k}_c)/\mathscr{I}^{\mathsf{T}})^K \arrow[r, "\eta_K, \cong"] & (S(\mathfrak{p}_c)\otimes \mathrm{End}\, V_\tau)^K \arrow[d, "\zeta_K"', "\cong"] \\
U(\mathfrak{g}_c)^K \arrow[rr, "\mu"] & & D(E_\tau)\,.
\end{tikzcd}$$

(ii) *The kernel of $\mu$ is $U(\mathfrak{g}_c)^K \cap U(\mathfrak{g}_c)\mathscr{I}^{\mathsf{T}}$.*

(iii) *Let $\mu_K$ be the algebra isomorphism of $U(\mathfrak{g}_c)^K/U(\mathfrak{g}_c)^K \cap U(\mathfrak{g}_c)\mathscr{I}^{\mathsf{T}}$ into $D(E_\tau)$ given by $\mu$ and* (ii). *Then $\mu_K$ is bijective if, and only if, $\xi_K$ is surjective.*

PROOF. (i) Take any $p \in (S(\mathfrak{p}_c)\otimes U(\mathfrak{k}_c))^K$, write it as $p=\sum q_i \otimes z_i$ ($q_i \in S(\mathfrak{p}_c)$, $z_i \in U(\mathfrak{k}_c)$) and set $D=\psi_K(p)$. Then for any $f \in C^\infty(G, \tau)$ and $g \in G$, we have

$$D=\sum \Lambda(q_i)z_i\,,$$
$$\eta_K(\xi_K(p))=\sum q_i \otimes \tau(z_i^{\mathsf{T}})$$

and

$$\begin{aligned}(\mu(D)f)(g) &= (\textstyle\sum \Lambda(q_i)z_i f)(g)\\ &= \textstyle\sum \tau(z_i^{\mathsf{T}})(\Lambda(q_i)f)(g)\,.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(\zeta_K(\eta_K(\xi_K(p)))f)(g) &= (\zeta_K(\textstyle\sum q_i \otimes \tau(z_i^{\mathsf{T}}))f)(g)\\ &= ((\textstyle\sum \Lambda(q_i)\otimes \tau(z_i^{\mathsf{T}}))f)(g)\\ &= \textstyle\sum \tau(z_i^{\mathsf{T}})(\Lambda(q_i)f)(g)\,.\end{aligned}$$

Therefore we get

$$\mu(D)f=\zeta_K(\eta_K(\xi_K(p)))f\,.$$

(ii) By the bijectivity of the maps $\psi_K$, $\eta_K$ and $\zeta_K$, we have $\ker \mu = U(\mathfrak{g}_c)^K \cap \psi(\ker \xi)$ by (i). Since $\ker \xi$ is equal to $S(\mathfrak{p}_c)\otimes \mathscr{I}^{\mathsf{T}}$, $\ker \mu$ is equal to $U(\mathfrak{g}_c)^K \cap U(\mathfrak{g}_c)\mathscr{I}^{\mathsf{T}}$ ([3, Proposition 2.4.15]).

(iii) This follows immediately from (i). This completes the proof.

Now we give some sufficient conditions for the surjectivity of $\mu_K$.

THEOREM 1.3. *Let $G$ be a connected Lie group and let $K$ be a closed subgroup of $G$ such that $G/K$ is reductive. Let $\tau$ be a finite-dimensional representation of $K$ such that its differential representation is irreducible and let $\mathscr{I}$ be the kernel of $\tau$ in $U(\mathfrak{k}_c)$. Let $E_\tau$ be the vector bundle over $G/K$ associated to $\tau$ and let $D(E_\tau)$ be the algebra of invariant differential operators on the vector bundle $E_\tau$. Then the algebra isomorphism $\mu_K$ of $U(\mathfrak{g}_c)^K/U(\mathfrak{g}_c)^K \cap U(\mathfrak{g}_c)\mathscr{I}^{\mathrm{T}}$ into $D(E_\tau)$ is surjective if one of the following conditions is satisfied*:

(i) *There exists a $K$-invariant subspace $\mathscr{J}$ in $U(\mathfrak{k}_c)$ such that $U(\mathfrak{k}_c)=\mathscr{I}\oplus\mathscr{J}$ (direct sum), or equivalently $U(\mathfrak{k}_c)=\mathscr{I}^{\mathrm{T}}\oplus\mathscr{J}$ (direct sum).*

(ii) *The adjoint representation of $K$ on $\mathfrak{k}_c$ is semisimple.*

(iii) Ad$(K)$ *is compact.*

PROOF. Clearly (iii) implies (ii) and (ii) implies (i) (cf. [1]). Now assume that $\mathscr{J}$ is a $K$-invariant subspace such that $U(\mathfrak{k}_c)=\mathscr{I}\oplus\mathscr{J}$. Then since Ad$(k)$ $(k\in K)$ commutes with T, we have $U(\mathfrak{k}_c)=\mathscr{I}^{\mathrm{T}}\oplus\mathscr{J}^{\mathrm{T}}$ and a $K$-homomorphism $\alpha: S(\mathfrak{p}_c)\otimes U(\mathfrak{k}_c)/\mathscr{I}^{\mathrm{T}}\to S(\mathfrak{p}_c)\otimes U(\mathfrak{k}_c)$ such that $\xi\cdot\alpha$ is identity. Therefore $\xi_K$ is surjective and consequently $\mu_K$ is surjective by Proposition 1.2. This finishes the proof.

