TOKYO J. MATH.
VOL. 15, NO. 1, 1992

The Involutions of Compact Symmetric Spaces, II

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Dedicated to Professor Ichiro Satake on his Sixtieth Birthday

Introduction. This is a part of our series on a geometric theory of the compact symmetric spaces M ([CN-1], [CN-2], [CN-3], [NS-1], [NS-2], [N]); especially we will give proofs to certain statements in [N] (as was promised there). The theory features the subspaces, called the polars and the meridians (1.5 for both) of M, which are significant building blocks of maximal size (i.e. larger than the cells in any reasonable decomposition such as that of Bruhat). The polars are directly related to the space M in topology; 1.11 is just one example. They are the critical submanifolds of a certain Bott-Morse function (at least if M is an R-space; see [T-1]), and so forth. More strikingly, even the signature (or index) of M equals the sum of that of the polars, of which we do not have a direct proof yet. We point out another intriguing fact about polars; Uhlenbeck [U] found that every harmonic map from the 2-sphere into $U(n)$ is a product of those into polars. On the other hand, the meridians have equal rank to that of M and their root systems are related to that of M with a simple rule (2.15); therefore their curvature is related to that of M in an equally simple way. The polars are paired with the meridians; they are the "orthogonal complements" to each other, while the polars are the connected components of the fixed points of any one of the involutions by which M becomes a symmetric space (1.5). Their theoretical significance lies in the fact that M is determined by a single pair of a polar and a meridian (1.15). Thus it is an easy corollary that a simple M is hermitian if and only if a polar and a meridian in a pair are hermitian (2.30). Also the theory aims at studies of interrelationship between symmetric spaces or morphisms $f: B \rightarrow M$ between them. The case of $\dim B=0$ was studied in [CN-3] and [T-3]. The case of $B=$sphere was done in [NS-1] fairly completely.

In section 1, we will explain basic concepts and facts about them as well as their relevance in our geometric theory. Theorem 1.8 gives the basic property of the meridian in connection with the maximal tori, whose proof includes a new proof of conjugacy

Received September 25, 1990

of the maximal tori. We begin Section 2 with formulating a variational problem to connect geometry with the root theory more directly; the roots describe the curvature and the root spaces describe certain elliptic subspaces. Thus Chow's arithmetic distance can be handled in a more general context (Section 11 of [N]; it was worked out by Takeuchi [T-2], of which he is proud). We will show how to determine the meridians (Theorem 2.15, which was Theorem 2.5 in [N]). In the second half of the section, we will look at the subspaces which any two roots define. An outcome is a more geometric classification of the symmetric spaces than those in [A] and [H]; it also explains the strong restrictions on the multiplicity of the root and its meaning. In Section 3, we will describe the involutions with emphasis on the interrelationship between involutions of different spaces; all the involutions will be defined with those of the orthogonal group (or its local isomorphism class).

In order to keep this paper short, we have to appeal to facts expounded in [B], [H] and [KN].

Notations. We use Cartan's notations for compact 1-connected symmetric spaces except that we use the symbol $G_r(K^n)$ to denote the Grassmann manifold of the r-dimensional linear subspaces of K^n over $K=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$ and GI for his $\mathrm{G}=\mathrm{G}_2/\mathrm{SO}(4)$. $\mathscr{L}G$ will denote the Lie algebra of a Lie group G. The symbol $\doteqdot$ means local isomorphism between symmetric spaces.

§1. Basics on symmetric spaces.

We will define the category of the compact symmetric spaces and discuss rudiments thereof.

1.1. Definition. A smooth manifold M is called a *symmetric space* when M admits an involutive smooth transformation $s_o : M \to M$ for every point o of M such that o is an isolated fixed point of s_o and there is a Riemannian metric which every symmetry s_o leaves invariant. M is called *affine symmetric* if M satisfies the above conditions with the metric replaced with an affine connection.

A symmetric space is affine symmetric. Moreover the invariant connection of an affine symmetric space is unique and it has no torsion, since the involution s_o leaves invariant no tensor $\neq 0$ of odd degree, in particular none of type $(1, 2)$, while the invariant Riemannian metrics are not quite unique.

1.2. Definition. A smooth map $f : M \to N$ between symmetric spaces M and N is called a *morphism* or a *homomorphism* if f commutes with the symmetries; $f \circ s_x = s_{f(x)} \circ f$ for every point x of M.

1.2a. A morphism f is an affine map, as is known [KN]. In fact, the exponential map $\exp : T_oM \to M$, restricted to any line $L \subset T_oM$ through 0, is affine and a morphism at once; hence f commutes with exp. Thus, in case M is connected, f is a morphism if and only if f is affine. In particular, the automorphism group $\mathrm{Aut}(M)$ is a Lie group ([KN]).

1.2b. If M is connected, then a morphism $f: M \to N$ gives rise to a Lie algebra homomorphism $\mathscr{L}f: \mathscr{L}\mathrm{Aut}(M) \to \mathscr{L}\mathrm{Aut}(N)$, since the composites $s_o \circ s_p$ for all the pairs of the points o, p of M generate the identity-component of $\mathrm{Aut}(M)$ (and hence one has a *local* homomorphism: $s_o \circ s_p \mapsto s_{f(o)} \circ s_{f(p)}$ of $\mathrm{Aut}(M)$ into $\mathrm{Aut}(N)$).

1.3. REMARK. With these morphisms, the symmetric spaces make a category, on which we will work. For instance, *a subspace B* of M means a symmetric space in M such that the inclusion map is a morphism. In case B is connected, this is equivalent to say that B is a totally geodesic submanifold of M. We assume that M is connected, generally. One easily sees that the intersection of two subspaces, the image and the inverse image under a homomorphism are all subspaces. Hence every homomorphism is the composite of a monomorphism and an epimorphism. The fixed point set of an automorphism t of M, $F(t, M)$, is another subspace.

1.3a. EXAMPLE AND REMARK. Every compact Lie group L is a symmetric space by the symmetry $s_x: y \mapsto xy^{-1}x$. Every automorphism of the group L is that of the symmetric space L, but not vice versa. For this reason, we use the term group-automorphism to emphasize that the action preserve the group structure also.

1.3b. EXAMPLE. Let G be a compact Lie group and let σ be a group involution of G. We then make G act on G itself by $c: x \mapsto cx\sigma(c)^{-1}$ for $c \in G$. We call it *the σ-action.* The orbit through the unit element 1 is a subspace of G. This subspace $\{c\sigma(c)^{-1} \mid c \in G\}$ is a homogeneous space $G/F(\sigma, G)$. Also it equals $F(s_1 \circ \sigma, G)_{(1)}$ (if G is connected); here and elsewhere the symbol $B_{(x)}$ for a topological space B with a point x on it denotes the connected component of B through x.

1.4. LEMMA. *If M is connected (which we usually assume), then every isometry f of M (in particular, every symmetry s_x) is an automorphism of M and the identity component $G := \mathrm{Aut}(M)_{(1)}$ of the automorphism group $\mathrm{Aut}(M)$ is transitive on M.*

PROOF. We will show $s_{f(x)} = f \circ s_x \circ f^{-1}$ for every point x of M. In fact, the transformations in the both hand sides are isometries and fix the point $f(x)$. They act on the tangent space $T_{f(x)}M$ accordingly, and we have only to show that they agree there. The left hand side $s_{f(x)}$ is -1 times the identity, while $f \circ s_x \circ f^{-1}$ is conjugate to s_x acting on the tangent space T_xM which is -1 times the identity on T_xM. So we obtain the equality. If c is a geodesic segment joining a point o to another point p, then the symmetry s_m at the midpoint m exchanges o and p. Therefore $\mathrm{Aut}(M)$ is locally (hence globally) transitive on M, and so is G (alternatively, one can prove the transitivity of G by trisecting c instead of bisecting it). (Of course $\mathrm{Aut}(M)$ is a Lie group. See [KN] quoted in 1.2a.) QED

1.5. DEFINITION. Each connected component of the fixed point set $F(s_o, M)$ is called *a polar of o in M* and denoted by M^+ or by $M^+(p)$ if it contains the point p. Through the point p, there is a unique connected subspace $F(s_p \circ s_o, M)_{(p)}$ whose tangent space at p is the orthogonal complement of the tangent space $T_pM^+(p)$ in T_pM. We call $F(s_p \circ s_o, M)_{(p)}$ the *meridian to* $M^+(p)$ through p and denote it by M^- or $M^-(p)$. More generally, if t is an involution of M which fixes a point o, the subspace $F(t \circ s_o, M)_{(o)}$ has the tangent space at o which is the orthogonal complement of the tangent space to the subspace $F(t, M)_{(o)}$ at o in T_oM. We call these subspaces *complementally orthogonal* or just *c-orthogonal* to each other at o. We call a polar $M^+(p)$ *a pole of o in M* if it is a singleton $\{p\}$. $\{p\}$ is a pole $\Leftrightarrow s_p = s_o$. (The point p is also called a pole if $\{p\}$ is. Usually, we do not mean that o itself is a pole or polar.)

1.5a. REMARKS. (i) A theoretical significance of the concepts of the polars and the meridians lies in the fact that M is (globally) determined by any one pair (M^+, M^-) if M is compact and connected; see Theorem 1.15 below. We assume M is compact, hereafter.

(ii) Every polar of o is a $K_{(1)}$-orbit, where K is the isotropy subgroup of $G := \mathrm{Aut}(M)_{(1)}$ at o. Indeed, every component of $F(t, M)$ is an orbit of the connected group $F(\mathrm{ad}(t), G)_{(1)}$ for an automorphism t of M, obviously. As a consequence, if M is a group, then $F(s_1, M)$ is the set of the involutive members and the polars are their conjugate classes, since K is the adjoint group, $\mathrm{ad}(M)$. Another consequence is that the isomorphism class (or the G-congruence class, more precisely) of the meridian $M^-(q)$ to $M^+(p)$ at q is independent of q in $M^+(p)$. Also $\{p\}$ is a pole of o in $M = G/K$, $K(o) = o$, if and only if $K_{(1)}$ fixes p.

(iii) The above subspace $F(t \circ s_o, M)_{(o)}$ meets every component of $F(t, M)$ for every involutive automorphism t of M which fixes the point o. In particular every meridian to a polar of o meets every polar of o (and $\{o\}$).

PROOF OF (iii). Let $M^t(q)$ denote the component of $F(t, M)$ through the point q. Let γ be a shortest geodesic arc joining $M^t(o)$ to $M^t(q)$. γ is orthogonal to $M^t(o)$ at a point p. Some member b of $F(\mathrm{ad}(t), G)_{(1)}$ carries p into o. b then carries γ into a geodesic $b\gamma$ which is orthogonal to $M^t(o)$ at o. This means $b\gamma$ is entirely contained in the orthogonal space $F(t \circ s_o, M)_{(o)}$; the other end of $b\gamma$ still lies on $M^t(q)$. QED

1.5b. REMARK. We like to review a few more concepts and facts from [CN-3] and [N] for later use. Every covering epimorphism $\pi : M \to M''$ carries the polars of o in M and their meridians onto those of $\pi(o)$ in M'' and the meridians except that some poles in M may fall onto the point $\pi(o)$. More specifically, if a point p is a pole of a point o in M, then there is a double covering epimorphism $\pi : M \to M''$ which carries o and p into a single point. Those polars of $\pi(o)$ in M'' which are not the π-images of polars in M are the projections of the connected components of what we call *the centrosome* $C(o, p)$; $C(o, p)$ is a subspace consisting of the midpoints of the geodesic arcs from o to p by definition. Next we recall the polars of *the dot product* $M \cdot N$ of two spaces M and N, where $M \cdot N$ denotes the orbit space $(M \times N)/\boldsymbol{Z}_k$ for the (obvious from the context) cyclic group $\boldsymbol{Z}_k$ of order k acting freely on M and N hence on the product $M \times N$ accordingly as an automorphism group. When k equals 2, the $\boldsymbol{Z}_k$-orbit through a point (o_M, o_N) has another point (p_M, p_N) in it. Let π denote the projection : $M \times N \to M \cdot N$. Then the polars of the point $z := \pi(o_M, o_N)$ is described by the formula: $F(s_z, M \cdot N) = F(s_{o_M}, M) \cdot F(s_{o_N}, N) \cup C(o_M, p_M) \cdot C(o_N, p_N)$. Typically, the dot product appears in connection with a tensor product; thus, SO(4) is isomorphic with the dot product $\mathrm{Sp}(1) \cdot \mathrm{Sp}(1)$, since SO(4) may be interpreted as the group $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$, made effective, acting on $\boldsymbol{C}^2 \otimes \boldsymbol{C}^2$ in the natural fashion. Similarly, $\mathrm{U}(n)$ is the dot product of U(1) and $\mathrm{SU}(n)$.

1.6. NOTATIONS AND DEFINITION. We denote $\mathrm{Aut}(M)_{(1)}$ by G or G_M. Let $\mathscr{L}G$ *or* $\mathfrak{g}$ denote its Lie algebra, which is thought of as a linear space of vector fields on M. Given a point o of M, we have an involution $\mathrm{ad}(s_o)$ of $\mathfrak{g}$. Its eigenspace decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ is called the *symmetry decomposition at o* for M. $\mathfrak{k}$ is the Lie algebra of the isotropy subgroup K of G at o. $\mathfrak{m}$ is linearly isomorphic with the tangent space T_oM to M at o by the evaluation map: $v \mapsto v(o)$.

1.6a. One knows $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$.

1.6b. DEFINITIONS. Given a point o in M, we call the map $Q : M \to G : x \mapsto s_x \circ s_o$ *the quadratic transformation* (of E. Cartan).

Q is a G-equivariant morphism, as is easily seen, if one chooses the σ-action of G on itself (1.3b). Q is a covering morphism of M onto the subspace $Q(M) = \{b\sigma(b)^{-1} \mid b \in G\}$; thus one has an "exact" sequence of morphisms of spaces: $\{o\} \to P \to M \to Q(M) \to \{1\}$, where P denotes $Q^{-1}(1)$. P consists of all the poles of o in M. Hence Q is a monomorphism if and only if M has no pole. One notes (1.14) that P has the structure of an abelian group.

1.7. DEFINITION. A space M is called *semisimple* if G is so. M is called *simple* if M is semisimple but not a local product of two spaces of positive dimensions. The *bottom* (space) (or the adjoint space [H]) M^* of a connected semisimple space M is the space such that every connected space which is locally isomorphic with M is a covering space of M^*. G^* is the adjoint group of G. And one sees the unique existence of M^*; in fact $M^* = F(s_1 \circ \mathrm{ad}(s_o), G^*)_{(1)} = G^*/F(\mathrm{ad}(s_o), G^*)$. Also $M^* \cong Q(M^*)$.

1.8. THEOREM. *Let M be a compact connected symmetric space. Then*

(i) *every meridian $M^-(p)$ contains a maximal torus of M, that is, a maximal subspace of M which is isomorphic with a torus.*

(ii) *Every maximal torus $\ni o$ meets every polar of o.*

(iii) *The maximal tori are G-congruent with each other; furthermore, if two maximal tori A and B contain a point o, then some member of $K_{(1)}$ at o carries A into B.*

The dimension of a maximal torus in M is called *the rank of M*, denoted by $r(M)$. Thus every meridian $M^-(p)$ has an equal rank to M.

PROOF. (i) Consider a geodesic γ through o and p. Then there is a member H of $\mathfrak{m}$ such that γ is the orbit $\{\exp(tH)(o) \mid t \in \boldsymbol{R}\}$ of the 1 dimensional group. If $\mathfrak{a}$ denotes a maximal abelian subalgebra in $\mathfrak{m}$ which contains γ, the orbit $\exp(\mathfrak{a})(o)$ is a maximal torus A in M which contains γ. Equivalently $Q(A)$ is a maximal abelian subgroup (maximal in $Q(M)$) that contains the abelian subgroup $Q(\gamma)$ of G (and A is connected). $\{p\}$ is a pole of o in A, since every polar in a torus is a pole. Hence s_p acts on the tangent space T_pA as -1 times the identity. Therefore A is contained in the meridian $M^-(p)$. We prove (ii) and (iii) simultaneously, although (iii) is well known. We may assume M is semisimple and further that M is the bottom M^*; thus M has no pole. We induct on the dimension of M; the case of $\dim M=0$ is trivial (and so is the case of $\dim M=2$). Also we may assume by 1.4 that A and B share the point o. Let p be a pole of o in A. Then A is a subspace of the meridian $M^-(p)$. Similarly, B is a subspace of another meridian $M^-(q)$. $M^+(q) \cap M^-(p)$ is not empty by 1.5a (iii). Every connected component $M^{-+}(q)$ of $M^+(q) \cap M^-(p)$ is a polar of o in $M^-(p)$. Hence A meets $M^{-+}(q)$ by the induction assumption on (ii). Hence some member b of $K_{(1)}$ carries A into a torus bA that contains the point q (and o) by 1.5a (ii). $M^-(q)$ contains bA by (i). Again by the induction assumption (this time on (iii)), another member c of $K_{(1)}$ carries bA into B. QED

1.8a. COROLLARY. *A connected M is 1-connected if some meridian is.*

PROOF. This follows from 1.8 (i), since every member of the fundamental group $\pi_1(M)$ contains a closed geodesic. QED

1.9. THEOREM. *Every compact connected symmetric space M is a subspace of a canonical finite covering group $G^\wedge$ of $G=\mathrm{Aut}(M)_{(1)}$ such that one has $M=F(s_1 \circ \sigma, G^\wedge)_{(1)}=\{b\sigma(b)^{-1} \mid b \in G^\wedge\}$, $\sigma:=\mathrm{ad}(s_o)$, and the inclusion induces an isomorphism of the fundamental group $\pi_1(M)$ onto $\pi_1(G^\wedge)$.* This way, Q lifts to a monomorphism $Q^\wedge : M \to G^\wedge$ (see 1.13). (The group $G^\wedge$ is not unique with the above condition; "canonical" means that 1.12 below obtains.)

PROOF. In case M is a torus, one takes the identity map 1_M as $Q^\wedge$; $M=G^\wedge$. We assume that $M=G/K$ is semisimple. The subspace $\{b\sigma(b)^{-1} \mid b \in G^\sim\}$ of the universal covering group $G^\sim$ is a homogeneous space $G^\sim/F(\sigma, G^\sim)$ and $F(\sigma, G^\sim)$ is connected (Theorem 8.2, [H], p. 320). Thus the subspace $M^\sim:=\{b\sigma(b)^{-1} \mid b \in G^\sim\}$ is 1-connected. On the other hand the projection π of $G^\sim$ onto the adjoint group G^* carries $M^\sim$ onto the bottom space $M^*=Q(M^*)$. Therefore the fundamental group $\pi_1(M^*)$ is a subgroup of $\pi_1(G^*)$ which is the center $C(G^\sim)$ of $G^\sim$. Hence $\pi_1(M)$ is also a subgroup of $C(G^\sim)$; moreover $\pi_1(M) \subset \pi_1(M^*)=C(G^\sim) \cap M^\sim$. Thus M is a subspace of $G^\wedge:=G^\sim/\pi_1(M)$; one has $M=\{b\sigma(b)^{-1} \mid b \in G^\wedge\}$. The construction of $G^\wedge$ is now obvious in the general case. QED

1.10. Corollary. *If $G^{\sim}$ is a compact* 1*-connected Lie group, then the subspace* $M=\{b\sigma(b)^{-1} \mid b\in G^{\sim}\}$ *is* 1*-connected for every group involution* σ. *Its bottom space* M^* *is* $\{b\sigma(b)^{-1} \mid b\in G^*\}$. *In particular, the fundamental group* $\pi_1(M^*)$ *is a subgroup of* $\pi_1(G^*)$ *which is the center* $C(G^{\sim})$ *of* $G^{\sim}$ *for every compact connected semisimple space* M.

1.11. Corollary. *The polars in a compact* 1*-connected Lie group are all* 1*-connected.* In particular they are semisimple.

1.12. Corollary. *The monomorphism* $Q^{\wedge}: M\mapsto G^{\wedge}$ *(or rather* $(M, o)\mapsto(G^{\wedge}, 1)$*) given in the theorem extends to a functor of the category of the compact connected symmetric spaces into that of the compact connected Lie groups. That is, a homomorphism* $f: M=G_M/K_M\to N$ *extends to a homomorphism* $G^{\wedge}f: G^{\wedge}_M\to G^{\wedge}_N$.

