TOKYO J. MATH.
VOL. 15, NO. 1, 1992

On the Subring of Universally Stable Elements in a mod-2 Cohomology Ring

Yoshito OGAWA

Tohoku Institute of Technology
(Communicated by J. Tomiyama)

§1. Introduction.

Let p be a prime and k be a field of p-elements. We fix a finite p-group P and denote by $H^*(P)$ the cohomology ring of P with coefficients in k. For a finite group G, $O^p(G)$ means the subgroup of G generated by the p'-elements.

Evens and Priddy [3] introduced the ring of *universally stable elements*:

$$I(P)=\bigcap_G \mathrm{Im}(\mathrm{res}: H^*(G)\to H^*(P)),$$

where G runs over all finite groups with P as a Sylow p-subgroup. Then $H^*(P)$ is a finitely generated $I(P)$-module and the above intersection can be taken over a finite collection of G's.

They observed two cases. First $I(P)=H^*(P)^{GL(P)}$, if P is an elementary abelian p-group. Secondly $I(P)=H^*(G)$ for some G with P as a Sylow p-subgroup, if P is a dihedral group, a quaternion group or a semi-dihedral group.

Extending the former result for $p=2$, we show that $I(P)$ is an invariant subring of $H^*(P)$, if P is an extension of a cyclic group of order 2 by an elementary abelian 2-group. (If $|P|$ is given and $|P|\geq 8$, there is only one exception.) The ring structure of $H^*(P)$ was determined by Quillen [6]. An example of P is an extra-special 2-group [5].

THEOREM A. *Let $p=2$. If P is an extension of a cyclic group of order 2 by an elementary abelian 2-group, then we have*

$$I(P)=H^*(P)^{O^2(\mathrm{Out}(P))},$$

except when P is isomorphic to the direct product of a dihedral group of order 8 and an elementary abelian 2-group. In this case $I(P)$ is not the subring of invariants of any subgroup of Out(P). *More precisely, there is an element of $H^1(P)^{\mathrm{Out}(P)}$ which does not belong to $I(P)$.*

Received February 20, 1991

Let P be an extension of a cyclic group of order 2 by an elementary abelian 2-group. Then it is well-known that P is isomorphic to an elementary abelian 2-group or the direct product of D^nC $(n\geq 0)$, D^n $(n\geq 1)$ or $D^{n-1}Q$ $(n\geq 1)$ and an elementary abelian 2-group. No two of these groups are isomorphic. Here C is a cyclic group of order 4, D is a dihedral group of order 8 and Q is a quaternion group of order 8. Also D^nC, D^n and $D^{n-1}Q$ are central products of these groups. We give this classification in §4 for completeness.

We list $O^2(\mathrm{Out}(P))$, when P is an elementary abelian 2-group or P has no direct factor of an elementary abelian 2-group. In the latter case, $\mathrm{Out}(P)$ can be found in Griess [4] and Huppert [5]. We refer to [1] for classical groups and quadratic forms. Here k means a field of two elements.

(i) $\mathrm{Out}(P)=GL_{n+1}(2)\simeq A_n(2)$, where P is an elementary abelian 2-group of rank $n+1$.

$$O^2(\mathrm{Out}(P))\simeq A_n(2) \qquad \text{for} \quad n\geq 2\,,$$

$$O^2(\mathrm{Out}(P))\simeq \boldsymbol{Z}/3\boldsymbol{Z} \qquad \text{for} \quad n=1\,.$$

(ii) $\mathrm{Out}(D^nC)\simeq \boldsymbol{Z}/2\boldsymbol{Z}\times Sp_{2n}(2)$ for $n\geq 1$, where $Sp_{2n}(2)$ is the symplectic group of dimension $2n$. We know $Sp_{2n}(2)\simeq B_n(2)\simeq C_n(2)$ for $n\geq 2$, $Sp_2(2)\simeq S_3$ and $Sp_4(2)\simeq S_6$. These groups $Sp_{2n}(2)$ are simple for $n\geq 3$.

$$O^2(\mathrm{Out}(D^nC))\simeq B_n(2)\simeq C_n(2) \qquad \text{for} \quad n\geq 3\,,$$

$$O^2(\mathrm{Out}(DC))\simeq \boldsymbol{Z}/3\boldsymbol{Z} \quad \text{and} \quad O^2(\mathrm{Out}(D^2C))\simeq A_6\,.$$