REMARK. Let $\tau$ be the trivial representation of $K$ and let $D(G/K)$ denote the algebra of invariant differential operators on $G/K$. Then, since $\mathscr{I}=U(\mathfrak{k}_c)\mathfrak{k}$, we have a $K$-invariant direct sum decomposition $U(\mathfrak{k}_c)=U(\mathfrak{k}_c)\mathfrak{k}\oplus C$, and by the above theorem, we have the well-known isomorphism

$$U(\mathfrak{g}_c)^K/U(\mathfrak{g}_c)^K\cap U(\mathfrak{g}_c)\mathfrak{k}\cong D(G/K)$$

for the reductive coset space $G/K$.

## 2. Eigensections and spherical sections.

In this section we assume that $G$ is a connected Lie group and $K$ is a compact subgroup of $G$. Then the coset space $G/K$ is automatically reductive ([7]). We keep to the notation in §1.

Let $R(D_\tau)$ denote the set of the equivalence classes of finite-dimensional representations of the algebra $D(E_\tau)$. We denote a representation $\chi$ of $D(E_\tau)$ on a vector space $H$ by $(\chi, H)$, and denote its equivalence class by $[\chi, H]$ (or simply by $[\chi]$).

Let $\mathscr{B}(E_\tau)$ be the space of hyperfunctional sections of $E_\tau$ over $G/K$. Let $G$ act on $\mathscr{B}(E_\tau)$ by $\pi(g)u(x)=gu(g^{-1}x)$ for $g\in G$, $u\in\mathscr{B}(E_\tau)$, and $x\in G/K$ (see [15]). Then $\mathscr{B}(E_\tau)$ is regarded as a $G$-module by $\pi$. For any $u$ in $\mathscr{B}(E_\tau)$, let $u^\sim$ denote the $V_\tau$-valued hyperfunction on $G$ which corresponds to $u$ by $u^\sim(g)=g^{-1}u(gK)$ $(g\in G)$ in the same way as §1. With the consideration of Lemma 3.4 we give the following

DEFINITION 2.1. Let $[\chi, H]$ be in $R(D_\tau)$. A (hyperfunctional) section $u$ in $\mathscr{B}(E_\tau)$ is called an eigensection of type $\chi$ if there exists a finite number of $D(E_\tau)$-invariant subspaces $H_i$ such that 1) $u$ belongs to $\sum_i H_i$, and 2) as a $D(E_\tau)$-module, each $H_i$ is isomorphic to a quotient $D(E_\tau)$-module of $H$.

Let $\mathscr{B}(E_\tau, \chi)$ be the space of eigensections of type $\chi$. Because of the $G$-invariance of $\Delta \in D(E_\tau)$, $\mathscr{B}(E_\tau, \chi)$ is a $G$-invariant subspace of $\mathscr{B}(E_\tau)$.

LEMMA 2.2. *All sections in $\mathscr{B}(E_\tau, \chi)$ are analytic.*

PROOF. Let $\mathfrak{p}$ be an $\mathrm{Ad}(K)$-invariant subspace of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. Then there exists an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{p}$ since $K$ is compact. Let $X_1, X_2, \cdots, X_m$ be an orthonormal basis of $\mathfrak{p}$ with respect to this inner product. Then the element $q=(\sum_i X_i^2)\otimes 1$ belongs to $(S(\mathfrak{p}_c)\otimes \mathrm{End}\, V_\tau)^K$. Put $\Delta=\zeta_K(q)$. Then $\Delta$ is an elliptic differential operator. Let $S$ be a subspace of $\mathscr{B}(E_\tau, \chi)$ which is isomorphic to a quotient of $H$ and take an arbitrary element $u$ in $S$. Then, since $\dim(S)$ is finite, there exists a monomial $P(t)$ in $t$ such that $P(\Delta)u=0$. Since $P(\Delta)$ is also elliptic, we see that $u$ is analytic. This finishes the proof.

Hereafter we write $\mathscr{A}(E_\tau, \chi)$ instead of $\mathscr{B}(E_\tau, \chi)$.

Let $d(\tau)$ be the dimension of $V_\tau$ and let $dk$ be the Haar measure on $K$ so normalized that the total volume of $K$ is equal to 1. Let $f$ be a $V_\tau$-valued analytic function on $G$ and consider the following three conditions:

(C1) $f(gk)=\tau(k^{-1})f(g) \quad (g\in G, k\in K)$,

(C2) $f(kgk^{-1})=f(g) \quad (g\in G, k\in K)$,

(C3) $d(\tau)\int_K \overline{\mathrm{tr}(\tau(k))}f(k^{-1}g)dk=f(g) \quad (g\in G)$.

Let $\mathscr{S}$ (resp. $\mathscr{T}$) be the space of such analytic functions on $G$ that satisfies (C1) and (C3) (resp. (C2) and (C3)). Let $e$ denote the identity element in $G$.