Proof. The given homomorphism f gives rise to a homomorphism $\mathscr{L}f: \mathfrak{g}_M\to\mathfrak{g}_N$ on the Lie algebra level by 1.6a; $\mathfrak{m}$ generates $\mathfrak{g}$. (Of course we use the symmetry decomposition of $\mathfrak{g}_N$ at $f(o)$.) $\mathscr{L}f$ lifts to a homomorphism $G^{\sim}f: G^{\sim}_M\to G^{\wedge}_N$ of the 1-connected group. Since the kernel of the projection of $G^{\sim}_M$ onto $G^{\wedge}_M$ is contained in $M^{\sim}$ as in the proof of the theorem, one obtains the desired $G^{\wedge}f$. The rest is obvious. QED

1.13. Remark. The significance of 1.12 or the concept of $G^{\wedge}$ lies in the fact that a morphism $f: M\to N$ does *not* necessarily give rise to a homomorphism of $\mathrm{Aut}(M)_{(1)}\to\mathrm{Aut}(N)_{(1)}$. If one writes $Q^{\wedge}$ for the monomorphism: $M\to G^{\wedge}$ in the theorem, then one has $Q=\pi\circ Q^{\wedge}$, where π is the projection: $G^{\wedge}\to G$. Theoretical importance of $Q^{\wedge}$ lies in that, in a way, $Q^{\wedge}$ replaces the projection: $G\to M=G/K$, which is *not* a homomorphism in our sense.

1.13a. Remark. If f is an epimorphism, then so is $G^{\wedge}f$. The analog is false for a monomorphism. Every automorphism t of M, however, extends to that of $G^{\wedge}$.

1.14. Corollary. *For a covering morphism* $\pi: M\to M''$, *one has an "exact" sequence*: $\{o\}\to M'\to M\to M''\to\{o''\}$ *and the "kernel"* $M':=\pi^{-1}(o'')$, $o'':=\pi(o)$, *has the structure of an abelian group with the unit element* $1=o$ *which is a subgroup of the center of* $G^{\wedge}$. We call M$'$ *the center* (through o) of M if $M''=M^*$, the bottom.

Proof. The group homomorphism $G^{\wedge}\pi: G^{\sim}/\pi_1(M)\to G^{\sim}/\pi_1(M'')$ has the kernel $N\cong\pi_1(M'')/\pi_1(M)$, which is abelian as a quotient group of a subgroup $\pi_1(M'')$ of the center of $G^{\sim}$. N may be identified with M' by the theorem. QED

1.14a. Corollary. *Every meridian through o contains the kernel* M' *in the above.*

Proof. This is immediate from 1.8 (i) and the fact that the center of a connected compact Lie group is contained in every maximal torus (which is a subgroup). QED

1.14b. REMARK. Homomorphisms between symmetric spaces are not too much involved; that is, if $f: M \to N$ is a homomorphism, then M is a local product $M' \times M''$ of subspaces such that $f(M')$ is a single point and the restriction of f to M'' is a covering map onto $f(M)$, a subspace of N. This is immediate from the fact that the compact connected group $G_M^{\wedge}$ is the local direct product of the kernel of $G^{\wedge}f$ and its image.

1.15. THEOREM. *A compact connected simple symmetric space M is determined by any one pair (M^+, M^-) of a polar M^+ and a meridian M^- to M^+. More precisely, another such space N is isomorphic with M if a pair (N^+, N^-) of a polar in N and a meridian to it is isomorphic with (M^+, M^-) in the following strong sense; M^+ is isomorphic with N^+ and M^- is isometric with N^- up to a constant multiple of the metric.*

PROOF. One could verify this case by case, since those pairs are all known for all the spaces. The corresponding local isomorphism theorem will be proven later (2.28). We assume 2.28 and will prove the theorem. If M and N satisfy the hypothesis for the pairs (M^+, M^-) and (N^+, N^-), then the bottoms do locally by 1.5b. Hence we have the isomorphism $M^\sim \cong N^\sim$ for the universal covering spaces by 2.28. The kernel (1.14) for the projection $\pi: M^\sim \to M$ lies within the corresponding meridian (1.14a), which can be compared with the counterpart for N by the isometry assumption. QED

1.15a. REMARK. Some comment may be due for the isomorphism in the stronger sense. For most spaces it is sufficient to assume that M^+ is isomorphic with N^+ and M^- with N^-, but, when M^- is a local product of a circle T and a semisimple space, the ratio of the diameter of T to that of M^- counts in locating M within its local isomorphism class, just as the groups in the local isomorphism class of the Lie group $\mathrm{SL}(2, \boldsymbol{R})$ are distinguished by the diameter of the maximal subgroup $\cong \mathrm{SO}(2) \cong T$ with respect to the standard pseudo-Riemannian metric (defined by the Killing form). One of the very few counterexamples is $(M^+, M^-) = (\mathrm{E\,III}, T\cdot\mathrm{SO}(10)^\sim)$ for $M = \mathrm{E}_6$ and $M^* = \mathrm{E}_6/\boldsymbol{Z}_3$.

§2. Observations of the root systems.

We will study subspaces which are the fixed point sets of involutions of compact symmetric spaces as Riemannian manifolds. The root systems will come up as ingredients of the curvature; the roots in our sense are called the restricted roots in the literature including [H]. In terms of the root system, we will describe the meridians in particular, which will allow us to determine them locally. The second half of the section will be devoted to detailed studies of the root systems; an introduction for them will be given between 2.24 and 2.25.

Let t be an involution of M which fixes a point o. We write M^t for $F(t, M)_{(o)}$, the connected component through o of the fixed point set of t acting on M. Let M^{-t} denote $F(t \circ s_o, M)_{(o)}$, the c-orthogonal space to M^t at o.

Consider a geodesic c in M^{-t} starting at $c(0)=o$. Let JF^t denote the space of the Jacobi fields v on c ([KN] vol. 2, p. 63) which are tangent to M^t at o and whose covariant derivatives $\nabla_{c'(0)}v$ at o are tangent to M^{-t}. A point $c(u)$ is called *a focal point* (with respect to JF^t) if there is a nonzero Jacobi field $v=v\circ c$ in JF^t which vanishes at $c(u)$, while the point $c(u)$ is called *a conjugate point* of o along c if there is a nonzero Jacobi field in JF^t which vanishes at $c(u)$ and at o. It is a known important fact that there is no focal point $c(u)$ on the arc $c|[0, u_o]$, $0<u<u_o$, if the arc is weakly stable (that is, it is not longer than any nearby curves from M^t to the point $c(u_o)$). The Jacobi fields are exactly the solutions of the following Jacobi equation, an ordinary linear differential equation of the second order:

2.1. $$\nabla_H\nabla_H v+K(v, H)H=0\,,$$

where H denotes the tangent vector $c'(u)$ and K is the curvature tensor. To solve this equation, one notes that the linear operator: $v\mapsto K(v, H)H$ is symmetric and constant in the sense $\nabla K=0$, which follows from the fact that every point symmetry fixes K.

Now the action of $\mathrm{Aut}(M)$ makes its Lie algebra $\mathfrak{g}$ a linear space of vector fields v on M. If one embeds M into $G^\wedge$ as the subspace $F(s_1\circ \mathrm{ad}(s_o), G^\wedge)_{(1)}$ by 1.9, then a member b of $G^\wedge$ (or G) carries a point x of M into $bx(\mathrm{ad}(s_o)b)^{-1}$. This convention has technical advantages; it frees us from distinction between the group manifolds and the other spaces in proving some theorems, for instance.

Since every $v\in\mathfrak{g}$ is a Killing vector field, the restriction $v\circ c$ to c is a Jacobi field. We employ the symmetry decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ at o and the one $\mathfrak{g}=\mathfrak{k}_{c(u)}+\mathfrak{m}_{c(u)}$ at every point $c(u)$ of c. We choose H in $\mathfrak{m}$ so that $H(o)=c'(0)$. Then we have $H(c(u))\in\mathfrak{m}_{c(u)}$; and hence $H(c(u))=c'(u)$ at every point. One recalls $K(v, H)=-[[v, H], H]$ at o and for every $v\in\mathfrak{m}$ ([H], p. 215) and hence for every $v\in\mathfrak{g}$; therefore one has $K(v, H)H=-[[v, H], H]$ everywhere on c for every $v\in\mathfrak{g}$. Thus 2.1 reads: $\nabla_H\nabla_H v-\mathrm{ad}(H)^2\, v=0$ for every $v\in\mathfrak{g}$. In terms of the Fermi coordinates, or equivalently, by identifying the tangent space $T_{c(u)}M$ with T_oM or $\mathfrak{m}$ by the transformation $\exp(uH)$ (which is a parallel displacement along c), this is converted to the equation

2.1a. $$\left(\frac{d}{du}\right)^2 v-\mathrm{ad}(H)^2v=0 \qquad \text{for a curve } v \text{ on } T_oM\,.$$

To go further, we decompose $\mathfrak{k}$ and $\mathfrak{m}$ by means of $\mathrm{ad}(t)$; $\mathfrak{k}=\mathfrak{k}^{-t}+\mathfrak{k}^t$, and $\mathfrak{m}=\mathfrak{m}^{-t}+\mathfrak{m}^t$, where $\mathfrak{k}^t=F(\mathrm{ad}(t), \mathfrak{k})$, $\mathfrak{k}^{-t}=F(-\mathrm{ad}(t), \mathfrak{k})$, $\mathfrak{m}^t=F(\mathrm{ad}(t), \mathfrak{m})$ and $\mathfrak{m}^{-t}=F(-\mathrm{ad}(t), \mathfrak{m})$. We can do this since $\mathrm{ad}(s_o)$ commutes with $\mathrm{ad}(t)$. The symmetry decompositions of M^t and M^{-t} are $\mathfrak{k}^t+\mathfrak{m}^t$ and $\mathfrak{k}^t+\mathfrak{m}^{-t}$ at o respectively.

We choose a maximal abelian subalgebra $\mathfrak{a}^{-t}$ in $\mathfrak{m}^{-t}$. Since $Q(\mathfrak{a}^{-t})$ generates a toral group A^{-t}, we have another decomposition $\mathfrak{g}=\sum\mathfrak{g}(\alpha)$ of $\mathfrak{g}$ through the adjoint action of A^{-t}, where α is a linear form on $\mathfrak{a}^{-t}$ and $\mathfrak{g}(\alpha)$ is a subspace on which $(\mathrm{ad}(H))^2$ is a scalar multiplication by $-\alpha(H)^2$ for every member H of $\mathfrak{a}^{-t}$. We agree $\mathfrak{g}(\alpha)=\mathfrak{g}(-\alpha)$ because of the ambiguity involved in the definition; whenever we need distinguish $-\alpha$

from α for a technical reason, we have only to work on the complexification of $\mathfrak{g}$, as in the usual Lie algebra theory.

Since every $(\mathrm{ad}(H))^2$ commutes with $\mathrm{ad}(s_o)$ and $\mathrm{ad}(t)$, we obtain a finer decomposition: $\mathfrak{k}^t=\sum\mathfrak{k}^t(\alpha)$, $\mathfrak{k}^{-t}=\sum\mathfrak{k}^{-t}(\alpha)$, $\mathfrak{m}^t=\sum\mathfrak{m}^t(\alpha)$, and $\mathfrak{m}^{-t}=\sum\mathfrak{m}^{-t}(\alpha)$, where $\mathfrak{k}^t(\alpha):=\mathfrak{k}^t\cap\mathfrak{g}(\alpha)$, etc. Note $\mathfrak{m}^{-t}(0)=\mathfrak{a}^{-t}$. One sees that $\mathfrak{k}^t(\alpha)$, $\alpha\neq 0$, is linearly isomorphic with $\mathfrak{m}^{-t}(\alpha)$ and $\mathfrak{k}^{-t}(\alpha)$ with $\mathfrak{m}^t(\alpha)$ by any $\mathrm{ad}(H)$, $\mathrm{H}\in\mathfrak{a}^{-t}$ satisfying $\alpha(H)\neq 0$, since $\mathrm{ad}(H)^2$ is then a linear automorphism on $\mathfrak{k}(\alpha)$ and $\mathfrak{m}(\alpha)$.

2.2. Notations. Let $R(M;\mathfrak{a}^{-t})$, $R(M^t;\mathfrak{a}^{-t})$ and $R(M^{-t};\mathfrak{a}^{-t})$ denote the set of all the linear forms $\pm\alpha\neq 0$ with $\mathfrak{g}(\alpha)\neq\{0\}$, its subset consisting of α with $\mathfrak{m}^t(\alpha)\neq\{0\}$, and the subset consisting of α with $\mathfrak{m}^{-t}(\alpha)\neq\{0\}$ respectively; thus one has $R(M;\mathfrak{a}^{-t})=R(M^t;\mathfrak{a}^{-t})\cup R(M^{-t};\mathfrak{a}^{-t})$. The linear forms $\pm\alpha$ are identified with members of $\mathfrak{a}^{-t}$ by means of the inner product on $\mathfrak{m}$.

2.3. Proposition. *$R(M;\mathfrak{a}^{-t})$ and $R(M^{-t};\mathfrak{a}^{-t})$ are root systems in the usual sense* (cf. [B]). *$R(M^t;\mathfrak{a}^{-t})$, or the union $R(M^t;\mathfrak{a}^{-t})\cup\{0\}$ in case $\mathfrak{m}^t(0)\neq\{0\}$, is a weight system in the sense to be explained below.*

Proof. This is known for the first two sets [OS]; the root theory for a Lie algebra $\mathfrak{g}$ remains valid to this extent and beyond even if the abelian subalgebra $\mathfrak{a}^{-t}$ is not maximal in $\mathfrak{g}$ (See 2.4c for another example). The point is that every member α of $R(\mathrm{M};\mathfrak{a}^{-t})$ together with a nonzero member x of $\mathfrak{g}(\alpha)$ generates a Lie subalgebra $\mathfrak{g}_x(\alpha)\cong\mathscr{L}\mathrm{Sp}(1)$ of $\mathfrak{g}$ if 2α is not a member of $R(M;\mathfrak{a}^{-t})$. $\mathfrak{g}_x(\alpha)$ acts on $\mathfrak{g}$ through the adjoint action. It acts on $\mathfrak{g}^{-t}:=\mathfrak{k}^t+\mathfrak{m}^{-t}$ if α is in addition a member of $R(M^{-t};\mathfrak{a}^{-t})$. We add that $R(M^{-t};\mathfrak{a}^{-t})$ spans $\mathfrak{a}^{-t}$ if M^{-t} is semisimple. From now on we will use the language of the root theory [B]; see [B] also for some notation, the numbering of the simple roots, etc. The set $R(M^t;\mathfrak{a}^{-t})$ is a *weight system* for $R(M^{-t};\mathfrak{a}^{-t})$ in the sense that (i) this finite set is invariant under the action of the Weyl group of $R(M^{-t};\mathfrak{a}^{-t})$; (ii) its members λ are weight forms, that is, $\langle\lambda,\alpha^\vee\rangle$ is an integer for every root α in $R(M^{-t};\mathfrak{a}^{-t})$, where $\alpha^\vee=2\|\alpha\|^{-2}\alpha$; and (iii) it contains $\lambda+m\alpha$ whenever it contains λ and $\lambda+p\alpha$, m an integer between 0 and the integer p; namely, the set is R-saturé, $R=R(M^{-t};\mathfrak{a}^{-t})$, in the sense of [B], Chap. 8. (The linear span of $R(M^{-t};\mathfrak{a}^{-t})$ does not necessarily contain $R(M^t;\mathfrak{a}^{-t})$.) This obtains since the Lie subalgebra $\mathfrak{g}_x(\alpha)$ stabilizes the linear subspace $\mathfrak{k}^{-t}+\mathfrak{m}^t$ of $\mathfrak{g}$. QED

2.3a. Remark. If M is a group and $t\circ s_o$ is a group automorphism, $o=1$, then $R(M^t;\mathfrak{a}^{-t})$ is the weight system of the isotropy representation for the space $M^t\risingdotseq M/M^{-t}$; $\risingdotseq$ means local isomorphism. By a later theorem 2.15, one can more easily determine the representation for a meridian than E.Cartan did years ago.

2.4. Definition. We call $R(M^{-t};\mathfrak{a}^{-t})$ the *root system* of M^{-t} and denote it by $R(M^{-t})$ also; its isomorphism class depends on the space M^{-t} only, independent of $\mathfrak{a}^{-t}$ and t. The members of $R(M^{-t};\mathfrak{a}^{-t})$ are called the *roots* of M^{-t} (with respect to $\mathfrak{a}^{-t}$), and those of $\mathfrak{m}^{-t}(\alpha)$ are *the root space for* α. We call the dimension of $\mathfrak{m}^{-t}(\alpha)$ *the multiplicity of* α, denoted by $m^{-t}(\alpha)$. One has $m^{-t}(\alpha)=\dim \mathfrak{k}^{t}(\alpha)$. *The rank of* M^{-t} is $r(M^{-t})=\dim \mathfrak{a}^{-t}$ as defined earlier. In case $t=s_o$, M^{-t} is M itself and, dropping the upper index $-t$, we write $\mathfrak{a}$ for $\mathfrak{a}^{-t}$ to have $R(M;\mathfrak{a})=R(M^{-t})$; also $m(\alpha)=m^{-t}(\alpha)$.

2.4a. REMARK. One gets some information about G from $R(M)$ and the action of G on $M=G/K$. Thus, if G is the bottom $G^*=\mathrm{ad}(G)$ acting on a semisimple M, one has $\exp(2\pi H)=1 \Leftrightarrow \mathrm{ad}(\exp(2\pi H))=1 \Leftrightarrow \cos(2\pi\alpha(H))=1$ for every α in $R(M)$, where $H\in\mathfrak{a}$; cf. 2.20. Other examples are 2.4b, 2.4c and 2.5.

The next lemma gives a geometric meaning of the root.

2.4b. LEMMA. *Let* α *be a root* $\in R(M)$, *M simple. Assume* 2α *is not a root. Then* (i) $\mathfrak{m}(\alpha)$ *and* α *together span the tangent space* $T_oM(\alpha)$ *to a connected subspace* $M(\alpha)$ *of positive constant curvature* $\|\alpha\|^2$ *(that is,* $M(\alpha)$ *is either a sphere or a real projective space).* (ii) *The bracket product*: $\mathfrak{m}(\alpha)\wedge\mathfrak{m}(\alpha)\to[\mathfrak{m}(\alpha),\mathfrak{m}(\alpha)]\cong\mathscr{L}O(\mathfrak{m}(\alpha))$ *is bijective. Also* $\mathfrak{m}(\alpha)+\boldsymbol{R}\alpha$ *generates* $\mathscr{L}O(m(\alpha)+2)$. (iii) $M(\alpha)$ *is a sphere unless* $M=M(\alpha)$ *or* $R(M)$ *is* $\mathbf{B}_r$ *in which* α *is a shorter root.* (iv) $M(\alpha)$ *is a sphere if* M *is* 1*-connected.* (v) $M(\alpha)$ *contains a subspace* $M'(\alpha)\ni o$ *whose tangent space* $T_oM'(\alpha)$ *is* $\mathfrak{m}(\alpha)$, *which is congruent with the polar* ($\cong$ *a real projective space*) *or the centrosome* ($\cong$ *a sphere*) *in* $M(\alpha)$. *And* (vi) *there is an involutive member b of* $K_{(1)}$ *which stabilizes* $\mathfrak{a}$ *and induces the reflection in the hyperplane with a normal vector* α *on* $\mathfrak{a}$. *Thus* $K_{(1)}$ *contains members which, restricted to* $\mathfrak{a}$, *generate the Weyl group* $W(R(M))$.