(iii) $\mathrm{Out}(D^n)$ is isomorphic to the orthogonal group of dimension $2n$ defined over k stabilizing a non-singular quadratic form with Witt index n. Except for $n=2$, the orthogonal group has a commutator subgroup of index 2. We denote this by $\Omega^+_{2n}(2)$. Then $\Omega^+_{2n}(2)$ is simple and isomorphic to $D_n(2)$ for $n\geq 3$.

$$O^2(\mathrm{Out}(D^n))\simeq D_n(2) \qquad \text{for} \quad n\geq 3\,,$$

$$O^2(\mathrm{Out}(D))=1 \quad \text{and} \quad O^2(\mathrm{Out}(D^2))\simeq \mathrm{Z}/3\boldsymbol{Z}\times \boldsymbol{Z}/3\boldsymbol{Z}\,.$$

(iv) $\mathrm{Out}(D^{n-1}Q)$ is isomorphic to the orthogonal group of dimension $2n$ defined over k stabilizing a non-singular quadratic form with Witt index $n-1$. The orthogonal group has a commutator subgroup of index 2. We denote this by $\Omega^-_{2n}(2)$. Then $\Omega^-_{2n}(2)\simeq {}^2D_n(2)$ for $n\geq 3$, $\Omega^-_2(2)\simeq \boldsymbol{Z}/3\boldsymbol{Z}$ and $\Omega^-_4(2)\simeq A_5$. These groups $\Omega^-_{2n}(2)$ are simple for $n\geq 1$.

$$O^2(\mathrm{Out}(D^{n-1}Q))\simeq {}^2D_n(2) \qquad \text{for} \quad n\geq 3\,,$$

$$O^2(\mathrm{Out}(Q))\simeq \boldsymbol{Z}/3\boldsymbol{Z} \quad \text{and} \quad O^2(\mathrm{Out}(DQ))\simeq A_5\,.$$

§ 2. Preliminaries.

In this section we summarize the facts needed for the proof of Theorem A. We begin by giving a criterion when $I(P)$ is an invariant subring of $H^*(P)$. The proof is an improvement of an argument by Evens and Priddy [3, Proposition B].

THEOREM B. *Suppose $H^*(G) \simeq H^*(N_G(P))$ for all finite groups G with P as a Sylow p-subgroup. Then*

$$I(P) = H^*(P)^{O^p(\mathrm{Out}(P))}.$$

PROOF. Let $\mathscr{C}$ be the set of p'-subgroups of Out(P). It is easy to see that $O^p(\mathrm{Out}(P)) = \langle C \mid C \in \mathscr{C} \rangle$ and that

$$H^*(P)^{O^p(\mathrm{Out}(p))} = \bigcap_{C \in \mathscr{C}} H^*(P)^C.$$

We show that $H^*(P)^C$ runs over all subrings of $H^*(P)$ of the form $\mathrm{res}\, H^*(G)$, where G is some finite group with P as a Sylow p-subgroup. Namely, we can lift each C to a p'-subgroup A of Aut(P) by the Schur-Zassenhaus theorem. Let G be the semi-direct product of P by A. Then we have $H^*(P)^C = \mathrm{res}\, H^*(G)$. Conversely, let G be any finite group with P as a Sylow p-subgroup. Put $C = \varphi(N_G(P))$, where $\varphi: N_G(P) \to \mathrm{Out}(P)$. Then the assumption gives $H^*(P)^C = \mathrm{res}\, H^*(N_G(P)) = \mathrm{res}\, H^*(G)$, as desired.

The hypothesis of Theorem B is satisfied, if P is abelian, which is a theorem of Swan [8]. As a generalization of this fact, we know the following.

THEOREM C. *Suppose P/M acts trivially on $H^*(M)$ for every maximal subgroup M of P. If G is a finite group with P as a Sylow p-subgroup, then we have $H^*(G) \simeq H^*(N_G(P))$.*

This is a special case of a result due to Yoshida [10, Lemma 4.4] and Sasaki [7, Theorem 1]. Under the hypothesis of Theorem C, it is easy to see that P has no proper singularity for P-functor H^*. Hence we can apply their result. For details, see the above references.

THEOREM D (Quillen [6]). *Let $p=2$. Suppose P is an extension of a cyclic group K of order 2 by an elementary abelian 2-group V. Let E be an elementary abelian 2-subgroup of P with maximum rank and put $|P:E| = 2^h$. Then*

$$H^*(P) \simeq S(V^*)/J \otimes k[w_{2^h}],$$

where J is a certain homogeneous ideal of $S(V^*)$ and w_{2^h} is any element of $H^{2^h}(P)$ with $\mathrm{res}_{P,K}(w_{2^h}) \neq 0$.