LEMMA 2.3. *Let $S$ and $T$ be the linear maps defined by*

$$(Sf)(g)=\int_K f(kgk^{-1})dk \qquad (f\in\mathscr{S},\ g\in G),$$

$$(Tf)(g)=d(\tau)^2\int_K \tau(k)f(gk)dk \qquad (f\in\mathscr{T},\ g\in G).$$

*Then*

(i) *$S$ is an isomorphism of $\mathscr{S}$ onto $\mathscr{T}$ and $T$ is its inverse.*

(ii) *Let $f$ be an element of $\mathscr{S}$ and let $D$ be an element of $U(\mathfrak{g}_c)^K$. Then $(Sf)(e; D)$ is equal to $f(e; D)$. In particular, if $f$ satisfies $f(e; D)=0$ for all $D\in U(\mathfrak{g}_c)^K$, then $f$ is equal to zero.*

PROOF. The first assertion (i) is clear if we notice that $\mathrm{tr}(\tau(k^{-1}))=\overline{\mathrm{tr}(\tau(k))}$ and that

$$d(\tau)\int_K \overline{\operatorname{tr}(\tau(k))}\tau(k)dk=\mathrm{Id}\,,$$

$$d(\tau)\int_K \tau(kk_1k^{-1})dk=(\operatorname{tr}(\tau(k_1)))\mathrm{Id}$$

(here Id denotes the identity operator on $V_\tau$), since $\tau$ is irreducible. Let $D$ be in $U(\mathfrak{g}_c)^K$. Then

$$(Sf)(e;\,D)=\int_K f(e;\,\mathrm{Ad}(k)D)dk=f(e;\,D)\,.$$

Now suppose $f(e;\,D)=0$ for all $D\in U(\mathfrak{g}_c)^K$. Given $D\in U(\mathfrak{g}_c)$, put

$$D^K=\int_K \mathrm{Ad}(k)Ddk\,.$$

Since $(Sf)(e;\,\mathrm{Ad}(k)D)=(Sf)(e;\,D)$, we have $(Sf)(e;\,D^K)=(Sf)(e;\,D)$, Then using (i), we have $(Sf)(e;D)=f(e;\,D^K)=0$ for all $D\in U(\mathfrak{g}_c)$. $G$ being connected, $Sf=0$. Hence $S=0$ by (i). This completes the proof.

Let $[\chi,\,H]\in R(D_\tau)$. Then from the definition of $\mathscr{A}(E_\tau,\,\chi)$, we have $\Delta u=0$ for $\Delta\in\ker\chi$ and $u\in\mathscr{A}(E_\tau,\,\chi)$. Therefore $u$ gives a map $D(E_\tau)/\ker\chi\to V_\tau$ by $\Delta\mapsto(\Delta u)(eK)$. Thus we can define a linear map $s_\chi$ of $\mathscr{A}(E_\tau,\,\chi)$ into $V_\tau\otimes(D(E_\tau)/\ker\chi)^*$ such that

$$\langle s_\chi(u),\,\Delta+\ker\chi\rangle=(\Delta u)(eK)$$

for $\forall u\in\mathscr{A}(E_\tau,\,\chi)$ and $\forall\Delta\in D(E_\tau)$.

DEFINITION 2.4. A section $u$ in $\mathscr{A}(E_\tau,\,\chi)$ is called spherical if $u$ satisfies the condition

$$d(\tau)\int_K \overline{\operatorname{tr}(\tau(k))}\pi(k)udk=u\,.$$

It is clear that $u\in\mathscr{A}(E_\tau,\,\chi)$ satisfies the above condition if and only if $u^\sim$ satisfies condition (C3). Let $\mathscr{A}(E_\tau,\,\chi)^\tau$ denote the space of spherical sections in $\mathscr{A}(E_\tau,\,\chi)$.

THEOREM 2.5. *Let $G$ be a connected Lie group and $K$ be a compact subgroup of $G$. Let $\tau$ be a finite-dimensional representation of $K$ and assume that the differential of $\tau$ is irreducible. Let $E_\tau$ be the vector bundle over $G/K$ associated to $\tau$ and let $\chi$ be a finite-dimensional representation of $D(E_\tau)$. Then the restriction of $s_\chi$ to $\mathscr{A}(E_\tau,\,\chi)^\tau$ is injective. In particular,*

$$\dim\mathscr{A}(E_\tau,\,\chi)^\tau\le d(\tau)\cdot\dim(D(E_\tau)/\ker\chi)\,.$$

PROOF. Suppose $u\in\mathscr{A}(E_\tau,\,X)^\tau$ satisfies $s_\chi(u)=0$. Then by the definition of $s_\chi$, for $\forall D\in U(\mathfrak{g}_c)^K$,

$$(\mu(D)u)^{\sim}(e)=0\,.$$

Since $(\mu(D)u)^{\sim}(e)=u^{\sim}(e;D)$, we get $u^{\sim}=0$ by using Lemma 2.2 and Lemma 2.3, which completes the proof.

## 3. Spherical sections and the Poisson transform on Riemannian symmetric spaces.

In this section we assume that $G$ is a connected semisimple Lie group with finite center. Let $K$ be a maximal compact subgroup of $G$ and let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $G$ and $K$, respectively as before. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. Since $\mathfrak{p}$ is $\mathrm{Ad}(K)$-invariant, $G/K$ is reductive. We keep to the notation in previous sections. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, then complexify $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{a}$ to $\mathfrak{g}_c$, $\mathfrak{k}_c$ and $\mathfrak{a}_c$ respectively, and introduce a linear order in the dual space $\mathfrak{a}^*$. Let $\Sigma$ and $\Sigma_+$ be the set of restricted roots and positive restricted roots, respectively. For any root $\alpha$ in $\Sigma$, we denote by $\mathfrak{g}_\alpha$ the root space in $\mathfrak{g}$ corresponding to $\alpha$. We put $\mathfrak{n}=\sum_{\alpha\in\Sigma_+}\mathfrak{g}_\alpha$ and $\rho=\frac{1}{2}\sum_{\alpha\in\Sigma_+}\dim(\mathfrak{g}_\alpha)\alpha$. Let $A$ and $N$ denote the analytic subgroups of $G$ corresponding to $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Then by the Iwasawa decomposition $G=KAN$, write any $g\in G$ as $g=\kappa(g)e^{H(g)}n$ with $\kappa(g)\in K$, $H(g)\in\mathfrak{a}$ and $n\in N$. Let $M$ be the centralizer of $A$ in $K$ and put $P=MAN$.