PROOF. Recall E.Cartan's result [H]: a linear subspace $\mathfrak{m}'$ of $\mathfrak{m}$ is tangent to a unique connected subspace M' of M if and only if $[[\mathfrak{m}',\mathfrak{m}'],\mathfrak{m}']\subset\mathfrak{m}'$. The condition is equivalent to say that the subalgebra $\mathfrak{k}'$ of $\mathfrak{g}$ generated by $[\mathfrak{m}',\mathfrak{m}']$ normalizes $\mathfrak{m}'$; the symmetry decomposition for M' is $\mathfrak{k}'+\mathfrak{m}'$. We will verify it for $\mathfrak{m}':=\mathfrak{m}(\alpha)+\boldsymbol{R}\alpha$. One notes

2.4b.1. $$[\mathfrak{g}(\alpha),\mathfrak{g}(\beta)]\subset\mathfrak{g}(\alpha+\beta)+\mathfrak{g}(\alpha-\beta)$$

for roots (or zeroes) α and β; in particular one has $[\mathfrak{g}(\alpha),\mathfrak{g}(\alpha)]\subset\mathfrak{g}(0)=\mathfrak{k}(0)+\mathfrak{a}$, since 2α is not a root. Hence one has $[\mathfrak{m}(\alpha),\mathfrak{m}(\alpha)]\subset\mathfrak{k}(0)$. One knows $[\mathfrak{m}(\alpha),\boldsymbol{R}\alpha]=\mathfrak{k}(\alpha)$ and $[\mathfrak{k}(\alpha),\boldsymbol{R}\alpha]=\mathfrak{m}(\alpha)$.

One obtains $[\mathfrak{k}(\alpha),\mathfrak{m}(\alpha)]=\boldsymbol{R}\alpha$ from the inner product $\langle[v,x],H\rangle=\langle v,[x,H]\rangle=\langle v,-\alpha(H)y\rangle=-\langle v,y\rangle\alpha(H)$ for every $v\in\mathfrak{k}(\alpha)$, $H\in\mathfrak{a}$ and $x\in\mathfrak{m}(\alpha)$ and for some $y\in\mathfrak{k}(\alpha)$, $[H,x]=\alpha(H)y$; one has $[v,x]=-\langle v,y\rangle\alpha$. Since $\mathfrak{k}(0)$ normalizes $\mathfrak{m}(\alpha)$, $\mathfrak{k}(\alpha)$ and $\boldsymbol{R}\alpha$, one has a subalgebra $(\mathfrak{k}(0)+\mathfrak{k}(\alpha))+(\mathfrak{m}(\alpha)+\boldsymbol{R}\alpha)$ of $\mathfrak{g}$ with this symmetry decomposition; we have established the unique existence of $M(\alpha)$. Now we compute the curvature by using the formula [H] for the one corresponding to the orthonormal vectors v and $x\in\mathfrak{m}'$: $\langle K(v,x)x,v\rangle=\langle-[[v,x],x],v\rangle=\|[x,v]\|^2$. One may assume that x is proportional to α; in fact one can rotate x into a unit vector $H\in\boldsymbol{R}\alpha$ fixing v by means of the group generated by $\mathrm{ad}(y)$, $[\alpha,x]=\|\alpha\|^2y\in\mathfrak{k}(\alpha)$, because of $\langle[y,v],\alpha\rangle=\langle v,[\alpha,y]\rangle=-\langle v,\|\alpha\|^2x\rangle=0$ and $[\mathfrak{k}(\alpha),\mathfrak{m}(\alpha)]=\boldsymbol{R}\alpha$. Now $\langle K(v,H)H,v\rangle=\langle-[[v,H],$

$H], v\rangle = \|[H, v]\|^2 = \|\alpha(H)v\|^2 = \|\alpha\|^2$. We turn to (ii). Now that $M(\alpha)$ is an elliptic space, the isotropy subgroup of its automorphism group has the Lie algebra $\mathfrak{k}_0(\alpha) + \mathfrak{k}(\alpha)$, where $\mathfrak{k}_0(\alpha) = [\mathfrak{k}(\alpha), \mathfrak{k}(\alpha)] = [\mathfrak{m}(\alpha), \mathfrak{m}(\alpha)]$ is that of the subgroup which fixes α. $\mathfrak{k}_0(\alpha) + \mathfrak{k}(\alpha)$ is the Lie algebra of the orthogonal group $O(\mathfrak{m}(\alpha) + \boldsymbol{R}\alpha)$. (Replacing $\mathfrak{k}(0)$ with $\mathfrak{k}_0(\alpha)$ in the above symmetry decomposition, one gets the effective Lie algebra acting on $M(\alpha)$. Also see the proof of 2.25 for a more algebraic proof of the above equalities involving $\mathfrak{k}_0(\alpha)$.) For (iii), we first assume that G is the bottom G^*. And let $G_x(\alpha)$ be the connected subgroup of G^* generated by $\mathfrak{g}_x(\alpha)$ which is, as before, generated in $\mathfrak{g}$ by a nonzero $x \in \mathfrak{m}(\alpha)$ and α. Then $G_x(\alpha)$ is isomorphic with $Sp(1)$ (not with $SO(3)$) if and only if $\langle \beta, \alpha^\vee \rangle$ is odd for some root β in $R(M)$, as one sees by letting $G_x(\alpha)$ act on the sum of $\mathfrak{g}(j\alpha + \beta)$, j integer, ([B] on the length of α-series of roots). Such a root β fails to exist if and only if either $M = M(\alpha)$ or α is a shorter root in $R(M)$ which is B_r. In case M is also the bottom M^*, $G_x(\alpha) \cap M^*$ is still a 2-sphere if such a root β exists, since M^* is a subspace $Q(M^*)$. Then the elliptic space $M(\alpha)$ of which $G_x(\alpha) \cap M^*$ is a subspace is also a sphere. Therefore one has (iii) for every covering space of M^*. Assume that M is 1-connected. If M has rank 1, then $M(\alpha)$ is the whole space M and of course it is a sphere. If $R(M)$ is B_r, then (let us assume one knows or one will find later that) M^* is the group $SO(2r+1)$ or the real Grassmann manifold $G_r(\boldsymbol{R}^{2r+m})$, $m \geqq 1$. $M^*(\alpha)$ is $SO(3)$ (embedded in the obvious way) or the real projective space $G_1(\boldsymbol{R}^{2+m})$ respectively. It is not hard to see that the lift of the circle with the tangent α passes through the pole in M. (iii) and (iv) are essentially due to Helgason, although he discusses the highest root only in [H]. (v) is obvious; $M'(\alpha)$ is c-orthogonal to the circle with a tangent α at o and $[\mathfrak{m}(\alpha), \mathfrak{m}(\alpha)] + \mathfrak{m}(\alpha)$ is the symmetry decomposition for the elliptic space $M'(\alpha)$. (vi) is elementary. Take y in the above and normalize y (and x) so that $[x, y] = \alpha$, hence $[y, \alpha] = x$. Then $\mathrm{ad}(y)$ acts trivially on the hyperplane with the normal α in $\mathfrak{a}$. And $b := \exp(\pi\, \mathrm{ad}(y))$ carries α into $-\alpha$. QED

2.4c. Remark. One has the multiplicity $m(\alpha) = m(\beta)$ if the length $\|\alpha\| = \|\beta\|$ and M is simple, by 2.4b. Moreover $m(\alpha) \leqq m(\beta)$ if $\|\alpha\| \geqq \|\beta\|$, as one sees by the usual arguments on α-series of a weight [B].

The construction of $M(\alpha)$ for a single root α may be generalized to that of a subspace for several roots.

2.4d. Lemma. *Given a set $S \subset R(M)$, one has a unique connected subspace $M(S)$ of M whose tangent space $T_oM(S)$ equals $\mathfrak{m}_S := \mathfrak{a}_S + \sum_{R(S)} \mathfrak{m}(\alpha)$, summed up for the roots in $R(S)$, where $\mathfrak{a}_S$ is the linear span of S and $R(S)$ is the minimal root subsystem of $R(M)$ satisfying $S \subset R(S) \subset R(M) \cap \mathfrak{a}_S$.*

Proof. $R(S)$ exists, since $R(M) \cap \mathfrak{a}_S$ is a root system (See [B]). Let $\mathfrak{k}_S$ be the subalgebra of $\mathfrak{k}$ generated by $[\mathfrak{m}_S, \mathfrak{m}_S]$. One knows $[\mathfrak{m}(\alpha), \mathfrak{m}(\beta)] \subset \mathfrak{k}(\alpha + \beta) + \mathfrak{k}(\alpha - \beta)$, and one sees $\alpha \pm \beta \in R(S)$ if $\mathfrak{k}(\alpha \pm \beta) \neq 0$ and $\alpha \pm \beta \neq 0$. Hence $\mathfrak{k}_S \subset \mathfrak{k}(0) + \sum_{R(S)} \mathfrak{k}(\alpha)$. While $\mathfrak{k}(0)$ normalizes every $\mathfrak{m}(\beta)$, one has $[\mathfrak{k}(\alpha), \mathfrak{m}(\beta)] \subset \mathfrak{m}(\alpha + \beta) + \mathfrak{m}(\alpha - \beta)$, and one concludes

$[\mathfrak{k}_S, \mathfrak{m}_S] \subset \mathfrak{m}_S$. This establishes the unique existence of $M(S)$; the symmetry decomposition for $M(S)$ is $\mathfrak{k}_S + \mathfrak{m}_S$. QED

2.5. PROPOSITION. *The following five conditions are equivalent.*

(i) *The multiplicity* $m(\alpha)=1$ *for every root* α *of* $M=G/K$.

(ii) *The rank* $r(M)=r(G)$.

(iii) *The root system* $R(M) \cong R(G)$.

(iv) *The subspace* $\mathfrak{k}(0)=\{0\}$.

(v) *The subalgebra* $\mathfrak{k}(0)$ *acts on* $\mathfrak{m}(\alpha)$ *reducibly for every root* α *of* M.

PROOF. One clearly sees $[\mathfrak{k}(0), \mathfrak{m}(\alpha)] \subset \mathfrak{m}(\alpha)$ for every α. Under the assumption of (i), this gives $[\mathfrak{k}(0), \mathfrak{m}(\alpha)]=\{0\}$, since $\mathfrak{k}(0)$ is the Lie algebra of a compact Lie group. Hence $\mathfrak{k}(0)=\{0\}$, since $\mathfrak{k}$ is effective on $\mathfrak{m}$. Obviously, (iv) $\Leftrightarrow \mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g} \Leftrightarrow$ (ii). Assume (ii). Then $\mathfrak{g}$ is the direct sum of $\mathfrak{a}$ and $\mathfrak{g}(\alpha)$ for the roots of M; hence every root α of M is that of $\mathfrak{g}$. The converse is trivial: (iii) $\Rightarrow$ (ii), since $\mathfrak{a}$ is the direct sum of the linear space spanned by $R(M)$ and the torus part of $\mathfrak{m}$. Under the equivalent assumptions of (ii), one recalls $\dim \mathfrak{g}(\beta)=2$ to derive $m(\beta)=\frac{1}{2}\dim \mathfrak{g}(\beta)=1$; (ii) $\Rightarrow$ (i). Trivially, (iv) implies (v). Finally assume (v). If 2α is not a root, then $[\mathfrak{m}(\alpha), \mathfrak{m}(\alpha)] \subset \mathfrak{k}(0)$ is $\mathscr{L}\mathrm{O}(\mathfrak{m}(\alpha))$ by 2.4b. Thus $m(\alpha)=1$ by (v). The case of a root 2α is handled similarly (See 2.26a).

2.5a. REMARK. One cannot generalize 2.5 by replacing M and G with M^{-t} and M respectively. For example, if $M=\mathrm{EI}=\mathrm{E}_6/\mathrm{Sp}(4)^*$, there is an involution for which $M^{-t}=\mathrm{CI}(4)^*=(\mathrm{Sp}(4)/\mathrm{U}(4))^*$; thus (i) does not imply (ii). However, (v) implies (i): $\mathfrak{k}^{+t}(0)$ *is irreducible on* $\mathfrak{m}^{-t}(\alpha)$ *if* $\dim \mathfrak{m}^{-t}(\alpha)>1$. *And similarly for its action on* $\mathfrak{m}^t(\lambda)$.

2.6. EXAMPLE. Let $M=\mathrm{SU}(n)$, $M^t=\mathrm{AI}(n):=\mathrm{SU}(n)/\mathrm{SO}(n)$ and $M^{-t}=\mathrm{SO}(n)$; thus t is the composite of the complex conjugation κ and the symmetry at 1. Then $R(M;\, \mathfrak{a}^{-t})$ is, if $n=2n'$ is even, $\mathrm{C}_{n'}(4,2)$, where $\mathrm{C}_{n'}$ is the root system, 4 is the multiplicity of the shorter roots and 2 is the common multiplicity of the longer roots. By convention, we indicate the multiplicities of the roots in the order of the length of the roots, based on the fact that the roots of equal length have equal multiplicity for simple symmetric spaces; this is true, because the members of the Weyl group represented by those of $K_{(1)}$ preserve the multiplicity. $R(M^{-t})$ is $\mathrm{D}_{n'}(2,0) \cong R(\mathrm{SO}(n))$ and the highest weight in $R(M^t;\, \mathfrak{a}^{-t})$ is $2\omega_1$, which indicates the action of the isotropy subgroup $\mathrm{SO}(n)$ on the tangent space to $\mathrm{AI}(n)$ at the unit element 1. $R(M^t;\, \mathfrak{a}^{-t})$ is actually $\mathrm{C}_{n'}(4,2)$. If $n=2n'+1$ is odd, then $R(M;\, \mathfrak{a}^{-t})$ is $\mathrm{BC}_{n'}(4,2,2)$; the longest roots have multiplicity 2. We also note that $\mathfrak{m}^t(0)+\mathfrak{a}^{-t}$ is a maximal abelian subalgebra of the Lie algebra $\mathscr{L}\mathrm{SU}(n)$. Now we exchange $\mathrm{AI}(n)$ and $\mathrm{SO}(n)$; $t=\kappa$ so that we consider $M^{-t}=\mathrm{AI}(n):=\mathrm{SU}(n)/\mathrm{SO}(n)$ and $M^t=\mathrm{SO}(n)$. Then $R(M^{-t}) \cong \mathrm{A}_{n-1}(1,0)$, $R(M^t;\, \mathfrak{a}^{-t}) \cong \mathrm{A}_{n-1}(1,0)$, $R(M;\, \mathfrak{a}^{-t}) \cong \mathrm{A}_{n-1}(2,0)$ of course.

Now we return to the geodesic c, $c'(0)=H \in \mathfrak{a}^{-t}$, $H \neq 0$, and JF^t; $c'(u)$ is (the restriction of) a member H of $\mathfrak{a}^{-t}$ so that one has $c(u)=\exp(uH)(o)$. If v is a member of $\mathfrak{g}(\alpha)$, then v solves 2.1a or, quite explicitly, $(d/du)^2v+\alpha(H)^2v=0$. Thus, if one notes $v(c(u))=v(\exp(uH)o)=\exp(uH)\circ\operatorname{ad}(\exp(-uH))(v)(o)$ for any value of the parameter u, then one obtains the next lemma.

2.7. LEMMA. *(i) v vanishes at $c(u)$ if and only if* $\operatorname{ad}(\exp(-uH))(v)$ *belongs to* $\mathfrak{k}$; *and* (ii) *under additional conditions of $v\in\mathfrak{k}(\alpha)$ and $\alpha(H)\neq 0$, v vanishes at $c(u)$ if and only if $u\alpha(H)$ is an integral multiple of π.*

In case $\alpha(H)=0$, one sees that not only $v\in\mathfrak{g}(\alpha)$ but the vector field: $u\mapsto uv=uv(c(o))$ is also a solution of the Jacobi equation. And similarly for the members of $\mathfrak{a}^{-t}$. Therefore we obtain the next lemma.

2.8. LEMMA. *JF^t is, in the natural fashion, isomorphic with the direct sum of the following linear spaces;* (1) $\mathfrak{m}^t$, (2) $\mathfrak{k}^t(\alpha)$ *for* $\alpha(H)\neq 0$, (3) $\{$*the vector fields*: $u\mapsto uv(c(u))\mid v\in\mathfrak{m}^{-t}(\alpha)\}$ *for* $\alpha(H)=0$, *and* (4) $\{$*the vector fields*: $u\mapsto uv(c(u))\mid v\in\mathfrak{a}^{-t}\}$.

PROOF. This must be obvious from the above, since JF^t has dimension $=\dim M$ and one has $\nabla_H v=0$ on c for v in (3) and (4). QED

The following four conditions are equivalent to each other. (i) The point $p=c(\pi)=\exp(\pi H)(o)$ is fixed by t; (ii) c meets another component $M^t(p)$ of $F(t, M)$ through p; (iii) s_o fixes p; and (iv) $\exp(2\pi H)(o)=o$. We assume $t(p)=p$. Then c is a circle with $c(2\pi)=o$; it may happen, however, that p coincides with o. By 2.7, this gives that $\alpha(2H)$ is an integer for every α in $R(M;\mathfrak{a}^{-t})$; alternatively, one sees this from the fact that $\operatorname{ad}(\exp(2\pi H))$ preserves $\mathfrak{k}$, which follows from $o=c(2\pi)=\exp(2\pi H)(o)$. We use these facts to prove the next proposition.

2.9. PROPOSITION. *Assume that t fixes the point $p=c(\pi)$ on the geodesic c: $u\mapsto\exp(uH)(o)$, $H\in\mathfrak{a}^{-t}$.*

(i) *If there is no conjugate point of o on the interval* $(0,\pi)$ *along the geodesic c, then one has $\alpha(H)=0$, $\pm\frac{1}{2}$ or ± 1 for every α in $R(M^{-t};\mathfrak{a}^{-t})$.*

(ii) *If there is no focal point of o on the interval* $(0,\pi)$ *along the geodesic c, then one has $\lambda(H)=0$ or $\pm\frac{1}{2}$ for every λ in $R(M^t;\mathfrak{a}^{-t})$.*

PROOF. (i) We saw that $\alpha(2H)$ is an integer for every α in $R(M;\mathfrak{a}^{-t})$. But the members of $\mathfrak{k}^t$ cannot vanish on the interval $(0,\pi)$ by the assumption. (ii) While $\operatorname{ad}(\exp(2\pi H))$ preserves $\mathfrak{m}^t$, the members of $\mathfrak{m}^t$ cannot vanish on the interval $(0,\pi)$ similarly. More details may be due. Let v be a (nonzero) member of $\mathfrak{m}^t(\lambda)$. Then there is a member w of $\mathfrak{k}^{-t}(\lambda)$ such that one has $\operatorname{ad}(\exp uH)(v)=\cos(u\lambda(H))\,v+\sin(u\lambda(H))\,w$ for every real number u by (2.1a). If $\lambda(H)\neq 0$, then $\lambda(H)=\pm\frac{1}{2}$ by (2.8) and the assumption, since v never vanishes ($\Leftrightarrow\cos(u\lambda(H))$ never vanishes) on the interval $(0,\pi)$ and yet $\lambda(2H)$ is an integer. We might add that a conjugate point is focal in our setting. QED

2.10. COROLLARY. (i) *In the setting of* (ii) *of* (2.9), $\mathfrak{m}^t(\lambda)$ *is contained in*

$F(\mathrm{ad}(s_p), \mathfrak{m}^t) = F(-\mathrm{ad}(Q(p)), \mathfrak{m}^t)$ *for* λ *in* $R(M^t; \mathfrak{a}^{-t})$ *if* $\lambda(H) \neq 0$; *and* $\mathfrak{m}^t(\lambda) \subset F(-\mathrm{ad}(s_p), \mathfrak{m}^t)$ *if* $\lambda(H) = 0$.

(ii) *In the setting of* (i) *of* (2.9), $\mathfrak{k}^t(\alpha)$ *is contained in* $F(\mathrm{ad}(s_p), \mathfrak{k}^t)$ *for* α in $R(M^{-t}; \mathfrak{a}^{-t})$ *if* $\alpha(H) = \pm 1$ *or* 0; *and* $\mathfrak{k}^t(\alpha) \subset F(-\mathrm{ad}(s_p), \mathfrak{k}^t)$ *if* $\alpha(H) = \pm\frac{1}{2}$.