Quillen constructed the element w_{2^h} topologically, while we give this by using

Evens' norm map [2]:

LEMMA E (Quillen [6, Lemma 4.5]). *If z is a non-zero element of $H^1(K)$, then* $z^{2^h} \in \mathrm{Im}(\mathrm{res}: H^*(P) \to H^*(K))$.

PROOF. We keep the notation in Theorem D. We know $K \subsetneqq E$, and so there is a projection $\mathrm{pr}_1: E \to K$. We define $\inf_{K,E}: H^*(K) \to H^*(E)$ by pr_1^*. Let $u = \inf_{K,E}(z)$ and $w_{2^h} = \mathrm{norm}_{E,P}(u)$, where $\mathrm{norm}_{E,P}: H^*(E) \to H^*(P)$ is Evens' norm map. Then $\mathrm{res}_{E,K}(u) = z$. We put $P = \bigcup_{g \in P/E} gE$ and $E^g = gEg^{-1}$. Under the assumption of Theorem D, we have $E^g = E$. Applying double coset rule, we have the following:

$$\begin{aligned}\mathrm{res}_{P,K}(w_{2^h}) &= \mathrm{res}_{P,K}(\mathrm{norm}_{E,P}(u)) \\ &= \prod_{g \in P/E} \mathrm{norm}_{K \cap E^g, K}(\mathrm{res}_{E^g, K \cap E^g}(u^g)) \\ &= \prod_{g \in P/E} \mathrm{res}_{E,K}(u^g) \\ &= \prod_{g \in P/E} (\mathrm{res}_{E,K}\, u)^g \\ &= \prod_{g \in P/E} z^g \\ &= z^{2^h}.\end{aligned}$$

This proves the lemma.

§3. Proof of Theorem A.

Let $p = 2$ and P be a finite 2-group. Before proving the theorem we recall the equivalence of the following three conditions.

(i) P has no homomorphism onto the dihedral group of order 8.

(ii) All subgroups of index 4 are normal in P.

(iii) For any maximal subgroup M of P, P/M acts trivially on $H^1(M) \simeq \mathrm{Hom}(M, k)$.

The equivalence of (i) and (ii) was observed by Yoshida [9, §3]. To show that of (ii) and (iii), we note that there is a one-to-one correspondence between the non-zero elements of $\mathrm{Hom}(M, k)$ and the maximal subgroups of M. This correspondence is compatible with the action of P.

PROOF OF THEOREM A. Let P be an extension of a cyclic group K of order 2 by an elementary abelian 2-group. Suppose P is not isomorphic to the direct product of a dihedral group of order 8 and an elementary abelian 2-group. Let M be any maximal subgroup of P. We show that P/M acts trivially on $H^*(M)$. We may assume that P is not an elementary abelian 2-group.

It is easy to see that P has no homomorphism onto the dihedral group of order 8. Note that P is isomorphic to the direct product of D^nC $(n \geq 0)$, D^n $(n \geq 2)$ or $D^{n-1}Q$ $(n \geq 1)$ and an elementary abelian 2-group. Therefore, by the preceding remark, P/M acts trivially on $H^1(M)$.

Next, let E be an elementary abelian 2-subgroup of M with maximum rank. Then we have $P' = K \subseteqq E \subseteqq M \subseteqq P$. Note that $K \subseteqq M$, for otherwise P is an elementary abelian 2-group. M is also an extension of K by an elementary abelian 2-group. For M and E, we define w_{2^h} as in the proof of Lemma E:

$$w_{2^h} = \mathrm{norm}_{E,M}(\mathrm{inf}_{K,E}(z)),$$

where $|M:E| = 2^h$ and z is a non-zero element of $H^1(K)$. If $g \in P - M$, then g normalizes E. Hence we have

$$\begin{aligned} w_{2^h}{}^g &= \mathrm{norm}_{E,M}(\mathrm{inf}_{K,E}(z))^g \\ &= \mathrm{norm}_{E,M}(\mathrm{inf}_{K,E}(z^g)) \\ &= \mathrm{norm}_{E,M}(\mathrm{inf}_{K,E}(z)) \\ &= w_{2^h}. \end{aligned}$$

Theorem D shows that $H^*(M)$ is generated by $H^1(M)$ and w_{2^h}, and so P/M acts trivially on $H^*(M)$. Theorem C and Theorem B immediately imply that $I(P)$ is the subring of invariants of $O^2(\mathrm{Out}(P))$.

For the exceptional case, we use the following notation:

S: the symmetric group of degree 4,

D: a dihedral Sylow 2-subgroup of S of order 8,

C: a unique cyclic subgroup of D of order 4,

V: an elementary abelian 2-group.