Let $\tau$ be an irreducible representation of $K$ on a vector space $V$ and let $E_\tau$ be the vector bundle over $G/K$ associated to $\tau$. In this section too, we denote the differential of $\tau$ by $\tau$. Let $\tau_M$ denote the restriction of $\tau$ to the group $M$. For a finite-dimensional representation $\sigma$ of $M$ on $V_\sigma$ and $\lambda\in\mathfrak{a}_c^*$, let $F_{\sigma,\lambda}$ denote the vector bundle over $G/P$ associated to the representation $P\ni man\mapsto e^{(-\lambda+\rho)H(a)}\sigma(m)$ $(m\in M, a\in A, n\in N)$. In particular, if $\sigma=\tau_M$, we write simply $F_{\tau,\lambda}$ instead of $F_{\tau_M,\lambda}$. Let $\mathscr{B}(E_\tau)$ and $\mathscr{B}(F_{\sigma,\lambda})$ be the spaces of hyperfunctional sections of $E_\tau$ and $F_{\sigma,\lambda}$ over $G/K$ and $G/P$, respectively. As in §2, $\mathscr{B}(E_\tau)$ and $\mathscr{B}(F_{\sigma,\lambda})$ are considered as $G$-modules and we denote the action of $G$ on $\mathscr{B}(E_\tau)$ and $\mathscr{B}(F_{\sigma,\lambda})$ by $\pi$ and $\pi_{\sigma,\lambda}$, respectively. Let $\mathscr{B}(G,\tau)$ and $\mathscr{B}(G,(\sigma,\lambda))$ denote the spaces of $V$-valued hyperfunctions $f$ on $G$ such that $f(gk)=\tau(k^{-1})f(g)$ $(g\in G, k\in K)$ and $f(gman)=e^{(\lambda-\rho)H(a)}\sigma(m^{-1})f(g)$ $(g\in G, m\in M, a\in A, n\in N)$, respectively. Then in the same way as $C^\infty(E_\tau)\cong C^\infty(G,\tau)$, we have the canonical isomorphisms $\mathscr{B}(E_\tau)\cong\mathscr{B}(G,\tau)$ and $\mathscr{B}(F_{\sigma,\lambda})\cong\mathscr{B}(G,(\sigma,\lambda))$. Throughout this section, we identify $\mathscr{B}(E_\tau)$ and $\mathscr{B}(F_{\sigma,\lambda})$ with $\mathscr{B}(G,\tau)$ and $\mathscr{B}(G,(\sigma,\lambda))$, respectively, by these isomorphisms.

Let $\hat{M}$ be the set of equivalence classes of irreducible representations of $M$ and $R_\tau$ be the set of $\sigma$ in $\hat{M}$ such that $[\tau:\sigma]$, the multiplicity of $\sigma$ in $\tau_M$, is positive. For $\sigma\in\hat{M}$ we denote by $V_\sigma$ the fixed $M$-module in $\sigma$, and for simplicity we denote the representation of $M$ on $V_\sigma$ also by $\sigma$.

Let $U(\mathfrak{g}_c)$, $U(\mathfrak{n}_c)$, $U(\mathfrak{a}_c)$ and $U(\mathfrak{k}_c)$ be the universal enveloping algebras of $\mathfrak{g}_c$, $\mathfrak{n}_c$, $\mathfrak{a}_c$ and $\mathfrak{k}_c$ respectively. For $D\in U(\mathfrak{g}_c)^K$ let $\omega(D)$ be the element in $U(\mathfrak{a}_c)U(\mathfrak{k}_c)$ determined by the condition $D-\omega(D)\in\mathfrak{n}U(\mathfrak{g}_c)$. Then identifying $U(\mathfrak{a}_c)U(\mathfrak{k}_c)$ with the algebra