(iii) *In the setting of* (ii) *of* (2.9), *the parallel transport of the tangent space* T_oM *along the circle* $c = c \mid [0, 2\pi]$ (*= the action of* $\exp(2\pi H) = Q(p)$ *on* T_oM) *is the identity on the sum*

$$\mathfrak{m}^t(0) + \sum_{\lambda(H)=0} \mathfrak{m}^t(\lambda) + \sum_{\alpha(H)=\pm 1 \text{ or } 0} \mathfrak{m}^{-t}(\alpha) + \mathfrak{a}^{-t}.$$

This is identified with $T_oM^-(p)$ *by evaluation at* o *and with* $T_pM^-(p)$ *at* p *by evaluation.*

2.11. Corollary. *In the setting of* (ii) *of* (2.9), *the tangent space* $T_pM^t(p)$ *at* p *is identified with the sum*

$$\mathfrak{m}^t(0) + \sum_{\lambda(H)=0} \mathfrak{m}^t(\lambda) + \sum_{\alpha(H)=\pm 1/2} \mathfrak{k}^t(\alpha)$$

by evaluation. This sum is $F(-\mathrm{ad}(s_p), \mathfrak{g}^t)$. *In particular the tangent space to the intersection* $M^{-t} \cap M^t(p)$ *at* p *is identified with the third sum* $\sum \mathfrak{k}^t(\alpha)$ *for* $\alpha(H) = \pm\frac{1}{2}$ *by evaluation.*

To state an important corollary, we denote by $(H^k)_{1 \leqq k \leqq r}$, $r := \dim \mathfrak{a}^{-t}$, the dual basis to a basis of simple roots $(\alpha_j)_{1 \leqq j \leqq r}$ for $R(M; \mathfrak{a}^{-t})$ under the assumption that M is simple; thus $\alpha_j(H^k) = \delta_j^k$ and (H^k) is a basis for $\mathfrak{a}^{-t}$, while the k-th fundamental weight ω_k equals $\frac{1}{2}\|\alpha_k\|^2 H^k$. The vector H in $\mathfrak{a}^{-t}$ is a unique linear combination $\sum h_k H^k$. We may assume that $h_k \geqq 0$, in our study. And the highest root $\alpha^\sim$ of $R(M; \mathfrak{a}^{-t})$ is written $\sum n^j \alpha_j$ with positive integers n^j as coefficients.

2.12. Example. We describe these for the root system $BC_r = B_r \cup C_r$. $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$ for $1 \leqq j < r$ and $\alpha_r = \varepsilon_r$, while $\alpha^\sim = 2\varepsilon_1 = 2(\alpha_1 + \cdots + \alpha_r)$. One sees $\omega_j = H^j = \varepsilon_1 + \cdots + \varepsilon_j$ for $1 \leqq j < r$ and $2\omega_r = H^r = \varepsilon_1 + \cdots + \varepsilon_r$.

2.13. Corollary. *Under the assumption of* 2.9 (i), *if* M *is simple, then* H *is congruent* (*by* $K_{(1)}$) *with the vector in* 1° *or* 2° *below*:

1° $H = \frac{1}{2}H^j$ *for some* j *with* $n^j = 1$ *or* 2; *or*

2° $H = \frac{1}{2}H^j + \frac{1}{2}H^k$ *for some* j *and* k *with* $n^j = 1 = n^k$.

The case 1° *occurs if* $\alpha^\sim$ *is not a member of* $R(M^{-t}; \mathfrak{a}^{-t})$.

Proof. $H = \sum h_k H^k$ is carried into a given Weyl domain in $\mathfrak{a}^{-t}$ by 2.4a. And the transform of H still satisfies the assumption of 2.9 (i). Thus we may assume every h_k is nonnegative. For every simple root α_j, one has $\alpha_j(H) = h_j = 0$, $\frac{1}{2}$ or 1 and $\alpha^\sim(H) = \sum n^j h_j = \frac{1}{2}$ or 1 by 2.9. If $\alpha^\sim(H) = \frac{1}{2}$ then one has the case 1° with $n^j = 1$ necessarily. If $\alpha^\sim(H) = 1$ then one has 1° or 2° as is easily seen. QED

2.14. Remark. By this corollary, one can find the length of the arc $c \mid [0, \pi]$ and the distance from o to $p=c(\pi)$ (and to $M^t(p)$), or its ratio if the invariant Riemannian metric is not specified; in fact the length of $H^j=(2/\|\alpha_j\|^2)\omega_j$ is readily available in [B]. The distance was computed for certain exceptional spaces by Atsuyama [At].

2.14a. REMARK. A weight ω_j with $n^j=1$ is called "minuscule" in [B].

2.14b. REMARK. The case 2° in 2.13 will be illustrated in 2.16 ① and 3.7.

2.15. THEOREM (*A local characterization of a meridian $M^-(p)$*). *Assume a point $p\neq o$ of a simple space M is fixed by the symmetry s_o. Then* (i) *the root system $R(M^-(p))$ of the meridian to the polar $M^+(p)$ is read off from the root system $R(M)$ as follows; one obtains the Dynkin diagram of $R(M^-(p))$ either by deleting a vertex α_j, $n^j=1$, from the Dynkin diagram of $R(M)$ or by deleting a vertex α_j, $n^j=2$, from the extended Dynkin diagram of $R(M)$, where n^j is a coefficient in $\alpha^\sim=\sum n^j\alpha_j$. In the first case (of $n^j=1$), $M^-(p)$ is a local product of a circle with a tangent H^j and a semisimple space with the obtained root system; and $M^-(p)$ is semisimple if and only if $n^j=2$.* (ii) *In all the cases (but that of the real projective space), the multiplicity of the roots is preserved.* (iii) *Moreover, if M is the bottom space M^*, then the converse is true* (Theorem 2.5, [N]).

PROOF. Letting $t=s_o$, we apply (2.13). We then have $M^{-t}=M$ and $M^t=\{0\}$. Thus $\mathfrak{a}^{-t}$ is a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{m}$. Let c be a shortest geodesic from $o=c(0)$ to $p=c(\pi)$. Then c lies in the meridian $M^-(p)$; in fact the arc $c\,|\,[0,\pi]$ is one of the shortest from o to the polar $M^+(p)=M^t(p)$ by 1.5a (ii) and hence c is orthogonal to $M^+(p)$. We may assume that $\mathfrak{a}$ is tangent to $M^-(p)$ (See 1.8) and that the initial tangent $H:=c'(0)$ lies in $\mathfrak{a}$ and further in a given Weyl chamber or its closure. Now we employ 2.10, which we can, since the assumption (i) of 2.9 is equivalent to (ii) because of $M^t=\{0\}$. By 2.10, the tangent space $T_oM^-(p)$ to the meridian is identified with $\sum\mathfrak{m}(\alpha)+\mathfrak{a}$ for $\alpha(H)=\pm1$ or 0, since $\mathfrak{m}^t(0)+\sum\mathfrak{m}^t(\lambda)$ is trivially zero by $M^t=\{0\}$. We will characterize the sets of the roots α satisfying this condition, which will be the root system of $M^-(p)$. By 2.13, we may assume 1° $H=\frac{1}{2}H^j$ for some j with $n^j=1$, 2° $H=\frac{1}{2}H^j$ for some j with $n^j=2$, or 3° $H=\frac{1}{2}H^j+\frac{1}{2}H^k$ for some j and k with $n^j=1=n^k$. In the case 1°, $R(M^-(p))$ consists of the roots which are orthogonal to $\frac{1}{2}H^j$; thus the Dynkin diagram of $R(M^-(p))$ is that of $R(M)$ less the vertex α_j and $M^-(p)$ is a local product of a semisimple subspace with $R(M^-(p))$ and the circle with the initial tangent $\frac{1}{2}H^j$. In the case of 2°, $R(M^-(p))$ contains the highest root and all the roots that are orthogonal to $\frac{1}{2}H^j$. Hence it has the extended Dynkin diagram of $R(M)$ less the vertex α_j. In the third case, it is easy to check that the Dynkin diagram of $R(M^-(p))$ is obtained from the extended Dynkin diagram of $R(M)$ by deleting the vertices α_j and α_k; alternatively one looks at the bottom for which the case 3° does not occur (See 2.21). Obviously the multiplicity is preserved; simply because $\mathfrak{m}(\alpha)$ is contained in the tangent space $T_oM^-(p)$ to the meridian whenever α is its root and $M^-(p)$ has an equal rank to M. Finally, assume that M is the bottom space M^*. Then, given a vector H in $\mathfrak{a}$, the point $c(2\pi)=\exp(2\pi H)(o)$ coincides with o if $\mathrm{ad}(\exp(2\pi H))$ preserves $\mathfrak{k}$, simply because M^* has no pole. This is the case if and only if $\sin(2\pi\alpha(H))=0$ on each $\mathfrak{k}(\alpha)+\mathfrak{m}(\alpha)$, $\alpha\in R(M)$. Now choose $H=\frac{1}{2}H^j$ for some j with $n^j=1$ or 2. And $2\pi\alpha(H)=\pi\alpha(H^j)$ is an integral multiple of π for every root α. Thus $c(2\pi)=o$, while $p:=c(2\pi)\neq o$ by $\alpha_j(\pi H)=\frac{1}{2}\pi$. Hence the point p lies in a polar $M^+(p)$. One has $Q(p)=\exp(2\pi H)$, which determines $M^-(p)$ as desired. QED

2.15a. REMARK. If one chooses the involution $t=s_p$ for a point p in a polar $M^+(p)$ (hence $t(o)=o$), one has $M^{-t}=M^-(p)$ and $\mathfrak{a}^{-t}$ is a maximal abelian subalgebra in $\mathfrak{m}$ by 1.8. The subspace M^t is congruent with $M^+(p)$ by the point symmetry s_m at the midpoint $m=c(\frac{1}{2}\pi)$, where c is a shortest geodesic from $o=c(0)$ to $p=c(\pi)$ as in the proof above. Since c is a shortest from o to the polar, the use of s_m shows that there is no focal point on $c|(0,\pi)$ and 2.10 applies as well as 2.11 and others; hence $\mathfrak{m}^t(0)+\sum\mathfrak{m}^t(\lambda)$ is zero. One concludes that (i) $R(M)$ is the disjoint union of $R(M^{-t})=R(M^-(p))$ and $R(M^t,\mathfrak{a}^{-t})$, (ii) $R(M^-(p))$ consists of the roots α of M such that $\alpha(H)=\pm1$ or 0, and (iii) $R(M^t)$ consists of the roots λ of M such that $\lambda(H)=\pm\frac{1}{2}$. In particular, $\mathfrak{k}^-:=\mathfrak{k}^{-t}=F(-\mathrm{ad}(t),\mathfrak{k})$ is exchanged with $\mathfrak{m}^+:=\mathfrak{m}^t=F(\mathrm{ad}(t),\mathfrak{m})$ by $\mathrm{ad}(s_m)$, while $\mathfrak{k}^+:=\mathfrak{k}^t=F(\mathrm{ad}(t),\mathfrak{k})$ and $\mathfrak{m}^-:=\mathfrak{m}^{-t}=F(-\mathrm{ad}(t),\mathfrak{m})$ remain invariant. The symmetry decomposition for $M^t\cong M^+(p)$ is $\mathfrak{g}^+=\mathfrak{k}^++\mathfrak{m}^+$ and the one for the meridian is $\mathfrak{g}^-=\mathfrak{k}^++\mathfrak{m}^-$. (These are thus graded at least if they are complexified.)

2.15b. COROLLARY. *Some coefficient n^j in $\alpha^\sim=\sum n^j\alpha_j$ is 1 or 2 for any root system.*

2.16. EXAMPLES. ① Let $M=\mathrm{SU}(n)$. Then every $n^j=1$. The circle in the direction of $\frac{1}{2}H^j+\frac{1}{2}H^{n-j}$ meets the polar $G_{2j}(C^n)$ as in [N] (3.4). But in $\mathrm{SU}(n)^*$ the circle in the direction of $\frac{1}{2}H^j$ meets the polar $G_j(C^n)$ (or $G_j(C^n)^*$ if $2j=n$) at $c(\frac{1}{2}\pi)$. Notice that the polars are closer in the bottom, since one has $\|\frac{1}{2}H^{2j}\|<\|\frac{1}{2}H^j+\frac{1}{2}H^{n-j}\|$. The theorem implies that similar phenomena occur to the spaces $\mathrm{AI}(n)$ and $\mathrm{AII}(n)$ of the same root system as well.

② Consider the case $R(M)=\mathrm{BC}_r=\mathrm{BC}_r(a,b,c)$, where a denotes the multiplicity of the shortest roots α, b that of the roots of medium length and c that of 2α, the longest. The meridians have the root systems $\mathrm{C}_p(b,c)\times\mathrm{BC}_{r-p}(a,b,c)$, $1\leqq p\leqq r$, by 2.12.

2.17. REMARK. The theorem 2.15 gives not only the local structure of $M^-(p)\risingdotseq G^-/K^+$ (together with 2.29) but also that of $M^+(p)\risingdotseq K_{(1)}/K^+$ by 2.15a, where $\risingdotseq$ means local isomorphism, with the reservation that K^+ is not necessarily almost effective on $M^+(p)$.

2.17a. REMARK. A few more words to the theorem 2.15. In the first case, the circle T lies in the direction of H^j or $\frac{1}{2}H^j+\frac{1}{2}H^k$. In the second case, H^j is orthogonal to all the simple roots of $M^-(p)$ but the added vertex $-\alpha^\sim$.

2.17b. REMARK. It is not quite always true that the circle T in the direction of H^j or $\frac{1}{2}H^j+\frac{1}{2}H^k$ is actually one of the shortest that reaches the polar. It depends on the spaces in the local isomorphism class of M, as the previous example ① shows. In E_7 and EV, the circles in the directions of H^2 and H^7 reach the pole, but the latter is shorter (See the proof of 3.13). In E_6, the circles in the directions of H^1, H^6 and H^1+H^6 reach the polar EIII; the third one is shorter in E_6 and longer in E_6^* (See the proof of 3.17).

2.17c. REMARK. One can immediately read off the isotropy representation of the isotropy subgroup K on the tangent space T_oM of the space $M=G/K$ from the root systems $R(K)$ and $R(G)$ if the rank $r(K)=r(G)$. In fact, if one removes the simple root α_j out of $R(G)$ to obtain the Dynkin diagram of $R(K)$ as in the theorem, then one has only to express α_j as a linear combination of the fundamental weights of $R(K)$; in case $n^j=1$, H^j should be included as a fundamental weight.

2.18. REMARK. Whether or not $p=c(\pi)$ is fixed by s_o, one has $\alpha(H)=\pm 1$ or 0 for every root α of $R(M)$ if $K_{(1)}$ fixes p, or equivalently ad(exp(πH)) preserves $\mathfrak{k}$ and if there is no conjugate point on $(0, \pi)$ along c. Since every homotopy class of closed smooth curves in M contains a circle of the shortest length, one can conclude that the nonzero members of the fundamental group $\pi_1(M^*)$ are in a one-to-one correspondence with those $\frac{1}{2}H^j$ with $n^j=1$, as is well known. We tabulate $\pi_1(M^*)$ below for each root system $R(M)$; $\boldsymbol{Z}_k$ denotes a cyclic group of order k.

$R(M)$	A_r	B_r	C_r	D_r, r odd	D_r, r even	E_6	E_7
$\pi_1(M^*)$	$\boldsymbol{Z}_{r+1}$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_4$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$	$\boldsymbol{Z}_3$	$\boldsymbol{Z}_2$

$\pi_1(M^*)$ is $\{1\}$ if $R(M)$ is G_2, F_4, E_8 or BC_r by the above. One obtains more information about $\pi_1(M^*)$, through the knowledge of the length of the circle c, $c(\frac{1}{2}\pi H^j)=o$ for $M=M^*$. The next proposition 2.19 illustrates the point.

2.18a. REMARK. Let $M^-(p)$ be a meridian in M. Then every covering morphism $\pi\colon M\to M''$ restricts to a covering morphism of $M^-(p)$ onto a meridian $M''^-(p'')$ in M''. We assert that the kernel of the restriction $\pi|M^-(p)$ coincides with that of π; see 1.14 for the kernel. This follows from the fact that the inclusion: $M^-(p)\to M$ induces a surjection: $\pi_1(M^-(p))\to\pi_1(M)$, which is a consequence of 1.14a. In particular, M covers M'' as many times as $M^-(p)$ covers $M''^-(p'')$. The assertion is true for any subspace of equal rank, not just for $M^-(p)$.

2.19. PROPOSITION. *Let t^* be an automorphism of a compact connected simple bottom space M^* which fixes a point o^*. Then*

(i) *t^* acts on the fundamental group $\pi_1(M^*)$ as the identity or $s_1: x\mapsto x^{-1}$ except when $R(M^*)=D_r$ and r is even.*

(ii) *t^* lifts to an automorphism of any covering space M of M^* with the same possible exceptions.*

(iii) *Assume $R(M^*)=D_r$ and r is even. If M is* SO(2r) *or $G_r(\boldsymbol{R}^{2r})$ with $r\neq 4$, then t^* lifts to an automorphism of M. In case $r\neq 4$, an automorphism t^* of $M^*=$* SO(2r)* *or*

$G_r(\boldsymbol{R}^{2r})^*$ lifts to an automorphism of $M=\mathrm{SO}(2r)^\#$ (=semi-spinor group) or $G_r(\boldsymbol{R}^{2r})^\#$ respectively if and only if t^* is inner (which is equivalent to say that t^* is homotopic to the identity map in this case) or the composite of an inner automorphism and a symmetry at a point. In case $r=4$, an inner automorphism t^* of M^* lifts to an automorphism of M, but this is not necessarily true if t^* is outer; see the proof for some details.

PROOF. t^* lifts to an automorphism $t^\sim$ of the universal covering space $M^\sim$, so that we have the "exact" sequences of homomorphisms:

$$\{o^\sim\}\to\pi_1(M)\to M^\sim\to M\to\{o\} \quad\text{and}\quad \{o\}\to C(M)\to M\to M^*\to\{o^*\},$$

where the projections: $M^\sim\to M\to M^*$ carry $o^\sim\mapsto o\mapsto o^*$, $t^\sim(o^\sim)=o^\sim$ (See 1.14). $t^\sim$ preserves $C(M^\sim)=\pi_1(M^*)$. The action may be identified with the above mentioned action of t^* on $\pi_1(M^*)$, which is the question in (i). In (ii) and (iii), the question is whether or not $t^\sim$ preserves the subgroup $\pi_1(M)$. There is no problem if $t^\sim$ is inner, simply because $t^\sim$ then acts on $\pi_1(M^*)$ trivially. In view of the table in 2.18, both (i) and (ii) are true obviously (by the group theory) unless the root system $R(M)$ is A_r or D_r. In the case of A_r, $t^\sim$ acts on the cyclic group $C(M^\sim)=\pi_1(M^*)$ as an automorphism. We will prove that $t^\sim$ stabilizes every subgroup of $C(M^\sim)$ in this case, by showing that $t^\sim$ either fixes any given generator g or reverses g. We have $H^j=\omega_j$ and $n^j=1$ for every j, $1\leqq j\leqq r$. One easily sees that the geodesic in the direction of H^j reaches a member of $C(M^\sim)$, whose inverse lies in the direction of H^{n-j} and that the length satisfies $\|\omega_j\|^2=j(n-j)/n$, $n:=r+1$. Therefore $t^\sim$ carries g into g or g^{-1}; (i) and (ii) are proven for A_r. There is no problem in the case of D_r, r odd. So assume $R(M)\cong\mathrm{D}_r$, r even. Then $\pi_1(M^*)$ is generated by two poles δ and ε ($=-1$ in the Clifford algebra); so $\pi_1(M^*)=\{1, \delta, \varepsilon, \delta\varepsilon\}$. δ (resp. ε and $\delta\varepsilon$) lies in the direction of H^r (resp. H^1 and H^{r-1}), and the respective distance from $o^\sim$ is proportional to the lengths of those vectors; one has $\|H^1\|^2=1$ and $\|H^{r-1}\|^2=\|H^r\|^2=r/4$. Thus, if $r=4$, all the poles are of equal distance from $o^\sim$; in fact there is an automorphism T which permutes these three cyclically (See [B]), which naturally lifts to $M^\sim$ only. If $r\neq 4$, then one has $\|H^1\|<\|H^{r-1}\|=\|H^r\|$ and indeed there is a well known outer involution which exchanges $\delta\varepsilon$ and δ which therefore cannot lift to $\mathrm{SO}(2r)^\sim/\{1, \delta\}\cong\mathrm{SO}(2r)^\#:=\mathrm{SO}(2r)^\sim/\{1, \delta\varepsilon\}$ if M^* is $\mathrm{SO}(2r)^*$ or to $G_r(\boldsymbol{R}^{2r})^\sim/\{1, \delta\}\cong G_r(\boldsymbol{R}^{2r})^\sim/\{1, \delta\varepsilon\}$ if M^* is $G_r(\boldsymbol{R}^{2r})^*$. QED

2.20. PROPOSITION. *Assume that M is the bottom space M^*. Then the "unit lattice" $\{H\in\mathfrak{a} \mid \exp(2\pi H)(o)=o\in M^*\}$ is $\{H\in\mathfrak{a} \mid (\exp(2\pi H))^2=1\in G^*\}=\{\sum c_jH^j\in\mathfrak{a} \mid 2c_j\in\boldsymbol{Z}\}$ spanned by $\frac{1}{2}H^j$, $1\leqq j\leqq r=$ rank of M, over $\boldsymbol{Z}$.*

PROOF. (This is known [H].) Recall that a point $p=\exp(\pi H)(o)$ lies in a polar of o if and only if $\exp(2\pi H)(o)=o$ and that a bottom space M^* is expressed as $G^*/F(\sigma, G^*)$, $\sigma=\mathrm{ad}(s_o)$. Let $H\in\mathfrak{a}$. Then one has $\sigma(H)=-H$. Thus $\exp(2\pi H)(o)=o\Leftrightarrow\exp(-2\pi H)=\sigma\exp(2\pi H)=\exp(2\pi H)\Leftrightarrow(\exp(2\pi H))^2=1\Leftrightarrow\mathrm{ad}(\exp(4\pi H))=1\Leftrightarrow\alpha(4\pi H)\in 2\pi\boldsymbol{Z}$ for every root $\alpha\in R(M)\Leftrightarrow 2\alpha(H)\in\boldsymbol{Z}$ for every root $\alpha\in R(M)\Leftrightarrow 2c_j=2\alpha_j(H)\in\boldsymbol{Z}$ for every simple root α_j, where $H=\sum c_j H^j$. QED

2.21. COROLLARY. *Assume $M=M^*$. Then, referring to* 2.17b, *the direction $H=\frac{1}{2}H^j+\frac{1}{2}H^k$ cannot give one of the shortest curve to a polar.*

PROOF. By the proposition, the point $\exp(\pi(\frac{1}{2}H^j-\frac{1}{2}H^k))(o)$ coincides with $\exp(\pi(\frac{1}{2}H^j+\frac{1}{2}H^k))(o)$. But the vector $\frac{1}{2}H^j-\frac{1}{2}H^k$ is shorter than $\frac{1}{2}H^j+\frac{1}{2}H^k$, since the inner product $\langle H^j, H^k\rangle$ is positive. This proves the corollary if $j\neq k$. If $j=k$, then $\exp(\frac{1}{2}\pi H^j)(o)$ is a point in the polar, since $\exp(\pi(\frac{1}{2}H^j+\frac{1}{2}H^k))(o)$ is then o, again by the proposition. QED

2.22. REMARK. The fact 2.21 shows that $R(M)$ is read off from $R(M^-(p))$ if $M^-(p)$ is semisimple. One can derive other various results on (M^+, M^-). The next proposition is an example.