Let $P = D \times V$ and $G = S \times V$. Then P is a Sylow 2-subgroup of G. We let $M = C \times V$ and define $\chi \in H^1(P)$ by $\chi: P \to P/M = k$. Clearly $\chi \in H^1(P)^{\mathrm{Out}(P)}$, for M is a characteristic subgroup of P. If $\theta \in H^1(G) = \mathrm{Hom}(G, k)$ and $M \subseteqq \mathrm{Ker}\,\theta$, then we have $\theta = 0$ by $G' = S'$. Hence χ does not belong to $\mathrm{res}\,H^1(G)$, and so $\chi \in H^1(P)^{\mathrm{Out}(P)} - I(P)$. This completes the proof of Theorem A.

§4. Appendix.

Let k be a field of two elements. We fix a finite dimensional vector space over k and denote it by V. On the lines of [6], we classify groups of the following type:

(4.1) an extension P of an additive group k by an elementary abelian group V.

First we give the complete invariants for the equivalence class of quadratic forms on V. Then we show that there is a one-to-one correspondence between the isomorphism classes of groups of type (4.1) and the equivalence classes of quadratic forms on V. Lastly we prove that every equivalence class of quadratic forms on V is realized by a group in §1.

A *quadratic form* is a function $F: V \to k$ such that

$$f(x, y) = F(x+y) + F(x) + F(y)$$

is bilinear. We call two quadratic forms are equivalent if one is transformed into the other by a non-singular linear transformation. Let

$$\mathrm{Ker} f = \{x \in V \mid f(x, y) = 0 \text{ for all } y \in V\},$$
$$\mathrm{Ker}\, F = \{x \in \mathrm{Ker} f \mid F(x) = 0\}.$$

The *nullity* and *rank* of F are the dimension and codimension of $\mathrm{Ker}\, F$, respectively. We say that F is *non-singular* if $\mathrm{Ker}\, F = 0$. A subspace is said to be *isotropic* for F if F is identically zero on that subspace. The *Witt index* of F is the greatest dimension of any isotropic subspace for F. (In fact, all maximal isotropic subspaces for F are of the same dimension. This can be shown in an elementary way below.)

We recall non-singular quadratic forms. When $\dim_k V = 2n+1$ is odd, then F is uniquely determined up to equivalence; Witt index is n. When $\dim_k V = 2n$ is even, then there are two possibilities up to equivalence; Witt index is n or $n-1$.

Let F be any quadratic form on V with nullity r. Then F is non-singular on a complement of $\mathrm{Ker}\, F$ in V. If the rank of F is $2n+1$, then Witt index is $r+n$. If the rank of F is $2n$, then Witt index is $r+n$ or $r+n-1$. Therefore the nullity and Witt index are the complete invariants for the equivalence class of quadratic forms on V.

A group P of type (4.1) defines a quadratic form F on V by $F(x) = \bar{x}^2$, where $x \in V$, $\bar{x} \in P$, $\pi(\bar{x}) = x$ and π is the canonical map. This gives the bijection $H^2(V, k) \simeq S^2(V^*)$ compatible with the action of $GL(V)$. Therefore two groups of type (4.1) are isomorphic, if the associated quadratic forms are equivalent.

Given a group P of type (4.1), we keep the above notation. For a subspace W of V, $\pi^{-1}(W)$ is an elementary abelian subgroup of P if and only if W is an isotropic subspace for F [6]. Also we have $\Omega_1(Z) = \pi^{-1}(\mathrm{Ker}\, F)$, where $\Omega_1(Z)$ is the greatest elementary abelian subgroup of the center of P. This follows from $Z = \pi^{-1}(\mathrm{Ker} f)$, where f is the bilinear form associated with F. Hence two groups of type (4.1) are not isomorphic, if one differs from the other in the equivalence class of the quadratic form. Note that the nullity and Witt index are the complete invariants for the equivalence class of quadratic forms on V.

It is straightforward to see that every quadratic form F on V is up to equivalence given by the group P which is an elementary abelian group or the direct product of D^nC $(n \geq 0)$, D^n $(n \geq 1)$ or $D^{n-1}Q$ $(n \geq 1)$ and an elementary abelian group of rank r. In the latter case, the groups D^nC $(n \geq 0)$, D^n $(n \geq 1)$ and $D^{n-1}Q$ $(n \geq 1)$ give three equivalence classes of non-singular quadratic forms mentioned above, respectively; while the rank r gives the nullity of F. This completes the classification of groups of type (4.1).

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Present Address:
TOHOKU INSTITUTE OF TECHNOLOGY
SENDAI 982, JAPAN