$U(\mathfrak{a}_c)\otimes U(\mathfrak{k}_c)$, we have an algebra anti-isomorphism $\omega$ of $U(\mathfrak{g}_c)^K$ into $U(\mathfrak{a}_c)\otimes U(\mathfrak{k}_c)^M$ ([3], [9], [12]). Let $\mathscr{I}$ be the kernel of $\tau$ in $U(\mathfrak{k}_c)$ and $\mathrm{End}_M V$ be the set of $T\in\mathrm{End}\ V$ such that $\tau(m)\cdot T=T\cdot\tau(m)$ for all $m\in M$. Let $\sharp$ denote the automorphism of $U(\mathfrak{a}_c)$ given by $\sharp(H)=H+\rho(H)$ $(H\in\mathfrak{a})$ and put $\omega_\tau=(\sharp\otimes(\tau\cdot\mathrm{T}))\cdot\omega$, $\omega'_\tau=(\sharp\otimes\tau)\cdot\omega$, where $\tau$ is regarded as a homomorphism of $U(\mathfrak{k}_c)$ onto $\mathrm{End}\ V$ and $\mathrm{T}$ is the canonical anti-automorphism of $U(\mathfrak{g}_c)$. Then $\omega_\tau$ (resp. $\omega'_\tau$) is an algebra homomorphism (resp. anti-homomorphism) of $U(\mathfrak{g}_c)^K$ into $U(\mathfrak{a}_c)\otimes\mathrm{End}_M V$, and the kernel of $\omega_\tau$ is equal to $U(\mathfrak{g}_c)^K\cap U(\mathfrak{g}_c)\mathscr{I}^{\mathrm{T}}$ ([9, Corollary 4.5]). Since $K$ is compact and connected and $\tau$ is irreducible, $\mu_K: U(\mathfrak{g}_c)^K/U(\mathfrak{g}_c)^K\cap U(\mathfrak{g}_c)\mathscr{I}^{\mathrm{T}}\to D(E_\tau)$ is a surjective algebra isomorphism by Theorem 1.3. Therefore there exists a unique algebra isomorphism $\chi_\tau$ of $D(E_\tau)$ into $U(\mathfrak{a}_c)\otimes\mathrm{End}_M V$ such that $\chi_\tau\cdot\mu=\omega_\tau$.

For $\lambda\in\mathfrak{a}_c$, let $e_\lambda$ be the evaluation map of $U(\mathfrak{a}_c)$ into $\boldsymbol{C}$ defined by $e_\lambda(H)=\langle\lambda, H\rangle$ $(H\in\mathfrak{a}_c)$. For $\sigma\in R_\tau$, let $H_\sigma$ denote $\mathrm{Hom}_M(V_\sigma, V)$. Since $V$ is isomorphic to $\sum_{\sigma\in R_\tau}V_\sigma\otimes H_\sigma$ as an $M$-module, we have $\mathrm{End}_M V\cong\sum_{\sigma\in R_\tau}\mathrm{End}H_\sigma$. Define $\omega_\sigma$ to be the projection of $\mathrm{End}_M V$ to $\mathrm{End}H_\sigma$ according to this decomposition. Put

$$\chi_{\tau,\lambda}=(e_\lambda\otimes\mathrm{Id})\cdot\chi_\tau\,,$$

$$\chi_{\tau,\sigma,\lambda}=(e_\lambda\otimes\omega_\sigma)\cdot\chi_\tau\,,$$

$$\omega_{\tau,\lambda}=\chi_{\tau,\lambda}\cdot\mu\,,$$

$$\omega_{\tau,\sigma,\lambda}=\chi_{\tau,\sigma,\lambda}\cdot\mu\,.$$

For $\sigma\in\hat{M}$ and $\lambda\in\mathfrak{a}_c^*$, let $V(\sigma,\lambda)$ denote the space of $K$-finite sections of $\mathscr{B}(F_{\sigma,\lambda})$ under $\pi_{\sigma,\lambda}$. Then by the differential representation of $\pi_{\sigma,\lambda}$, which we shall denote also by $\pi_{\sigma,\lambda}$, $V(\sigma,\lambda)$ is regarded as an admissible $(\mathfrak{g}_c, K)$-module and hence $\mathrm{Hom}_K(V, V(\sigma,\lambda))$ is regarded as a $U(\mathfrak{g}_c)^K$-module canonically. Let $\tau^*$ and $\sigma^*$ $(\sigma\in R_\tau)$ denote the contragredient representation of $\tau$ and $\sigma$ on $V^*$ and $V_\sigma^*$, respectively, and put $\omega'_{\tau^*,\sigma^*,\lambda}=(e_\lambda\otimes\omega_{\sigma^*})\cdot\omega'_{\tau^*}$. Since $\sigma^*$ is irreducible, the bilinear form

$$\langle b, a\rangle=\mathrm{tr}(a^t\cdot b)\qquad(b\in H_{\sigma^*}, a\in H_\sigma)$$

is non-singular, where $a^t$ denotes the transpose of $a$.

LEMMA 3.1. *Let $\sigma$ be in $R_\tau$ and $\lambda$ be in $\mathfrak{a}_c^*$, and regard $H_\sigma$ and* $\mathrm{Hom}_K(V^*, V(\sigma^*, -\lambda))$ *as $U(\mathfrak{g}_c)^K$-module by $\omega_{\tau,\sigma,\lambda}$ and $\pi_{\sigma^*,-\lambda}$, respectively.*

(i) *For $D\in U(\mathfrak{g}_c)^K$, $b\in H_{\sigma^*}$ and $a\in H_\sigma$ we have*

$$\langle\omega'_{\tau^*,\sigma^*,\lambda}(D)b, a\rangle=\langle b, \omega_{\tau,\sigma,\lambda}(D)a\rangle\,,$$

*and as a $U(\mathfrak{g}_c)^K$-module, $H_\sigma$ is isomorphic to* $\mathrm{Hom}_K(V^*, V(\sigma^*, -\lambda))$.