2.23. PROPOSITION. *Let M be a Kaehlerian (hermitian) symmetric space with $R(M)=\mathrm{C}_r$. Then* (i) *M has a connected centrosome $C(o, o')$.* (ii) *$C(o, o')$ is congruent with its c-orthogonal space $C^\perp$.* (iii) *Indeed one can rotate $C(o, o')$ onto $C^\perp$ at any point x of $C(o, o')$ with the complex structure $J_x\colon T_xM\to T_xM$ which extends to a global transformation of TM in the natural fashion. In particular $C(o, o')$ is totally real (and hence an R-space). And* (iv) *$C(o, o')$ is a dot product of a circle and a simple space of root system A_{r-1} with the equal multiplicity to the shorter ones in $R(M)$.*

PROOF. We will work on the bottom M^*, of which M is the double covering space. $C(o, o')$ will project onto a polar $M^{*+}(p^*)$ of the point o^*. We choose the meridian $M^{*-}(p^*)$ that corresponds to H^r (so the simple root α_r is the longer) by (2.15). $M^{*-}(p^*)$ is the dot product of the circle with a tangent H^r and a subspace as stated. Now the point symmetry at p^*, taken as t, gives the decompositions $\mathfrak{m}=\mathfrak{m}^++\mathfrak{m}^-$ and $\mathfrak{k}=\mathfrak{k}^++\mathfrak{k}^-$; $\mathfrak{m}^-$ gives the tangent space to $M^{*-}(p^*)$ at o^*. The longer roots $2\varepsilon_j$, $1\leq j\leq r$, belong to $\mathfrak{m}^+$ and make a system of strongly orthogonal roots. Recall (or see 2.26 and 2.27) that they have multiplicity 1. By using a Chevalley system (Cf. Proposition 7 about it in [B]), one can find a vector $I\in\sum\mathfrak{m}^+(2\varepsilon_j)\subset\mathfrak{m}^+$ such that $\{H^r, I\}$ generates a 3-dimensional subalgebra. And $J:=[H^r, I]\subset\mathfrak{k}^-$ generates a 1-parameter group whose orbit through H^r contains I. Hence, making it act on M^*, one sees its orbit through p^* contains a pole q^* of o^* in M^{*t}. This means that M^{*t} is a meridian $M^{*-}(q^*)$ to the polar $M^{*+}(p^*)$. QED

2.23a. REMARK. The proposition applies to $M=G_2^0(R^{4+m})$, $m>0$, $\mathrm{CI}(r)$, $G_r(C^{2r})$, $\mathrm{DIII}(2r)$ and EVII. A similar proposition obtains for the quaternion-Kaehlerian spaces in which the so-to-speak "totally complex" meridians $S^2\cdot B$ are congruent with the polars; details will appear elsewhere.

2.23b. REMARK. A meridian is a maximal subspace, generally. More precisely, let M be a simple space of rank >1. Then a proper connected subspace of M coincides with a meridian M^- if it contains M^-. This should be more or less obvious from various known facts such as 2.17c together with (ii) of 2.15. Namely, if any one root α of $R(M)$ (outside $R(M^-)$) is added to $R(M^-)$, then $R(M)$ will be the only root system that contains these, $\{\alpha\} \coprod R(M^-)$, by (2.3). (This argument at once shows that the isotropy representation of K for a simple space $M=G/K$ is simple with $r(K)=r(G)$ and hence its Lie algebra $\mathscr{L}K$ is a maximal subalgebra of $\mathscr{L}G$.)

2.24. LEMMA. *Let q be a pole of o in $M^-(p)$, the meridian to a polar $M^+(p)$ of o at p. Then there are at most three cases: $1°$ q is a pole of o in M; $2°$ $q=p$; or $3°$ q is a pole of p in M. In the case $3°$, the polar $M^+(q)$ is congruent with $M^+(p)$ by G.*

PROOF. Assume $1°$ is not the case. Then $M^-(q)$ is not the whole space M. The quadratic transform $Q(q)$ is the identity on $M^-(p)$ but not on the whole space M. We now assume that M is simple and has the rank $r(M) > 1$, in order to use 2.23b. Since $M^-(p)$ is then a maximal connected subspace of M, it follows that $M^-(p)$ is $F(Q(q), M)_{(o)}$. But one has $M^-(p)=F(Q(p), M)_{(o)}$ in general. Hence one obtains $Q(q)=Q(p)$; that is, $s_q=s_p$. Therefore q is in the case $2°$ or $3°$. If $r(M)=1$, then $M^-(p)$ is a sphere (of dimension 1, 2, 4 or 8), and one has $q=p$ certainly. In case M is not simple, M is the local product (i.e. a finite covering is a product) of simple spaces (and a torus). $M^-(p)$ is then a maximal connected subspace of that, M', of some of them (and the intersection with the torus). A pole in M remains to be one in M' and vice versa. The congruence is easy to see from the fact that $M^+(p)$ is carried onto the c-orthogonal space to $M^-(p)$ by the point symmetry s_m at the midpoint m of a geodesic arc joining o to p within $M^-(p)$, since the composite of two point symmetries is a member of G. QED

In the rest of the section, we will scrutinize the root system of a compact symmetric space of rank $\leqq 2$ (in continuation of Lemma 2.4b) to study its structure. As a result, a local classification of the symmetric spaces would come out easily (Remark 2.27a). Also the proof of Theorem 1.15 on (M^+, M^-) determining M will be completed (Lemma 2.28). As a consequence, it will be shown that a space M is locally determined by its root system $R(M)$ with multiplicity (2.29). To illustrate the use of the pair (M^+, M^-), we will show that M is hermitian $\Leftrightarrow$ M^+ and M^- are hermitian for some pair (M^+, M^-) $\Leftrightarrow$ this is true for every pair (M^+, M^-) (Proposition 2.30).

We introduce a few symbols. As before, $\mathfrak{k}_0(\alpha)$ denotes the subalgebra generated by $[\mathfrak{m}(\alpha), \mathfrak{m}(\alpha)]$ for a root $\alpha \in R(M)$. $\mathfrak{k}M(\alpha):=\mathfrak{k}_0(\alpha)+\mathfrak{k}(\alpha)$ and $m(\alpha):=\dim \mathfrak{k}(\alpha)=\dim \mathfrak{m}(\alpha)$, the multiplicity of α, $\mathfrak{g}_y(\alpha)$ is the subalgebra generated by α and a nonzero member y of $\mathfrak{g}(\alpha)$; $\mathfrak{g}_y(\alpha) \cong \mathscr{L}\mathrm{O}(3)$. $\mathfrak{g}(\beta++\alpha)$ denotes the linear subspace $\sum_j \mathfrak{g}(\beta+j\alpha)$ corresponding to the α-series of the root β (summed up for all the linear subspaces of this type); similarly for $\mathfrak{m}(\beta++\alpha)$. As usual, $n(\beta, \alpha):=\langle\beta, \alpha^\vee\rangle:=2\|\alpha\|^{-2}\langle\beta, \alpha\rangle$.

2.25 LEMMA. *Assume α is a root but not 2α. Then*

(a) *$\mathfrak{k}_0(\alpha)$ is isomorphic with $\mathscr{L}\mathrm{O}(m)$, $m=\dim \mathfrak{m}(\alpha)$. Moreover $\mathfrak{k}_0(\alpha)$ acts on $\mathfrak{m}(\alpha)$*

and on $\mathfrak{k}(\alpha)$ *as* $\mathscr{L}\mathrm{O}(m)$ *through the standard representation* ω_1 *of* $\mathrm{O}(m)$; $\mathfrak{k}_0(\alpha)+\mathfrak{k}(\alpha)$ *acts on* $\mathfrak{m}(\alpha)+\boldsymbol{R}\alpha$ *as* $\mathscr{L}\mathrm{O}(m+1)$ *similarly. Also one has* $\mathfrak{k}_0(\alpha)=[\mathfrak{k}(\alpha), \mathfrak{k}(\alpha)]=[\mathfrak{m}(\alpha), \mathfrak{m}(\alpha)]$.

(b) $\mathfrak{k}_0(\alpha)$ *is an ideal in* $\mathfrak{k}(0)$. *Hence* $\mathfrak{k}_0(\alpha)$ *stabilizes* $\mathfrak{m}(\beta)$ *for every root* β.

(c) $\mathfrak{g}_y(\alpha)$ *stabilizes* $\mathfrak{g}(\beta++\alpha)$. *If* $n(\beta, \alpha)=1$, *then* $\mathfrak{g}(\beta++\alpha)$ *is* $\mathfrak{g}(\beta)+\mathfrak{g}(\beta-\alpha)$ *and is a direct sum of simple* $\mathfrak{g}_y(\alpha)$*-modules* (*of dimension* 4) *on which* $\mathfrak{g}_y(\alpha)$ *acts through the spin representation of the highest weight* ω_1. *Thus they are direct sums of simple modules over every line in* $\mathfrak{g}_y(\alpha)$ *and, in particular, they meet every linear subspace* $\mathfrak{g}(\beta+j\alpha)\neq\{0\}$ *in a subspace of dimension* 2.

(d) *The subgroup* $G_y(\alpha)$ *with Lie algebra* $\mathfrak{g}_y(\alpha)$ *is isomorphic with* $\mathrm{SU}(2)$ (*not with* $\mathrm{SO}(3)$), *if the integer* $n(\beta, \alpha)$ *is odd for some root* β.

(e) *If* $n(\beta, \alpha)=1$ *and the multiplicity* $m(\alpha)>1$, *then* $\mathfrak{m}(\beta)$ *is a direct sum of simple* (*and nontrivial*) $\mathfrak{k}_0(\alpha)$*-modules; thus* $[\mathfrak{k}_0(\alpha), \mathfrak{m}(\beta)]=\mathfrak{m}(\beta)$. $\mathfrak{k}_0(\alpha)$ *acts on them through the spin or a half-spin representation; in case* $m(\alpha)=2$, *this simply means that* $\mathfrak{m}(\beta)$ *is a direct sum of isomorphic* 2*-dimensional* $\mathfrak{k}_0(\alpha)$*-modules, giving rise to a* $\mathfrak{k}_0(\beta)$*-invariant complex structure on* $\mathfrak{m}(\beta)$. *In case* $m(\alpha)=4$, $\mathfrak{k}_0(\alpha)$ *may act through the spin representation of* $\mathscr{L}\mathrm{O}(3)$, *the other ideal of* $\mathfrak{k}_0(\alpha)$ *acting trivially.* (*In particular,* $m(\beta)$ *is even.*)

PROOF. The linear subspace $[\mathfrak{m}(\alpha), \mathfrak{m}(\alpha)]\subset\mathfrak{k}(0)=\mathfrak{k}(0)+\mathfrak{k}(2\alpha)$ normalizes itself and hence it equals $\mathfrak{k}_0(\alpha)$. Since $\mathrm{ad}(\alpha)$ gives a bijection of $\mathfrak{m}(\alpha)$ onto $\mathfrak{k}(\alpha)$, the equality $[\mathfrak{k}(\alpha), \mathfrak{k}(\alpha)]=[\mathfrak{m}(\alpha), \mathfrak{m}(\alpha)]$ follows from a stronger fact

$$[x, x']=[y, y']$$

where $[\alpha, x]=\|\alpha\|^2 y\in\mathfrak{k}(\alpha)$ and $[\alpha, x']=\|\alpha\|^2 y'\in\mathfrak{k}(\alpha)$. To prove this, observe $[\alpha, [y', [\alpha, y]]]=[[\alpha, y'], [\alpha, y]]+[y', [\alpha, [\alpha, y]]]=[[\alpha, y'], [\alpha, y]]-\|\alpha\|^4[y', y]$, in which the two terms in the right hand side are alternating in y', y while the left hand side is symmetric in y', y by $[y', [\alpha, y]]=[\alpha, [y', y]]-[[\alpha, y'], y]=[y, [\alpha, y']]$ because $[y', y]\in\mathfrak{k}(0)$; hence $[[\alpha, y'], [\alpha, y]]=\|\alpha\|^4[y', y]$ as asserted. The rest of (a) follows from 2.4b or its proof: the $\mathfrak{k}_0(\alpha)$-module $\mathfrak{m}(\alpha)$ is isomorphic with $\mathfrak{k}(\alpha)=[\alpha, \mathfrak{m}(\alpha)]$ by $[\mathfrak{k}_0(\alpha), \alpha]\subset[\mathfrak{k}(0), \alpha]=\{0\}$. One has (b) by (a), since $\mathfrak{k}(0)\supset\mathfrak{k}_0(\alpha)$ normalizes every $\mathfrak{k}(\beta)$ and $\mathfrak{m}(\beta)$. We turn to (c). Clearly $\mathfrak{g}_y(\alpha)$ stabilizes $\mathfrak{g}(\beta++\alpha)$ by $[\mathfrak{g}(\alpha), \mathfrak{g}(\beta)]\subset\mathfrak{g}(\beta+\alpha)+\mathfrak{g}(\beta-\alpha)$ and $[\alpha, \mathfrak{m}(\beta)+\mathfrak{k}(\beta)]\subset\mathfrak{k}(\beta)+\mathfrak{m}(\beta)$. Thus, if $n(\beta, \alpha)$ is 1, one sees $\mathfrak{g}(\beta++\alpha)=\mathfrak{g}(\beta)+\mathfrak{g}(\beta-\alpha)$, another well known fact about the roots [B]. If the inner product $\langle\alpha, \beta+j\alpha\rangle$ is not zero, then $\mathrm{ad}(\alpha)$ carries $\mathfrak{m}(\beta+j\alpha)$ onto $\mathfrak{k}(\beta+j\alpha)$ bijectively. Hence $\mathfrak{g}_y(\alpha)$ acts effectively on each simple $\mathfrak{g}_y(\alpha)$-submodule of $\mathfrak{g}(\beta++\alpha)$ through the weight ω_1; the submodules thus have dimension 4. Since the lines in $\mathfrak{g}_y(\alpha)$ are conjugate to each other, their actions are known from that of $\boldsymbol{R}\alpha$. If $n(\beta, \alpha)$ is odd, $\mathfrak{g}_y(\alpha)$ acts similarly through the highest weight $n(\beta, \alpha)\omega_1$, and one obtains (d). For (e), one first notes that any two orthonormal vectors y_1, y_2 in $\mathfrak{k}(\alpha)$ generate a subalgebra $\mathfrak{k}_{12}$ which is isomorphic with $\mathscr{L}\mathrm{O}(3)$; in fact one knows $[y_1, y_2]\neq 0$ by 2.4b and that $\mathrm{ad}([y_1, y_2])$ exchanges the two lines spanned by y_1, y_2. $\mathfrak{k}_{12}$ stabilizes $\mathfrak{m}(\beta)+\mathfrak{m}(\beta-\alpha)$, which does not contain a trivial $\mathfrak{k}_{12}$-submodule by (c). One also notes $m(\beta)\geqq m(\alpha)>1$ by 2.4c.