(ii) *If $V(\sigma^*, -\lambda)$ is an irreducible $(\mathfrak{g}_c, K)$-module, then $H_\sigma$ is irreducible.*

PROOF. (i) Take any $D\in U(\mathfrak{g}_c)^K$ and write $D$ as

$$D=\sum_i h_i z_i+w\qquad(h_i\in U(\mathfrak{a}_c),\ z_i\in U(\mathfrak{k}_c),\ w\in\mathfrak{n}U(\mathfrak{g}_c))\,.$$

Then for $b\in H_{\sigma^*}$ and $a\in H_\sigma$, we have

$$\begin{aligned}\langle\omega'_{\tau^*,\sigma^*,\lambda}(D)b, a\rangle &= \mathrm{tr}(a^t\omega'_{\tau^*,\sigma^*,\lambda}(D)b)\\ &= \mathrm{tr}(\textstyle\sum a^t e_{\lambda+\rho}(h_i)\tau^*(z_i)b)\\ &= \mathrm{tr}(\textstyle\sum e_{\lambda+\rho}(h_i)a^t\tau^t(z_i^{\mathsf{T}})b)\\ &= \mathrm{tr}((\textstyle\sum e_{\lambda+\rho}(h_i)\tau(z_i^{\mathsf{T}})a)^t b)\\ &= \mathrm{tr}((\omega_{\tau,\sigma,\lambda}(D)a)^t b)\\ &= \langle b, \omega_{\tau,\sigma,\lambda}(D)a\rangle\,.\end{aligned}$$

On the other hand, $H_{\sigma^*}$ is adjoint to $\mathrm{Hom}_K(V^*, V(\sigma^*, -\lambda))$ by [9, Theorem 5.5 and Theorem 7.2]. Therefore $H_\sigma$ is isomorphic to $\mathrm{Hom}_K(V^*, V(\sigma^*, -\lambda))$ as a $U(\mathfrak{g}_c)^K$-module.

(ii) This follows immediately from [10, Theorem 5.5]. This finishes the proof.

Now we are in the position to give the definition of the Poisson transform for the vector bundle $E_\tau$ following Okamoto [11]. Let $dk$ be the normalized Haar measure on $K$. For $\phi\in\mathscr{B}(F_{\tau,\lambda})$, consider the function $\mathscr{P}_{\tau,\lambda}\phi$ on $G$ given by

$$(\mathscr{P}_{\tau,\lambda}\phi)(g)=\int_K \tau(k)\phi(gk)dk \qquad (g\in G)\,.$$

Then one can show that

$$(\mathscr{P}_{\tau,\lambda}\phi)(g)=\int_K e^{-(\lambda+\rho)(H(g^{-1}k))}\tau(\kappa(g^{-1}k))\phi(k)dk$$

and that $\mathscr{P}_{\tau,\lambda}\phi$ belongs to $\mathscr{B}(E_\tau)$. Put $\Psi_{\tau,\lambda}(g)=e^{-(\lambda+\rho)(H(g^{-1}))}\tau(\kappa(g^{-1}))$. Since $\Psi_{\tau,\lambda}$ is analytic and $K$ is compact, $\mathscr{P}_{\tau,\lambda}\phi$ is an analytic section of $E_\tau$. Let $\mathscr{A}(E_\tau)$ denote the space of analytic sections of $E_\tau$ on $G/K$.

DEFINITION 3.2. The map $\mathscr{P}_{\tau,\lambda}$ from $\mathscr{B}(F_{\tau,\lambda})$ into $\mathscr{A}(E_\tau)$ is called the Poisson transform for $E_\tau$.

LEMMA 3.3. *For any* $D\in U(\mathfrak{g}_c)^K$, $\Psi_{\tau,\lambda}(g; D)=\Psi_{\tau,\lambda}(g)\cdot\omega_{\tau,\lambda}(D)$ $(g\in G)$.

PROOF. Write $D\in U(\mathfrak{g}_c)^K$ as

$$D=\sum_i h_i z_i+\sum_j w_j h'_j z_j\,,$$

where $w_j\in\mathfrak{n}U(\mathfrak{n}_c)$, $h_i$ and $h'_j\in U(\mathfrak{a}_c)$, and $z_j\in U(\mathfrak{k}_c)$. Then for $n\in N$, $a\in A$ and $k\in K$, we have

$$\begin{aligned}\Psi_{\tau,\lambda}(nak; D)&=\tau(k^{-1})\Psi_{\tau,\lambda}(na; D)\\ &=\tau(k^{-1})\left\{\sum_i\Psi_{\tau,\lambda}(na; h_i z_i)+\sum_i\Psi_{\tau,\lambda}(na; w_j h_j z_j)\right\}\end{aligned}$$

$$=\tau(k^{-1})e^{(\lambda+\rho)(H(a))}\sum_i e_{\lambda+\rho}(h_i)\tau(z_i^{\mathrm{T}})\,,$$

since $\Psi_{\tau,\lambda}(na;X)=0$ for $X\in\mathfrak{n}$. Therefore we get

$$\begin{aligned}\Psi_{\tau,\lambda}(nak;D)&=\Psi_{\tau,\lambda}(nak)\sum_i e_\lambda(\sharp h_i)\tau(z_i^{\mathrm{T}})\\&=\Psi_{\tau,\lambda}(nak)\cdot\omega_{\tau,\lambda}(D)\,,\end{aligned}$$

which completes the proof.

Now for $\sigma\in R_\tau$, regard $H_\sigma$ as a trivial bundle over $G/K$. Then, since $F_{\tau,\lambda}$ is isomorphic to the direct sum of $F_{\sigma,\lambda}\otimes H_\sigma$ ($\sigma\in R_\tau$), we have

$$\mathscr{B}(F_{\tau,\lambda})\cong\sum_{\sigma\in R_\tau}\mathscr{B}(F_{\sigma,\lambda})\otimes H_\sigma \qquad \text{(direct sum)}$$

and $\mathscr{B}(F_{\sigma,\lambda})\otimes H_\sigma$ is regarded as a subspace of $\mathscr{B}(F_{\tau,\lambda})$ by the $G$-isomorphism

$$\mathscr{B}(F_{\sigma,\lambda})\otimes H_\sigma\ni\sum_i\phi_i\otimes a_i\longmapsto\sum_i a_i\phi_i\in\mathscr{B}(F_{\tau,\lambda})\,.$$

Define the map $\mathscr{P}_{\tau,\sigma,\lambda}$ to be the restriction of $\mathscr{P}_{\tau,\lambda}$ to $\mathscr{B}(F_{\sigma,\lambda})\otimes H_\sigma$.

Let $H_{\tau,\sigma}$ ($\sigma\in R_\tau$) denote $\mathrm{Hom}_M(V,\ V_\sigma)$. For $\lambda\in\mathfrak{a}_c^*$ and $w=\sum_i v_i\otimes b_i\in V\otimes H_{\tau,\sigma}$ ($v_i\in V$, $b_i\in H_{\tau,\sigma}$), the function $\psi_w$ given by

$$\psi_w(g)=\sum_i e^{(\lambda-\rho)(H(g))}b_i\cdot\tau(\kappa(g)^{-1})v_i$$

clearly belongs to $\mathscr{B}(F_{\sigma,\lambda})$. Let $\mathscr{B}(F_{\sigma,\lambda})^\tau$ be the set of the section $u\in\mathscr{B}(F_{\sigma,\lambda})$ satisfying

$$d(\tau)\int_K\overline{\mathrm{tr}(\tau(k))}\pi_{\sigma,\lambda}(k)u\,dk=u\,.$$

Then the map $\Gamma_{\sigma,\lambda}\colon w\mapsto\psi_w$ is an isomorphism of $V\otimes H_{\tau,\sigma}$ onto $\mathscr{B}(F_{\sigma,\lambda})^\tau$ by the Frobenius reciprocity theorem. Hereafter we identify $V\otimes H_{\tau,\sigma}$ with $\mathscr{B}(F_{\sigma,\lambda})$ by $\Gamma_{\sigma,\lambda}$.

Let $(\chi, H)$ be a subrepresentation of $(\chi_{\tau,\sigma,\lambda}, H_\sigma)$. Since $\sigma$ is irreducible, the bilinear form $\langle b,a\rangle=d(\tau)^{-1}\mathrm{tr}(b\cdot a)$ ($b\in H_{\tau,\sigma}$, $a\in H_\sigma$) is non-singular. By this bilinear form we can identify $H_\sigma$ with $H_{\tau,\sigma}^*$, dual of $H_{\tau,\sigma}$. Then we have canonical isomorphisms

$$D(E_\tau)/\ker\chi\cong\chi(D(E_\tau))\subset\mathrm{End}\,H\cong H\otimes H^*\subset H_\sigma\otimes H^*\cong(H_{\tau,\sigma}\otimes H)^*\,.$$

Let $p_\chi$ be the linear map from $H_{\tau,\sigma}\otimes H$ onto $(D(E_\tau)/\ker\chi)^*$ defined to be the transpose of the above inclusion $D(E_\tau)/\ker\chi\subset(H_{\tau,\sigma}\otimes H)^*$. We regard $\mathscr{B}(F_{\sigma,\lambda})\otimes H$ as a subspace of $\mathscr{B}(F_{\sigma,\lambda})\otimes H_\sigma$ canonically.

LEMMA 3.4. *Let $(\chi, H)$ be a subrepresentation of $(\chi_{\tau,\sigma,\lambda}, H_\sigma)$ ($\sigma\in R_\tau$, $\lambda\in\mathfrak{a}_c^*$).*

(i) *$\mathscr{P}_{\tau,\sigma,\lambda}$ is a $G$-homomorphism of $\mathscr{B}(F_{\sigma,\lambda})\otimes H$ into $\mathscr{A}(E_\tau)$ and the diagram*

$$\begin{tikzcd} \mathscr{B}(F_{\sigma,\lambda})\otimes H \arrow[r, "\mathrm{Id}\otimes\chi(\Delta)"] \arrow[d, "\mathscr{P}_{\tau,\sigma,\lambda}"] & \mathscr{B}(F_{\sigma,\lambda})\otimes H \arrow[d, "\mathscr{P}_{\tau,\sigma,\lambda}"] \\ \mathscr{A}(E_\tau) \arrow[r, "\Delta"] & \mathscr{A}(E_\tau) \end{tikzcd}$$

*is commutative for all* $\Delta\in D(E_\tau)$. *Therefore we have*

$$\mathscr{P}_{\tau,\sigma,\lambda}: \mathscr{B}(F_{\sigma,\lambda})\otimes H\to\mathscr{A}(E_\tau,\chi)\,,$$

$$\mathscr{P}_{\tau,\sigma,\lambda}: \mathscr{B}(F_{\sigma,\lambda})^\tau\otimes H\to\mathscr{A}(E_\tau,\chi)^\tau\,.$$