$\mathfrak{k}_{12}$ acts on each $\mathfrak{k}_{12}$-submodule through the weight ω_1 again by (c); $\mathrm{ad}(y_j): \mathfrak{m}(\beta)\to \mathfrak{m}(\beta-\alpha)$ is bijective. Therefore $\mathrm{ad}([y_1, y_2])$: $\mathfrak{m}(\beta)\rightleftarrows\mathfrak{m}(\beta)$ is bijective, which immediately implies that $\mathfrak{k}_0(\alpha)$ acts on each simple submodule of $\mathfrak{m}(\beta)$ through the spin or a half-spin representation; it may help to note that $([y_{2j-1}, y_{2j}])_j$ makes a basis for a maximal abelian subalgebra of $\mathfrak{k}_0(\alpha)$ for an orthogonal basis $(y_j)_j$ for $\mathfrak{k}(\alpha)$. In case $m(\alpha)=1$, $\mathfrak{m}(\beta)$ is a direct sum of a simple Spin(2)-submodules of dimension 2. In case $m(\alpha)=4$, a 3-dimensional ideal in $\mathfrak{k}_0(\alpha)$ may act trivially on $\mathfrak{m}(\beta)$, as one will see later. QED

2.25a. LEMMA. *Let L be a subset of the root system $R(M)$. Then the subspace $\mathfrak{m}'(L)=\sum_L\mathfrak{m}(\lambda)$, summed up for λ in L, is the tangent space $T_oM'(L)$ to a subspace $M'(L)$ of M if and only if* (1) *the set $R(L)=\{\lambda\pm\mu \mid \lambda, \mu\in L\}\cap R(M)$ is a root subsystem of $R(M)$ or empty,* (2) *L is a weight system for $R(L)$, and* (3) *$L\cap R(L)=\varnothing$.*

PROOF. Assume $\mathfrak{m}'(L)$ is tangent to a subspace $M'(L)$. Then, as in the proof of 2.4b, one has the symmetry decomposition $\mathfrak{k}'+\mathfrak{m}'$ for $M'(L)$, where $\mathfrak{m}'=\mathfrak{m}'(L)$ and $\mathfrak{k}'=[\mathfrak{m}', \mathfrak{m}']$ (1.6a). Hence $\mathfrak{k}'\subset\sum_L\mathfrak{k}_o(\lambda)+\sum_{R(L)}\mathfrak{k}(\alpha)$, summed up for λ in L and α in $R(L)$, where $\mathfrak{k}_0(\lambda):=[\mathfrak{m}(\lambda), \mathfrak{m}(\lambda)]$. Here the equality obtains; in fact, if $n(\mu, \lambda)=1$ as we may assume, then $\mathfrak{k}(\lambda-\mu)\subset\mathfrak{k}'$ by 2.25(c). Trivially $n(\beta, \alpha)$ is an integer for every pair of members of $R(L)$. $R(L)$ is left invariant by the reflections in its members by the proof of 2.4b (vi). Therefore $R(L)$ is a root system in the linear span $\mathfrak{a}_L$, unless $R(L)$ is empty (in which case the members of L are strongly orthogonal to each other). Since L is a subset of $R(M)$ and one has $[\mathfrak{k}', \mathfrak{m}']\subset\mathfrak{m}'$, L is a weight system for R_L in the sense of 2.3; the Weyl group there means that of R_L in the present context. If $L\cap R(L)$ contains a member α, then $\mathfrak{m}'$ must contain $[\mathfrak{m}(\alpha), \mathfrak{k}(\alpha)]=\boldsymbol{R}\alpha$, contrary to the assumption. Conversely, assume (1) through (3). Then $S:=R(L)\cup L$ is a root subsystem of $R(M)$ too. By 2.4d, there is a subspace $M(S)$, which admits an involution τ such that $M'(L)=F(\tau, M(S))_{(o)}$; $\mathrm{ad}(\tau\circ s_o)$ induces the identity on $\mathfrak{k}'$ and on $\mathfrak{a}_L+\sum_{R(L)}\mathfrak{m}(\alpha)$, summed up for α in $R(L)$. QED

2.26. LEMMA. *Assume α and 2α are roots $\in R(M)$, M simple. Let $M(\alpha, 2\alpha)$ denote the subspace $M(S)$ for the root subsystem $S=\{\alpha, 2\alpha\}$ (See* 2.4d). *Then* (i) *$M(\alpha, 2\alpha)$ has rank* 1, *and the ratio of the maximum of its sectional curvature to the minimum is* 4 : 1. (ii) *$M(\alpha, 2\alpha)$ is one of the complex, quaternion and Cayley projective spaces, denoted by $G_1(C^{2+k})$, $G_1(H^{2+k})$ and* FII *respectively. Its root system is* $\mathrm{BC}_1(2k, 1)$, $\mathrm{BC}_1(4k, 3)$ *or* $\mathrm{BC}_1(8, 7)$ *with multiplicity, the first number in each being that of the shorter ones; the space has dimension* $2k+2$, $4k+4$ *or* 16 *accordingly.* (iii) *$M(\alpha, 2\alpha)$ is* 1*-connected.*

PROOF. The stated ratio is obtained with the formula in the proof of 2.4b. We now assume $M=M(\alpha, 2\alpha)$ for simplicity and will prove (ii). Let $M'(\alpha)$ be the subspace with $\mathfrak{m}(\alpha)=T_oM'(\alpha)$; $M'(\alpha)=M'$ for $L=\{\alpha\}$ in 2.25a. $M'(\alpha)$ is c-orthogonal to the meridian $M(2\alpha)$ (See 2.15; $M'(\alpha)$ is congruent with the polar), which is in fact a sphere of dimension >1 by 2.4b (iii). The space M of rank 1 is thus 1-connected. The subalgebra

$\mathfrak{k}(0)+\mathfrak{k}(2\alpha)$ of $\mathfrak{k}$ is $\mathfrak{k}^+=F(\operatorname{ad} Q(p), \mathfrak{k})$ and hence it acts irreducibly on $T_oM'(\alpha)$ and $T_oM(2\alpha)$ and the isotropy subalgebras; the actions are irreducible because $M'(\alpha)$ and $M(2\alpha)$ have rank 1. We want to rotate M around o with a member b of $K_{(1)}$ so that b carries $M(2\alpha)$ into $M'(\alpha)$. Let p be the pole of o in $M(2\alpha)$. Choose b of $K_{(1)}$ which carries p into a point q in $M'(\alpha)$; q lies necessarily on the polar $M^+(p)$, the only polar of o in $M'(\alpha)$. Then $b(M(2\alpha))$ is entirely contained in $M'(\alpha)$, since q is the pole of o in the sphere $b(M(2\alpha))$. Therefore $M'(\alpha)$ has the same root system $\mathrm{BC}_1=\mathrm{BC}_1(a, s-1)$ as M with the multiplicity a of α less than that, $m(\alpha)$, of the root α of M by $s:=\dim M(2\alpha)=m(2\alpha)+1$; and $a=m(\alpha)-s$. Repeating rotations, one sees that $\dim M'(\alpha)=m(\alpha)$ is an integral multiple, js, of s. One has $n(\alpha, 2\alpha)=1$; hence $\dim \mathfrak{m}(\alpha)$ is even by 2.25 (c) and so is s. Also one observes that $\mathfrak{k}(2\alpha)$ normalizes $\mathfrak{g}(\alpha)$.

We begin with the case $s=2$. By 2.25 (c), $\mathfrak{k}(2\alpha)\cong\mathscr{L}\mathrm{U}(1)$ defines a complex structure on $\mathfrak{m}(\alpha)$ as well as $\mathfrak{m}$ which is invariant under the action of $\mathfrak{k}=\mathfrak{k}_0(\alpha)+\mathfrak{k}(\alpha)+\mathfrak{k}(2\alpha)$. One easily sees that M with rank 1 is a complex projective space $G_1(\boldsymbol{C}^{2+j})$, by showing that the holomorphic sectional curvature of M is constant (which boils down to the fact that any $\mathfrak{k}(2\alpha)$-invariant 2-plane in $\mathfrak{m}$ is tangent to some Riemann sphere $b(M(2\alpha))$) or showing that $\mathfrak{g}\cong\mathscr{L}\mathrm{SU}(j+2)$ and $\mathfrak{k}\cong\mathscr{L}\mathrm{U}(j+1)$ as in the next case.

In case $s=4$, we induct on j to show that M is a quaternion projective space $G_1(\boldsymbol{H}^{2+j})$. $\mathfrak{k}_0(2\alpha)\cong\mathscr{L}\mathrm{Sp}(1)$ gives the invariant $\boldsymbol{H}$-structure on $\mathfrak{m}(\alpha)$ by 2.25 (e) and one on $\mathfrak{m}(2\alpha)$, since $\mathfrak{k}_0(2\alpha)$ is an ideal in $\mathfrak{k}M(2\alpha):=\mathfrak{k}_0(2\alpha)+\mathfrak{k}(2\alpha)\cong\mathscr{L}\mathrm{O}(4)$ (See 2.4b). Assume $M'(\alpha)$ is $G_1(\boldsymbol{H}^{1+j})$; one has $\mathfrak{k}_0(\alpha)\cong\mathscr{L}\mathrm{Sp}(j)\cdot\mathrm{Sp}(1)$ and $\mathfrak{k}_0(\alpha)\cap\mathfrak{k}(0)\cong\mathscr{L}\mathrm{Sp}(j)$. The rotation $\operatorname{ad}\exp(t\alpha)$ acting on $\mathfrak{g}$ exchanges $\mathfrak{m}(\alpha)$ and $\mathfrak{k}(\alpha)$ and stabilizes $\mathfrak{k}(2\alpha)+\mathfrak{k}(0)$ for some $t\in\boldsymbol{R}$, since its angular velocity on $\mathfrak{m}(2\alpha)+\mathfrak{k}(2\alpha)$ is twice as much as the one on $\mathfrak{m}(\alpha)+\mathfrak{k}(\alpha)$. Hence one has $\mathfrak{k}\cong\mathscr{L}\mathrm{Sp}(j+1)\cdot\mathrm{Sp}(1)$. To conclude $M\cong G_1(\boldsymbol{H}^{2+j})$ or $\mathfrak{g}\cong\mathscr{L}\mathrm{Sp}(j+2)$, one has to verify certain uniqueness or that the curvature of M or the Lie algebra structure of $\mathfrak{g}$ is determined by the root system with multiplicity (and the induction assumption); more specifically one has to determine the bracket product $[\ ,\ ]: \mathfrak{m}\wedge\mathfrak{m}\to\mathfrak{k}$. The $\mathfrak{k}M(2\alpha)$-module $\mathfrak{m}(\alpha)$ is a (not unique) direct sum $\sum_P \mathfrak{m}_P$ of simple submodules $\mathfrak{m}_P$, $\mathfrak{m}_P\perp\mathfrak{m}_Q$ for $P\neq Q$, $\dim\mathfrak{m}_P=4$. Now the restriction $[\ ,\ ]: \mathfrak{m}_P\wedge\mathfrak{m}_Q\to\mathfrak{k}$ is known for $P\neq Q$, since $[\mathfrak{m}_P, \mathfrak{m}_Q]$ is contained in $\mathfrak{k}M'(\alpha)\cong\mathscr{L}\mathrm{Sp}(j)\cdot\mathrm{Sp}(1)$ by $\langle\mathfrak{k}M(2\alpha), [\mathfrak{m}_P, \mathfrak{m}_Q]\rangle=\langle[\mathfrak{k}M(2\alpha), \mathfrak{m}_P], \mathfrak{m}_Q\rangle=\langle\mathfrak{m}_P, \mathfrak{m}_Q\rangle=0$. For the remaining case of $P=Q$, one observes that $b(M(2\alpha))=\mathfrak{m}_P$ and $b(\mathfrak{m}_P)=M(2\alpha)$ for some $b\in K$; $b:=\operatorname{ad}(\exp(t[\alpha, x]))$ will do for any nonzero member x of $\mathfrak{m}_P$ and some real number t. Thus one concludes $\mathfrak{g}\cong\mathscr{L}\mathrm{Sp}(j+2)$ and M is $G_1(\boldsymbol{H}^{2+j})$.

Finally, assume $s>4$. Since $\mathfrak{k}_0(2\alpha)$ acts on every simple $\mathfrak{k}M(2\alpha)$-submodule $\mathfrak{m}_P$ of $\mathfrak{m}$ through the spin representation by 2.25 (e), one has $s=8$; in fact, writing $2r+1$ for $s-1=\dim\mathfrak{m}(2\alpha)$, one knows $2^r=s=\dim\mathfrak{m}_P$ for the spin representation and one finds $s=8$ as the only possibility. Furthermore one will see $j=1$ at the end. If $j=1$, $\mathfrak{k}M(2\alpha)\cong\mathscr{L}\mathrm{O}(8)$ acts on $\mathfrak{m}(\alpha)$ through a half-spin representation (which is the composite of the standard representation with an outer automorphism T of $\mathscr{L}\mathrm{O}(8)$, $\mathrm{T}^3=1$). Hence the ideal $\mathfrak{k}_0(2\alpha)\cong\mathscr{L}\mathrm{O}(7)$ in $\mathfrak{k}(0)$ must coincide with $\mathfrak{k}(0)$; that is, $\mathfrak{k}M(2\alpha)=\mathfrak{k}^+$. Thus one has $\mathfrak{k}=\mathfrak{k}^{+}+\mathfrak{k}(\alpha)\cong\mathscr{L}\mathrm{O}(9)$, which acts on $\mathfrak{m}$ through the spin representation just as $\mathfrak{k}_0(2\alpha)$ acts on $\mathfrak{m}(\alpha)$ through it. Hence $\mathfrak{g}$ is F_4 and M is FII; in fact one has the rank $r(\mathfrak{g})=r(\mathfrak{a})+r(\mathfrak{k}(0))=r(\mathfrak{a})+r(\mathfrak{k}_0(2\alpha))=1+3=r(\mathfrak{k})$ and the roots of $\mathfrak{g}$ are the union of those of $\mathfrak{k}$ and the weights of its spin representation. Finally, suppose $j>1$. Then $\mathfrak{k}(0)$ contains a subalgebra $\mathrm{B}_4\cong\mathscr{L}\mathrm{O}(9)$, which acts on $\mathfrak{m}_P+\mathfrak{m}_Q$, $P\neq Q$, irreducibly through the spin representation and normalizes $\mathfrak{k}_0(2\alpha)$ by 2.25 (b). Since B_4 acts on the 8-dimensional space $T_oM(2\alpha)$ trivially, B_4 and $\mathfrak{k}_0(2\alpha)$ centralize each other; $\mathrm{B}_4\times\mathfrak{k}_0(2\alpha)$ is a subalgebra of $\mathfrak{k}(0)$ which acts on $\mathfrak{m}_P+\mathfrak{m}_Q$ effectively. But this is impossible obviously. QED

2.26a. COROLLARY. *Assume both α and 2α are roots. Then the actions of $\mathfrak{k}_0(\alpha)$ and $\mathfrak{k}(0)$ on $\mathfrak{m}(\alpha)$ are irreducible and depend on the multiplicities $m(\alpha)$ and $m(2\alpha)$ only. More precisely, in case $s:=m(2\alpha)+1=2$, they act on $\mathfrak{m}(\alpha)$ as $\mathscr{L}\mathrm{U}(j)$, where $js:=m(\alpha)$. In case $s=4$, $\mathfrak{k}_0(\alpha)$ and $\mathfrak{k}(0)$ act on $\mathfrak{m}(\alpha)$ as $\mathscr{L}\mathrm{Sp}(1)\times\mathrm{Sp}(j)$, while $\mathfrak{k}_0(\alpha)$ acts on $\mathfrak{k}(\alpha)$ as $\mathscr{L}\mathrm{Sp}(j)$; $[\mathfrak{m}(\alpha),\mathfrak{m}(\alpha)]\neq[\mathfrak{k}(\alpha),\mathfrak{k}(\alpha)]$. In case $s=8$, $\mathfrak{k}_0(\alpha)$ acts on $\mathfrak{m}(\alpha)$ as $\mathscr{L}\mathrm{O}(8)$ and $\mathfrak{k}(0)$ does as $\mathscr{L}\mathrm{O}(7)$ through the spin representation; and $\mathfrak{k}_0(\alpha)=\mathfrak{k}(0)+\mathfrak{k}(2\alpha)$.*

2.26b. COROLLARY. *There exist monomorphisms $G_1(\boldsymbol{R}^{2+k})\to G_1(\boldsymbol{C}^{2+k})\to G_1(\boldsymbol{H}^{2+k})$ and $G_1(\boldsymbol{H}^3)\to\mathrm{FII}\leftarrow S^8$, which are the inclusion maps of the fixed point sets of involutions, which correspond to the extensions of the coefficient rings and which induce monomorphisms $\mathrm{A}_1(k)\to\mathrm{BC}_1(2k,1)\to\mathrm{BC}_1(4k,3)$ and $\mathrm{BC}_1(4,3)\to\mathrm{BC}_1(8,7)\leftarrow\mathrm{A}_1(7)$ of their root systems. There is another family of monomorphisms $G_1(F^{2+m})\to G_1(F^{2+k})$, $F=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$, $0\leqq m<k$, and $\mathrm{FII}\leftarrow S^8$ inducing $\mathrm{BC}_1(8,7)\leftarrow\mathrm{A}_1(7)$.*

2.27. LEMMA. *Let α and β be linearly independent roots $\in R(M)$. Let $R(\alpha,\beta)$ and $M(\alpha,\beta)$ denote $R(S)$ and $M(S)$ for $S=\{\alpha,\beta\}$ (See 2.4d). Then* (i) *$R(\alpha,\beta)$ is one of A_2, $B_2=\mathrm{C}_2$, G_2 and BC_2 unless $M(\alpha,\beta)$ is a local product $M''(\alpha)\times M''(\beta)$, where $M''(\gamma)$ denotes $M(S)$ for $S=\{\gamma\}$, $\{\gamma,2\gamma\}$ or $\{\frac{1}{2}\gamma,\gamma\}$.* (ii) *If the multiplicity is included, the root systems are $\mathrm{A}_2(m)$ for $m=1,2,4$ or 8, $\mathrm{B}_2(m,1)$ for $m\geqq 1$, $\mathrm{B}_2(2,2)$, $\mathrm{B}_2(4,3)$, $\mathrm{BC}_2(2k,2,1)$, $\mathrm{BC}_2(4,4,1)$, $\mathrm{BC}_2(8,6,1)$, $\mathrm{BC}_2(4k,4,3)$, $\mathrm{G}_2(1,1)$ and $\mathrm{G}_2(2,2)$.* (iii) *If $M(\alpha,\beta)$ is 1-connected, $M(\alpha,\beta)$ is uniquely determined by the root system with multiplicity (as made explicit in the proof).*

PROOF. The subspace $M(\alpha,\beta)$ has rank 2. If α and β are strongly orthogonal ($\Leftrightarrow[\mathfrak{m}(\alpha),\mathfrak{m}(\beta)]=\{0\}$), then $R(\alpha,\beta)$ is $\{\pm\alpha,\pm\beta\}$ possibly with $\pm 2\alpha$ (or $\pm\frac{1}{2}\alpha$) and/or $\pm 2\beta$ (or $\pm\frac{1}{2}\beta$) added, hence $M(\alpha,\beta)$ is a local product of $M''(\alpha)$ and $M''(\beta)$. If not, $M(\alpha,\beta)$ is simple and hence $R(\alpha,\beta)$ is one of 1° A_2, 2° $\mathrm{B}_2=\mathrm{C}_2$, 3° BC_2 and 4° G_2. Again we assume $M=M(\alpha,\beta)$ for simplicity.

Consider the case 1°: $R(\alpha,\beta)=\mathrm{A}_2$. One may assume $\gamma:=\alpha+\beta$ is another root. There is a subspace $M'(\alpha,\beta)$ with $T_oM'(\alpha,\beta)=\mathfrak{m}(\alpha)+\mathfrak{m}(\beta)$ by 2.25a. The subspace $M'(\alpha,\beta)$ is c-orthogonal at o to the meridian (2.15) $T\cdot M(\gamma)$ with $T_o(T\cdot M(\gamma))=\mathfrak{a}+\mathfrak{m}(\gamma)$; hence $M'(\alpha,\beta)$ is congruent with the polar and has the symmetry decomposition $\mathfrak{k}=\mathfrak{k}^{+}+\mathfrak{k}^{-}$ at p. $M'(\alpha,\beta)$ has rank$=1$ by 2.25 (c). Therefore the root system $R(M''(\alpha,\beta))$ is $\mathrm{BC}_1=\mathrm{BC}_1(m,m-1)$ unless $m:=m(\alpha)=1$. If $m>1$, one has $m=2,4$ or 8 by 2.26. We now prove (iii) in this case by using the symmetry decompositions $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$, $\mathfrak{k}=\mathfrak{k}^++\mathfrak{k}^-$ and $\mathfrak{k}^+=\mathfrak{k}(0)+\mathfrak{k}(\gamma)$ (modulo the centralizer $C(\mathfrak{k}(\gamma), \mathfrak{k}(0))$) in the reversed order. Thus, if $m=1$, one sees $M=\mathrm{AI}(3)$, since $\mathfrak{k}^+=\mathfrak{k}(\gamma)\cong\mathscr{L}\mathrm{O}(2)$, $\mathfrak{k}\cong\mathfrak{k}^+ + T_oM'(\alpha, \beta)\cong\mathscr{L}\mathrm{O}(2)+T_oS^2\cong\mathscr{L}\mathrm{O}(3)$ and one has its action $4\omega_1$ on $\mathfrak{m}\cong\mathfrak{m}(\mathrm{AI}(3))$, where $\mathfrak{m}(\mathrm{AI}(m))$ is, as one recalls, the space of the symmetric bilinear forms of trace 0 on $\boldsymbol{R}^m$, to see that $\mathfrak{g}$ is $\mathscr{L}\mathrm{SU}(3)$ having the root spaces $\mathfrak{g}(\alpha)$, etc. for $\mathrm{A}_2(2)$ with respect to the same $\mathfrak{a}$. If $m=2$, one has $M=\mathrm{SU}(3)$. If $m=4$, one has $M=\mathrm{AII}(3)$; $\mathfrak{k}$ is $\mathscr{L}\mathrm{Sp}(3)$, since $M'(\alpha, \beta)$ with $\mathrm{BC}_1(4, 3)$ is congruent with the polar $G_1(\boldsymbol{H}^3)$. One finds $\mathfrak{g}\cong\mathscr{L}\mathrm{SU}(6)$ from this and the action of $\mathfrak{m}$ on itself: $\mathfrak{m}\wedge\mathfrak{m}\to\mathfrak{k}$ which is determined by 2.25 and 2.4b. Let us add that $\mathfrak{k}(0)=\mathfrak{k}_0(\alpha)+\mathfrak{k}_0(\beta)+\mathfrak{k}_0(\gamma)$ is isomorphic with $A\times B\times C$, where A, B and C are all $\cong\mathscr{L}\mathrm{Sp}(1)$, $B\times C$ acts on $\mathfrak{m}(\alpha)$, $C\times A$ on $\mathfrak{m}(\beta)$, and $A\times B$ on $\mathfrak{m}(\gamma)$ all as $\mathscr{L}\mathrm{O}(4)$. In case $m=8$, M is EIV. First, $\mathfrak{k}$ is F_4, since FII is the polar $\cong M'(\alpha, \beta)$ (the $\cong$ by 2.15a) with $\mathrm{BC}_1(8, 7)$. $\mathfrak{g}$ is E_6, as is seen without difficulty. Let us add a few facts. (1) $\mathfrak{k}(0)\cong\mathscr{L}\mathrm{O}(8)$; (2) since $\mathfrak{k}(\gamma)$ acts on $\mathfrak{m}(\alpha)+\mathfrak{m}(\beta)$ effectively and $\mathfrak{k}(0)$ stabilizes these linear spaces, one has an embedding of $\mathfrak{k}(\gamma)$ into $\mathfrak{m}(\alpha)\otimes\mathfrak{m}(\beta)$ as a $\mathfrak{k}(0)$-module; (3) if $\mathfrak{k}(0)\cong\mathscr{L}\mathrm{O}(8)$ acts on $\mathfrak{m}(\alpha)$ by the representation $\omega_3=\omega_3(\mathrm{D}_4)$, say, on $\mathfrak{m}(\beta)$ by ω_4, on $\mathfrak{k}(\gamma)$ by ω_1, then $\mathfrak{m}(\alpha)\otimes\mathfrak{m}(\beta)$ is the direct sum of a simple module with the highest weight $\omega_3+\omega_4$ and another simple submodule with ω_1. (Recall that these are obtained from each other by composing with an outer automorphism T of $\mathscr{L}\mathrm{O}(8)$ or T^2; $\mathrm{T}^3=1$. There is a member t of F_4 which stabilizes $\mathfrak{a}$ and induces a cyclic permutation of $\{\alpha, \beta, \gamma\}$ by 2.4b (vi). This t then permutes $\{\mathfrak{m}(\alpha), \mathfrak{m}(\beta), \mathfrak{m}(\gamma)\}$ cyclically and stabilizes $\mathfrak{k}(0)$. Hence t induces the outer automorphism T on $\mathscr{L}\mathrm{O}(8)$.) The observation at once allows to describe the bracket: $\mathfrak{m}(\gamma)\wedge\mathfrak{m}(\alpha)\to\mathfrak{k}(\beta)$ completely, although this is found also by the cyclic automorphism and $M'(\alpha, \beta)=\mathrm{FII}$.

Now we turn to the case 2° of $\mathrm{B}_2=\mathrm{B}_2(m, n)$, $m\geqq n$. Assume α and β are shorter roots, $\pm\alpha\pm\beta$ being the longer. We first prove that the multiplicity n of $\alpha\pm\beta$ cannot be greater than 3. One recalls that $\mathfrak{k}(0)$ acts on $\mathfrak{m}(\alpha)$, $\mathfrak{k}(\alpha)$ and $\mathfrak{m}(\beta)$ as $\mathscr{L}\mathrm{O}(m)$ by 2.25 (a) and (b). In the adjoint representation, ad $\mathfrak{k}(\alpha\pm\beta)$ exchanges $\mathfrak{m}(\alpha)$ and $\mathfrak{m}(\beta)$ effectively by 2.25 (c), and hence one can embed $\mathfrak{k}(\alpha\pm\beta)$ into $\mathfrak{m}(\alpha)\otimes\mathfrak{m}(\beta)$ as a $\mathfrak{k}(0)$-module. From this, one immediately sees that $\mathfrak{k}(0)$ acts on $\mathfrak{m}(\alpha)\otimes\mathfrak{m}(\beta)$ as a single algebra $\mathscr{L}\mathrm{O}(m)$ (as opposed to $\mathscr{L}\mathrm{O}(m)\times\mathscr{L}\mathrm{O}(m)$ whose factors act on $\mathfrak{m}(\alpha)$ and $\mathfrak{m}(\beta)$ independently); in other words, $\mathfrak{m}(\alpha)\otimes\mathfrak{m}(\beta)$ is isomorphic with $\mathfrak{m}(\alpha)\otimes\mathfrak{m}(\alpha)$ as a $\mathfrak{k}(0)$-module. Except for the case of $m=8$, $\mathfrak{m}(\alpha)\otimes\mathfrak{m}(\alpha)$ is the direct sum of the trivial one, $\boldsymbol{R}$, a submodule $\mathfrak{m}(\alpha)\wedge\mathfrak{m}(\alpha)\cong\mathscr{L}\mathrm{O}(m)$ and a simple one $\mathfrak{m}(\mathrm{AI}(m))$ (explained earlier). They have dimensions 1, $\frac{1}{2}m(m-1)$ and $\frac{1}{2}m(m+1)-1$ respectively and $\mathfrak{k}(\alpha\pm\beta)$ must be one of these submodules except when $m=4$ and hence $\mathscr{L}\mathrm{O}(m)$ is the sum of two copies of $\mathscr{L}\mathrm{O}(3)$. Thus (m, n) is (2, 2), (3, 3) or (4, 3) unless $n=1$; the case (3, 3) contradicts 2.25 (e). In case $m=8$, the half-spin representations cannot appear as in 1° for the following reason. Since the roots $\alpha-\beta$ and $\alpha+\beta$ are strongly orthogonal, $\mathfrak{k}_0(\alpha-\beta)$ centralizes $\mathfrak{k}_0(\alpha+\beta)$ in acting on each of $\mathfrak{m}(\alpha)$ and $\mathfrak{m}(\beta)$. Therefore $R(M)$ is $\mathrm{B}_2(m, 1)$, $\mathrm{B}_2(2, 2)$ or $\mathrm{C}_2(4, 3)$. Thus we have determined all the possible embeddings of $\mathfrak{k}(\alpha\pm\beta)$ into

$\mathfrak{m}(\alpha)\otimes\mathfrak{m}(\beta)$, which one notices means that the action of, say, ad $\mathfrak{k}(\alpha)$: $\mathfrak{m}(\alpha\pm\beta)\to\mathfrak{m}(\beta)$ has been determined simultaneously. Suppose $n=1$; $R(M)$ is $B_2(m, 1)$. Then the 1-dimensional subalgebras $\mathfrak{k}(\alpha\pm\beta)$ centralizes each other and $\mathfrak{k}(0)$. Hence $\mathfrak{k}(\alpha\pm\beta)$ define a $\mathfrak{k}(0)$-invariant complex structure J' on $T_oM'(\alpha, \beta)=\mathfrak{m}(\alpha)\oplus\mathfrak{m}(\beta)$; $\{\pm J'\}$ is unique by the above. Thus one has $M'(\alpha, \beta)\fallingdotseq G_2(\boldsymbol{R}^{2+m})=\mathrm{SO}(2+m)/(\mathrm{SO}(2)\cdot\mathrm{SO}(m))$, $\mathfrak{k}(0)$ acting on $T_oM'(\alpha, \beta)\cong\boldsymbol{R}^2\otimes\mathfrak{m}(\alpha)$ as $\mathscr{L}\mathrm{O}(2)\times\mathscr{L}\mathrm{O}(m)$. Moreover $\mathfrak{k}(\alpha+\beta)+\mathfrak{k}(\alpha-\beta)$ is the unique sum $\mathfrak{k}_{(+)}+\mathfrak{k}_{(-)}$ such that $[\mathfrak{k}_{(+)}, \mathfrak{k}(\alpha)+\mathfrak{k}(\beta)]=0$ and $[\mathfrak{k}_{(-)}, \mathfrak{m}(\alpha)+\mathfrak{m}(\beta)]=0$ by the above; in fact, if a member y of $\mathfrak{k}(\alpha+\beta)+\mathfrak{k}(\alpha-\beta)$ satisfies $[y, \mathfrak{m}(\alpha)+\mathfrak{m}(\beta)+\mathfrak{k}(\alpha)+\mathfrak{k}(\beta)]=0$, then one has $[y, \mathfrak{m}(\alpha\pm\beta)]\subset[y, [\mathfrak{k}(\alpha), \mathfrak{m}(\beta)]=0$ and $y=0$ therefore. Hence $\mathfrak{k}_{(+)}$ centralizes $\mathfrak{k}$. Since $\mathfrak{k}$ is irreducible on $\mathfrak{m}$, $\mathfrak{k}_{(+)}$ thus defines a $\mathfrak{k}$-invariant complex structure on $\mathfrak{m}$; here one sees $\mathfrak{k}=\mathfrak{k}_{(+)}+\mathscr{L}\mathrm{O}(2+m)$, $\mathscr{L}\mathrm{O}(2+m)\supset\mathfrak{k}_{(-)}$, and that $\mathfrak{k}$ acts on $\mathfrak{m}\cong\boldsymbol{R}^2\otimes\boldsymbol{R}^{2+m}$ as $\mathscr{L}\mathrm{O}(2)\times\mathscr{L}\mathrm{O}(2+m)$. Therefore one concludes $M\fallingdotseq G_2(\boldsymbol{R}^{4+m})$. In the case of $B_2(2, 2)$, $\mathfrak{k}(\alpha\pm\beta)$ embeds as $\mathfrak{m}(\mathrm{AI}(2))$ and defines an invariant quaternion structure on $\mathfrak{m}^+$; $M^+\fallingdotseq G_1(\boldsymbol{H}^2)$. Thus $\mathfrak{k}=\mathscr{L}\mathrm{Sp}(2)$ and M is a group $\fallingdotseq\mathrm{Sp}(2)$. If $R(M)$ is $C_2(4, 3)$, $\mathfrak{k}(\alpha\pm\beta)$ defines an invariant quaternion structure on $\mathfrak{m}^+$ as in the case of $B_2(m, 1)$; $M^+\fallingdotseq G_1(\boldsymbol{H}^2)\times G_1(\boldsymbol{H}^2)$ and $M\fallingdotseq G_2(\boldsymbol{H}^4)$.

Now suppose $R(\alpha, \beta)=BC_2=BC_2(a, b, c)$, $a\geqq b\geqq c$. Let $R(\alpha, \beta)$ have α, β, $\alpha\pm\beta$, 2α, and 2β. Since 2α and $\alpha\pm\beta$ generate a root system $R_{\mathrm{B}}\cong B_2=B_2(b, c)$, there is a subspace $M^-=M(R_{\mathrm{B}})$ by 2.4d; M^- is a meridian. Hence the multiplicity $c=m(2\alpha)$ is 1, 2 or 3 by our result on B_2. But c cannot be 2, by 2.26 applied to the subspace $M(\alpha, 2\alpha)$. Assume $c=1$. Then a member J of $\mathfrak{k}(2\alpha)+\mathfrak{k}(2\beta)$ defines an invariant complex structure on $\mathfrak{m}$ as in the case of $B_2(m, 1)$; M will be hermitian. The subspace $M(\alpha, 2\alpha)$ is a complex projective space; hence $\mathfrak{m}(\alpha)$ has an even dimension $2k=a$. Besides $\mathfrak{k}_0(\alpha)$ acts on $\mathfrak{m}(\alpha)$ as $\mathscr{L}\mathrm{U}(k)$ (2.26a), normalizing $\mathfrak{k}_0(\alpha+\beta)$ by 2.25 (b), while $\mathfrak{k}_0(\alpha+\beta)$ is $\mathscr{L}\mathrm{O}(b)$ by 2.25 (a). This greatly limits the possibility of $(2k, b)$. Indeed, if $\mathrm{O}(b)$ is simple so that $\mathfrak{k}_0(\alpha+\beta)$ is effective on $\mathfrak{m}(\alpha)$, then this gives $\mathscr{L}\mathrm{O}(b)=\mathscr{L}\mathrm{SU}(k)$, and hence $(b, k)=(6, 4)$. For this, $\mathfrak{k}^+$ is $\mathscr{L}T\cdot T\cdot\mathrm{O}(8)=\boldsymbol{R}J+\mathscr{L}T\cdot\mathrm{O}(8)$ by $M^-\cong G_2^\circ(\boldsymbol{R}^{10})$, and $\mathfrak{k}$ is $\boldsymbol{R}J+\mathscr{L}\mathrm{O}(10)$; $M'(\alpha, \beta)$ is also $G_2^\circ(\boldsymbol{R}^{10})$. Since $\dim\mathfrak{m}=32$ obviously, $\mathscr{L}\mathrm{O}(10)$ acts on $\mathfrak{m}$ through a half-spin representation. Therefore $G\fallingdotseq E_6$ and $M\fallingdotseq\mathrm{EIII}$. If $\mathrm{O}(b)$ is not simple, $b=1, 2$ or 4, but $b=1$ is impossible, since otherwise $\mathfrak{k}(0)$ would act trivially on $\mathfrak{k}(\alpha+\beta)+\mathfrak{k}(\alpha-\beta)$ which must be a $\mathfrak{k}(0)$-submodule of $\mathfrak{m}(\alpha)\otimes\mathfrak{m}(\beta)$. If $b=2$, M is $G_2(\boldsymbol{C}^{4+k})$; M^- is $G_2(\boldsymbol{C}^4)\cong G_2^\circ(\boldsymbol{R}^6)$, while the c-orthogonal space $M^+:=M''(\alpha, \beta)$ to M^- is $G_2(\boldsymbol{C}^{2+k})$. If $b=4$, M^- is $\mathrm{DIII}(4)\cong G_2^\circ(\boldsymbol{R}^8)$ and $\mathscr{L}\mathrm{SU}(k)$ acting on $\mathfrak{m}(\alpha+\beta)$ must be an ideal in $\mathscr{L}\mathrm{O}(b)$; one finds $k=2$. Thus $M'(\alpha, \beta)$ is $G_4(\boldsymbol{C}^5)\cong G_1(\boldsymbol{C}^5)$. One concludes $M\cong\mathrm{DIII}(5)$. Now assume $c=3$. Then $\mathfrak{k}(2\alpha)+\mathfrak{k}(2\beta)\cong\mathscr{L}\mathrm{O}(4)$ acts on $\mathfrak{m}^+=T_oM^+$ as $\mathscr{L}\mathrm{Sp}(1)$, defining a quaternion structure. M^- is $G_2(\boldsymbol{H}^4)$ and M^+ is $G_2(\boldsymbol{H}^{2+k})$ by induction. M is $G_2(\boldsymbol{H}^{4+k})$.

Finally we have come to the case of $R(\alpha, \beta)=G_2$. Let α and β be the simple roots with $\alpha^\sim=2\alpha+3\beta$; α is longer. The multiplicity $m(\alpha)=1$, 2, 4 or 8, since the longer roots form the system A_2. But $m(\alpha)$ cannot be 4 or 8; in fact, since $\alpha^\sim$ and β are strongly orthogonal, $\mathfrak{k}_0(\alpha^\sim)\cong\mathscr{L}\mathrm{O}(m(\alpha))$ thus centralizes $\mathfrak{k}_0(\beta)\cong\mathscr{L}\mathrm{O}(m(\beta))$, $m(\alpha)\leqq m(\beta)$, and both act on $\mathfrak{m}(\alpha)$ nontrivially if $m(\alpha)>1$. Hence either $m(\alpha)=m(\beta)=2$ or $m(\alpha)=1$. If $m(\alpha)=1$, so does $m(\beta)=1$; in fact, $\mathfrak{k}(\alpha^{\sim})+\mathfrak{k}M(\beta)$ would act on $\mathfrak{m}^{+}=\sum_{0\leq j\leq 3}\mathfrak{m}(\alpha+j\beta)$ as complex transformations which properly contain $\mathscr{L}\mathrm{O}(m(\beta)+1)$. In case $m(\alpha)=2$, the meridian $M^{-}=M(\alpha^{\sim},\beta)$ is SO(4) and the polar M^{+} has the root system $\mathrm{G}_2(1,1)$ as is easily seen. $\mathfrak{k}^{+}\cong\mathscr{L}\mathrm{O}(4)$ acts on $\mathfrak{m}^{+}$ through (ω_1,ω_3). M is thus the group G_2. M^{+} is the space GI. In case $m(\alpha)=1$, both M^{+} and M^{-} are $S^2\cdot S^2$ and $\mathfrak{k}$ is $\mathscr{L}\mathrm{O}(4)$. Hence M is GI. The lemma 2.27 has been proven at last. QED

2.27a. Corollary. *There are* ① *monomorphisms*: AI(3)→SU(3)→AII(3)→EIV *for the* 1-*connected spaces of* $\mathrm{A}_2(m)$, *inducing* $\mathrm{A}_2(1)\to\mathrm{A}_2(2)\to\mathrm{A}_2(4)\to\mathrm{A}_2(8)$, ② *monomorphisms*: CI(2)→Sp(2)→$G_2(\boldsymbol{H}^4)$, *inducing* $\mathrm{B}_2(1,1)\to\mathrm{B}_2(2,2)\to\mathrm{B}_2(4,3)$, ③ *a monomorphism*: $G_2(\boldsymbol{R}^{4+m})\to G_2(\boldsymbol{R}^{4+k})$, $m<k$, *of the bottom space, inducing* $\mathrm{B}_2(m,1)\to\mathrm{B}_2(k,1)$, ④ *a monomorphism*: $G_2(\boldsymbol{C}^{4+k})\to G_2(\boldsymbol{H}^{4+k})$, *inducing* $\mathrm{BC}_2(2k,2,1)\to\mathrm{BC}_2(4k,4,3)$, ⑤ *monomorphisms*: $G_2(\boldsymbol{C}^5)\to$DIII(5)→EIII←$G_2^{\circ}(\boldsymbol{R}^{10})$ *inducing* $\mathrm{BC}_2(2,2,1)\to\mathrm{BC}_2(4,4,1)\to\mathrm{BC}_2(8,6,1)\leftarrow\mathrm{C}_2(6,1)$, *which restrict to* $G_2(\boldsymbol{R}^5)\to$SO(5)→$G_2(\boldsymbol{H}^4)^*\leftarrow S^4\cdot S^4$, ⑥ *the monomorphism*: EIII←$G_2^{\circ}(\boldsymbol{R}^{10})$ *also restricting to* FII←S^8 *with* $\mathrm{BC}_1(8,0,7)\leftarrow\mathrm{A}_1(7)$ *as well as* ⑦ *monomorphisms*: GI$=\mathrm{G}_2/\mathrm{SO}(4)\to\mathrm{G}_2\leftarrow\mathrm{SO}(4)$, *inducing* $\mathrm{G}_2(1,1)\to\mathrm{G}_2(2,2)\leftarrow\mathrm{A}_1(1)\times\mathrm{A}_1(1)$, *but NOT those which induce* $\mathrm{B}_2(2,1)\to\mathrm{B}_2(2,2)$ *or* $\mathrm{B}_2(3,1)\to\mathrm{B}_2(4,3)$; *see* [CN-3] *for the last impossibilities.*

2.27b. Remark on classification. These lemmas put together allow one to find the root systems with multiplicity of all the possible compact simple symmetric spaces. The details would be easy to work out; for instance, $\mathrm{A}_3(8)$ is impossible in view of our results on $\mathrm{A}_2(8)$, since $\mathfrak{k}_0(\alpha_1)$ and $\mathfrak{k}_0(\alpha_3)$, centralizing each other, must act on $\mathfrak{m}(\alpha_2)$ as $\mathscr{L}\mathrm{O}(8)$, which is impossible of course, where α_1, α_2 and α_3 are simple roots. Similarly for $\mathrm{C}_3(8,1)$. A complete list is found in Table VI of Chapter X, [H].

The lemmas give more together with their proofs; next we will complete the proof of Theorem 1.15 and show that a simple symmetric space M is locally determined by its root system $R(M)$ with multiplicity, thereby we will have given another classification of the symmetric spaces which is different from [A] and [H]. The above lemmas give the classification for the spaces of rank ≤ 2.