(ii) *Let* $s_\chi$ *be the linear map from* $\mathscr{A}(E_\tau,\chi)$ *into* $V\otimes(D(E_\tau)/\ker\chi)^*$ *defined in* §2. *Then we have the following commutative diagram*:

$$\begin{tikzcd} \mathscr{B}(F_{\sigma,\lambda})^\tau\otimes H \arrow[d, "\mathscr{P}_{\tau,\sigma,\lambda}"] & V\otimes H_{\tau,\sigma}\otimes H \arrow[l, "\Gamma_{\sigma,\lambda}\otimes\mathrm{Id}"'] \arrow[d, "\mathrm{Id}\otimes p_\chi"] \\ \mathscr{A}(E_\tau,\chi)^\tau \arrow[r, "s_\chi"] & V\otimes(D(E_\tau)/\ker\chi)^*. \end{tikzcd}$$

PROOF. (i) The $G$-equivariance of $\mathscr{P}_{\tau,\sigma,\lambda}$ is clear from the definition of $\mathscr{P}_{\tau,\lambda}$. Let $a\in H$ and $\phi\in\mathscr{B}(F_{\sigma,\lambda})$. Put $u=\mathscr{P}_{\tau,\sigma,\lambda}(\phi\otimes a)$. For $\Delta\in D(E_\tau)$, take a $D\in U(\mathfrak{g}_c)^K$ such that $\mu(D)=\Delta$. Then by Lemma 3.3,

$$\begin{aligned}(\Delta u)(g)&=u(g;D)\\ &=\int_K \Psi_{\tau,\lambda}(k^{-1}g)\cdot\omega_{\tau,\sigma,\lambda}(D)\cdot a\phi(k)dk\\ &=\int_K \Psi_{\tau,\lambda}(k^{-1}g)\cdot(\chi_{\tau,\sigma,\lambda}(\Delta)a)\phi(k)dk\\ &=\int_K \Psi_{\tau,\lambda}(k^{-1}g)\cdot(\mathrm{Id}\otimes\chi(\Delta))(\phi\otimes a)(k)dk\,,\end{aligned}$$

which shows that the above diagram is commutative and that $u$ belongs to $\mathscr{A}(E_\tau,\chi)$. Hence, by the $G$-equivariance, the image of $\mathscr{B}(F_{\sigma,\lambda})^\tau\otimes H$ is contained in $\mathscr{A}(E_\tau,\chi)^\tau$.

(ii) Let $v\in V$, $b\in H_{\tau,\sigma}$ and $a\in H$. Put $u=\mathscr{P}_{\tau,\sigma,\lambda}\,(v\otimes b\otimes a)$. Then for $\forall\Delta\in D(E_\tau)$,

$$\begin{aligned}(\Delta u)(e) &= \int_K \Psi_{\tau,\lambda}(k^{-1})\chi_{\tau,\sigma,\lambda}(\Delta)a\Gamma_{\sigma,\lambda}(v\otimes b)(k)dk \\ &= \int_K \tau(k)(\chi_{\tau,\sigma,\lambda}(\Delta)a)b\tau(k^{-1})vdk \\ &= d(\tau)^{-1}\mathrm{tr}(\chi_{\tau,\sigma,\lambda}(\Delta)ab)v \\ &= \langle b, \chi(\Delta)a\rangle v\,.\end{aligned}$$

On the other hand by the definition of $p_\chi$, we have

$$\langle p_\chi(b\otimes a), \Delta + \ker\chi\rangle = \langle b, \chi(\Delta)a\rangle\,.$$

Therefore we obtain that $s_\chi \cdot \mathscr{P}_{\tau,\sigma,\lambda}(v\otimes b\otimes a) = v\otimes p_\chi(b\otimes a)$. This completes the proof.

THEOREM 3.5. *Let $G$ be a connected semisimple Lie group with finite center and let $K$ be a maximal compact subgroup of $G$. Let $\tau$ be a finite-dimensional representation of $K$ and assume that the differential of $\tau$ is irreducible, and let $E_\tau$ be the vector bundle over $G/K$ associated to $\tau$. Let $(\chi, H)$ be a subrepresentation of $(\chi_{\tau,\sigma,\lambda}, H_\sigma)$ for $\sigma$ in $R_\tau$ and $\lambda$ in $\mathfrak{a}_c^*$. Then we have the followings*:

(i) *The map $s_\chi$ is an isomorphism of $\mathscr{A}(E_\tau, \chi)^\tau$ onto $V\otimes(D(E_\tau)/\ker\chi)^*$.*

(ii) *Identifying $\mathscr{B}(F_{\sigma,\lambda})^\tau\otimes H$ with $V\otimes H_{\tau,\sigma}\otimes H$ we have*

$$\mathscr{P}_{\tau,\sigma,\lambda}(V\otimes H_{\tau,\sigma}\otimes H) = \mathscr{A}(E_\tau, \chi)^\tau\,,$$

$$\ker\mathscr{P}_{\tau,\sigma,\lambda}\cap V\otimes H_{\tau,\sigma}\otimes H = V\otimes\ker p_\chi\,.$$

(iii) *The following three conditions are mutually equivalent*:

1) *$\chi_{\tau,\sigma,\lambda}$ is irreducible.*

2) $\dim \mathscr{A}(E_\tau, \chi_{\tau,\sigma,\lambda})^\tau = d(\tau)[\tau : \sigma]^2$.

3) *$\mathscr{P}_{\tau,\sigma,\lambda}$ is injective on $V\otimes H_{\tau,\sigma}\otimes H_\sigma$.*

PROOF. This theorem follows immediately from Theorem 2.5, Lemma 3.4 and the surjectivity of $p_\chi$.

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*Present Address*:
Japan Women's University
Mejirodai, Bunkyo-ku, Tokyo 112, Japan