2.28. Lemma. *A simple space M is locally determined by any polar M^{+} of a point o in M and a meridian M^{-} to it.*

Proof. The lemma asserts, on the local level, that two simple spaces M and N are isomorphic if a polar $M^{+}(p)$ in M is isomorphic with a polar N^{+} in N and the meridian M^{-} to $M^{+}=M^{+}(p)$ is isomorphic with a meridian N^{-} to N^{+}; the fact on the global level was stated in 1.15. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{m}=T_oM$ which is contained in $\mathfrak{m}^{-}:=T_oM^{-}$ (See 1.8), and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ be the symmetry decomposition for M at o. We have the root space decomposition of $\mathfrak{m}$ with respect to $\mathfrak{a}$. $\mathfrak{m}^{-}$ is spanned by $\mathfrak{a}$ and the root spaces $\mathfrak{m}(\alpha)$, $\alpha\in R(\mathfrak{m}^{-}):=R(M^{-})\subset R(M)$, by 2.15. Hence the orthogonal complement $\mathfrak{m}^{+}$ to $\mathfrak{m}^{-}$ in $\mathfrak{m}$ is the sum of the other root spaces $\mathfrak{m}(\lambda)$, $\lambda \in R(\mathfrak{m}^+) := R(M) - R(M^-)$. $\mathfrak{m}^+$ is the tangent space to a subspace which is isomorphic with M^+ (2.15a). Let $\mathfrak{k}^+$ denote $F(\operatorname{ad} Q(p), \mathfrak{k})$; $\mathfrak{k} = \mathfrak{k}^+ + \mathfrak{k}^-$. $\mathfrak{k}^+$ is generated by all the spaces $\mathfrak{k}(\alpha)$, $\alpha \in R(\mathfrak{m}^-)$, and $\mathfrak{k}_0(\lambda)$, $\lambda \in R(\mathfrak{m}^+)$. (It does not matter if there is some ambiguity about $\mathfrak{k}(0) \subset \mathfrak{k}^+$.) Given M^+ and M^-, one knows $R(M^-)$, the actions of $\mathfrak{k}^+$ on $\mathfrak{m}^-$ and $\mathfrak{m}^+$ and the curvature maps $[\ ,\ ]: \mathfrak{m}^- \otimes \mathfrak{m}^- \to \mathfrak{k}^+$ and $\mathfrak{m}^+ \otimes \mathfrak{m}^+ \to \mathfrak{k}^+$. We have to show that these data alone allow one to determine the other part of the curvature map $[\ ,\ ]: \mathfrak{m}^+ \otimes \mathfrak{m}^- \to \mathfrak{k}$ along with $[\ ,\ ]: \mathfrak{k}^- \otimes \mathfrak{m} \to \mathfrak{m}$. We may work on the bottom M^* and use 2.21.

To do it, we will use facts found in preceding lemmas and their proofs. $R(\mathfrak{m}^+)$ is a weight system (See 2.3 and 2.25a); indeed $R(\mathfrak{m}^+)$ is the weight system of $F(\operatorname{ad} Q(p), \mathfrak{g})$ acting on $\mathfrak{k}^- + \mathfrak{m}^+$, restricted to $\mathfrak{a}$. The decomposition of $\mathfrak{m}^+$ into the sum of the weight spaces $\mathfrak{m}(\lambda)$, $\lambda \in R(\mathfrak{m}^+)$, is obtained as the simple $\mathfrak{k}(0)$-module decomposition; $\mathfrak{k}(0)$ is the sum of $\mathfrak{k}_0(\alpha)$, $\alpha \in R(\mathfrak{m}^-)$, and the ideal in $\mathfrak{k}^+$ which acts trivially on $\mathfrak{m}^-$. To determine the weight λ of each weight space $\mathfrak{m}(\lambda)$ out of the sum $\sum \mathfrak{m}(\lambda) = \mathfrak{m}^+$, one looks at the action of $\mathfrak{k}M(\alpha) = \mathfrak{k}_0(\alpha) + \mathfrak{k}(\alpha)$ on $\mathfrak{m}^+$. From its action on $\mathfrak{m}(\lambda + + \alpha)$, the α series of λ, one finds the numbers $n(\lambda, \alpha) = 2\|\alpha\|^{-2}\langle\lambda, \alpha\rangle$ for $\lambda \in R(\mathfrak{m}^+)$ and $\alpha \in R(\mathfrak{m}^-)$ and thereby pinpoints λ in $\mathfrak{a}$; this is possible because $R(\mathfrak{m}^-)$ spans $\mathfrak{a}$ or its hyperplane. In the second case, the normalized normal vector H^j has the inner product $\langle\lambda, H\rangle = \pm 1$ for every λ; normalization can be done by finding the pole in the circle with the tangent H^j. Thus every $\mathfrak{m}(\lambda)$ has the name λ now. As to the curvature maps $[\ ,\ ]$ in question, one observes that $\mathfrak{m}(\lambda) \otimes \mathfrak{m}(\alpha) \to \mathfrak{k}^-$ is converted to $\mathfrak{m}(\lambda) \otimes \mathfrak{k}(\alpha) \to \mathfrak{m}^+$ by applying $\operatorname{ad}(H)$, $H \in \mathfrak{a}$ with $\lambda(H) = 0 \neq \alpha(H)$. Similarly $\mathfrak{k}^-(\lambda) \otimes \mathfrak{m}(\alpha) \to \mathfrak{m}^+$ is converted to the same $\mathfrak{m}(\lambda) \otimes \mathfrak{k}(\alpha) \to \mathfrak{m}^+$ (which is a part of the action at hand); an alternative, geometric proof for this is to use the symmetry s_m at the midpoint m on the geodesic arc joining o to p in $M^-(p)$ as in 2.15a and to see the obvious fact that the automorphism $\operatorname{ad}(s_m)$ of $\mathfrak{g}$ exchanges $\mathfrak{m}^+$ and $\mathfrak{k}^-$. QED

2.29. Corollary. *A compact symmetric space is locally determined by its root system with multiplicity.*

Proof. We may assume the space is semisimple and a bottom space. Then it has a polar and its meridian of lower dimension. These are known from the given root system with multiplicity. And one can apply the second half of the proof of the lemma. QED

In the same vein, one obtains such results as the next proposition; some properties of M are faithfully reflected to those of the pairs (M^+, M^-) of the polars M^+ and the meridians M^- to them.

2.30. Proposition. *Let M be a simple 1-connected (compact) space. Then M is hermitian if and only if some polar $M^+(p)$ and a meridian $M^-(p)$ to it are hermitian.* (If M is hermitian, all the polars and the meridians are hermitian, simply because the point symmetries s_p are holomorphic.)

PROOF. A simple space $M=G/K$ is hermitian if and only if M is 1-connected and K contains a one dimensional center [H]. From the above, one sees that this is equivalent to say that M is 1-connected and $R(M)$ is $BC_r = BC_r(a, b, 1)$ with the understanding that $BC_r(0, b, 1)$ means $C_r(b, 1)$ and $C_r(0, 1)$ is $A_1(1)$; see the proof of 2.27 in the case of B_2 for somewhat detailed analysis of this point. Assume $M^+(p)$ and $M^-(p)$ are hermitian. M is then 1-connected by 1.8a. If M^- has a simple factor with the root system $BC_s(*, *, 1)$, then $R(M)$ is necessarily of the form $BC_r(a, b, 1)$ by 2.15 and hence M is hermitian, since a space with BC_r is 1-connected. Thus we may assume that M^- is the product of spaces with the root systems of type $C_s(b, 1)$. Then the longer roots $2\varepsilon_j$, $1 \leq j \leq r$, form a basis for the maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{m}^-$; one agrees that (ε_j) is an orthonormal basis. Hence every root λ in $R(M^+, \mathfrak{a}) := R(M) - R(M^-)$ is shorter than $2\varepsilon_j$; there is no room for another root of that length. Thus $\|\lambda\|^2 = 1$ or 2, for otherwise $R(M)$ would be G_2, contrary to the above in view of the highest root $\alpha^\sim(G_2) = 3\alpha_1 + 2\alpha_2$ (See the paragraph above 3.21). If $\|\lambda\|^2 = 1$, then $R(M)$ is of type $BC_r(a, b, 1)$ and so M is hermitian. So we assume $\|\lambda\|^2 = 2$ for every root λ in $R(M^+, \mathfrak{a})$. One has $n(\lambda, 2\varepsilon_j) = \pm 1$ or 0. One sees $\lambda = \pm\varepsilon_j \pm \varepsilon_k$ and hence $R(M)$ is of type $C_r(b, 1)$. QED

§3. The involutions of the groups.

The purpose of this section is to describe the group involutions, G-Inv(M), of compact simple Lie groups M (as opposed to the involutions of M as spaces) from a new view point, although they are known in a way. Using interrelationship between the groups (such as Corollary 2.10 and Theorem 2.11), we will try to describe them in a unified way. Thus, as to the exceptional groups, we will show (3.10 through 3.21) that (i) every group involution of E_8 is conjugate with the extended adjoint action ad(b) of a member b of a fixed subgroup $SO(16)^\#$ of E_8; and (ii) all the group involutions (inner or outer) of the other 1-connected exceptional groups are restrictions of those of E_8, made explicit with fixed monomorphisms $E_8 \supset E_7 \supset E_6 \supset F_4 \supset G_2$. All the involutions of the exceptional Lie groups are given by a few members of $SO(16)^\#$; the results are summarized with diagrams in 3.22 through 3.29. $SO(16)^\#$ is locally isomorphic with $SO(16)$ of course, and all the group involutions of all the locally isomorphic groups to $SO(n)$ will be determined as well as those of the classical groups in terms of fixed standard monomorphisms $O(n) \subset SO(n+1)$, $U(n) \subset SO(2n)$ and $Sp(n) \subset SU(2n)$; so we will begin with $SO(n)$. All the group involutions will be placed in a single panorama.

The involutions of the spaces which are not groups will be systematically discussed in a forthcoming paper. There some monomorphisms such as the one: $D_r \to C_r$ which cannot be realized by Lie group monomorphisms but by symmetric space ones: $D_r \to C_r$ is realized by $G_r(\boldsymbol{R}^{2r}) \to G_r(\boldsymbol{C}^{2r})$, for instance. New Satake diagrams are necessary to describe them [OS].

3.1. NOTATIONS. $I_k = I_{k,n-k}$ denotes the diagonal matrix whose first k diagonal entries are -1 and the rest are equal to 1; that is, I_k corresponds to $((-1)1_k) \oplus 1_{n-k}$ in $O(n)$, where 1_k is the $k \times k$ unit matrix, and similarly 0_k denotes the zero matrix of that size. We will use the following symbols for 2×2 matrices:

$$3.2. \qquad I = \begin{Vmatrix} -1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad J = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix} \quad \text{and} \quad K = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}.$$

Also we denote by $K \otimes A$, say, the $2n \times 2n$ matrix which one gets by substituting the $n \times n$ matrix A into 1 at the two locations in K; thus $I \otimes 1_n$ equals $I_{n,n}$ for another example. We write P_k for the matrix $1_2 \otimes I_k$, which has a pair of I_k as diagonal blocks; P_k is conjugate with I_{2k} in $O(2n)$. Also we write J_k or simply J for $J \otimes 1_k$ and similarly K_k or simply K for $K \otimes 1_k$.

We begin with the orthogonal groups, aiming at Propositions 3.4 and 3.5. Our explanation will be brief for the classical groups, for linear algebra is more or less directly available.

Every involutive member of $O(n)$ is conjugate with some I_k. These involutive members I_k, $1 \leqq k \leqq n$, make a commutative system, which is in one-to-one correspondence with the conjugate classes of the involutive members of $O(n)$, i.e. with the polars of 1 in $O(n)$.

If $n = 2r+1$ is odd, every group involution of $SO(n)$ is conjugate with $\mathrm{ad}(I_k)$. And $\mathrm{ad}(I_k)$ is conjugate with $\mathrm{ad}(I_j)$ if and only if $j = k$ or $j + k = n$. To prove these, we use Theorem 2.11 after introducing necessary notations.

3.3. MORE NOTATIONS. We fix the maximal torus $A(SO(2r+1))$ which is also a maximal torus in $SO(2r)$ identified with $SO(2r) \oplus \{1_1\}$ and whose Lie algebra consists of $J \otimes D$, D arbitrary diagonal $r \times r$ matrices. We write ε_k for $J \otimes D$ in which the j-th diagonal member of D is δ^j_k. In conform with the notation of [B], the k-th fundamental weight is $\omega_k = \omega_k(B_r) = \omega_k(D_r) = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k$ for $k < r-1$ and $\omega_r = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r)$, while $\omega_{r-1}(B_r) = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{r-1}$ for $SO(2r+1)$ and $\omega_{r-1}(D_r) = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{r-1} - \varepsilon_r)$ for $SO(2r)$. We have identified the Cartan subalgebra with its dual vector space; the corresponding simple roots are $\alpha_k = \alpha_k(B_r) = \alpha_k(D_r) = \varepsilon_k - \varepsilon_{k+1}$ for $k < r$, while $\alpha_r(B_r) = \varepsilon_r$ and $\alpha_r(D_r) = \varepsilon_{r-1} + \varepsilon_r$. We also denote by the symbol $\omega \mapsto b$ the fact that $\exp(\pi\omega) = b \in G$ for a member ω of the Cartan subalgebra $\mathfrak{a}$ of G. For instance, one has $\omega_k \mapsto P_k$ if $G = SO(2r)$ and $k < r-1$. Also $\omega_r(D_r) \mapsto J_r = J \otimes 1_r$ for $SO(2r)$.

Back to the involutions of $SO(2r+1)$, one recalls the highest root $\alpha^\sim$ is $\varepsilon_1 + \varepsilon_2 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_r)$. Hence there is no outer involution (because the Dynkin diagram admits no nontrivial automorphism; Cf. [B] Chap. 8, p. 110); the group-automorphism group G-Aut$(SO(2r+1))$ is connected. Since $SO(2r+1)$ is the adjoint group $SO(2r+1)^*$, Corollary 2.13 tells us that H^k, $1 \leqq k \leqq r$, are tangent to the shortest geodesics to the polars of 1 in $SO(2r+1)$; here $H^k = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k$, $1 \leqq k \leqq r$. One has $H^k \mapsto P_k$. Therefore $\mathrm{ad}(P_k)$, $1 \leqq k \leqq r$, give a representative system of the conjugate classes of the group involutions of $SO(2r+1)$; no pair among these are conjugate simply because the different H^k have different lengths. By Theorem 2.15, the corresponding meridians are $SO(2k)\times SO(2r+1-2k)$. The universal covering group $SO(2r+1)^{\sim}$ has the center of order 2, $\{1, \varepsilon\}$, (since ω_1 is the only minuscule). Hence there is no other connected group in the local isomorphism class of $SO(2r+1)$. Every $\mathrm{ad}(P_k)$ can act on $SO(2r+1)^{\sim}$.

3.4. PROPOSITION (Case of B_r). $\mathrm{ad}(P_k\oplus 1_1)$, $1\leqq k\leqq r$, *form a representative system of the conjugate classes of the group involutions of* $SO(2r+1)$ *and* $SO(2r+1)^{\sim}$. *This is commutative.*

We turn to the class of $SO(2r)$. The adjoint group is $SO(2r)/\{\pm 1\}$. The highest root $\alpha^{\sim}$ is $\varepsilon_1+\varepsilon_2=\alpha_1+2(\alpha_2+\cdots+\alpha_{r-2})+\alpha_{r-1}+\alpha_r$. Hence the inner automorphism group $\text{G-Aut}(SO(2r)^*)_{(1)}$, $r\neq 4$, has index 2 in $\text{G-Aut}(SO(2r)^*)$. One has $H^k=\omega_k$, $1\leqq k\leqq r$, and $H^k\mapsto P_k$ for $k<r-1$ in $SO(2r)$, while one has $\omega_r\mapsto J_r=J\otimes 1_r$ and $\omega_{r-1}\mapsto J_r' := \mathrm{ad}(I_{2r-1})J_r=-J\otimes I_{r-1}$ in $SO(2r)$. By taking "$H^k\mapsto P_k$" as the definition of P_k, one sees what P_k is in the locally isomorphic groups $SO(n)^*$, $SO(n)^{\sim}$ and $SO(4m)^{\#}$. One notes that $\mathrm{ad}(P_k)$ is conjugate with $\mathrm{ad}(P_{r-k})$ in $SO(2r)^*$ and that $\mathrm{ad}(J_r)$ is conjugate with $\mathrm{ad}(J_r')$ in it in case r is odd; more generally, $\mathrm{ad}(J\otimes I_k)$ is conjugate with $\mathrm{ad}(J\otimes I_j)$ if and only if $k-j$ is even. To P_k there corresponds the meridian $SO(2k)\times SO(2r-2k)$ in $SO(2r)$, and $U(r)/\boldsymbol{Z}_2$ is isomorphic with the meridians corresponding to $\mathrm{ad}(J_r)$ and $\mathrm{ad}(J_r')$ in $SO(2r)^*$. Otherwise no pair of $\mathrm{ad}(P_k)$, $1\leqq 2k\leqq r$, $\mathrm{ad}(J_r)$ and $\mathrm{ad}(J_r')$ is conjugate to each other except for the case $r=4$ (in which $SO(8)^*$ admits an outer automorphism T of order 3, which permutes the 3 polars). If $n=2r$ is even, the adjoint group $O(n)^*$ has two connected components. The outer involutions come from the polars of 1 in it outside $SO(n)^*$. Thus one has $\mathrm{ad}(I_k)$, k odd, which are conjugate with $\mathrm{ad}(I_{2r-k})$ and nothing else. They commute with every $\mathrm{ad}(P_k)$ and no conjugate of $\mathrm{ad}(J_r)$. $\mathrm{ad}(I_1)$ carries ω_r into a conjugate of ω_{r-1}.

Next we look at $SO(n)^{\sim}$. It has the center of order 4 for $n=2r$ consisting of 1, δ_r, ε and $\delta_r\varepsilon$; one has $2H^k=2\omega_k\mapsto\varepsilon$ for k odd and $<r-1$, $2\omega_r\mapsto\delta_r$ and $2\omega_{r-1}\mapsto\delta_r\varepsilon$ in $SO(n)^{\sim}$. δ_r and $\delta_r\varepsilon$ projects to -1 in $SO(n)$, while ε does to 1 in it. ε is involutive, as is elucidated in Remark 3.6; thus $4\omega_1\mapsto 1$. Hence $4\omega_r\mapsto\varepsilon$ if r is odd, while δ_r is involutive if r is even. Therefore the center $\{1, \delta_r, \varepsilon, \delta_r\varepsilon\}$ is cyclic for r odd and isomorphic with $\{1, \delta_r\}\times\{1, \varepsilon\}\cong\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ for r even. Every automorphism of $SO(n)^{\sim}$ fixes ε because of the distance $\|2\omega_1\|$ from 1 compared with $\|2\omega_r\|$ except for T acting on $SO(8)^{\sim}$ which permutes $\{\delta_r, \varepsilon, \delta_r\varepsilon\}$ cyclically. In the general case including $SO(8)^{\sim}$, $\mathrm{ad}(P_k)$ is the identity on the center as well as on the maximal torus which projects to $A(SO(n))$ defined earlier, since P_k is a member of $A(SO(n))$ and $\mathrm{ad}(I_1)$ exchanges δ_r and $\delta_r\varepsilon$, in particular they fix ε. Hence every involution either fixes δ_r (in case it is inner) or carries it into $\delta_r\varepsilon$ (in case it is outer). Thus $SO(n)^{\sim}/\{1, \delta_r\}$ is isomorphic with $SO(n)^{\sim}/\{1, \delta_r\varepsilon\}=: SO(2r)^{\#}$, called semispinor group sometimes. Finally an involutive automorphism of $SO(2r)$ can act on $SO(2r)^{\#}$ if and only if it is inner. In finding the group involutions, one does not

have to consider the case 2° in Corollary 2.10, since it does not occur to the adjoint group (See Remark 3.7). We summarize these in a proposition.

3.5. PROPOSITION (Case of D_r). (i) $\mathrm{ad}(P_k)$, $1 \leqq 2k \leqq r$, *together with* $\mathrm{ad}(J_r)$ *and* $\mathrm{ad}(J'_r)$ *form a representative system of the conjugate classes of the inner group involutions of* $SO(2r)$, $SO(2r)^*$ *and* $SO(2r)^{\sim}$, *except that* (J_r) *is conjugate with* $\mathrm{ad}(J'_r)$ *in case r is odd. The system is commutative. Furthermore* $\mathrm{ad}(I_1) \circ \mathrm{ad}(P_k)$, $1 \leqq 2k \leqq r$, *form that of the outer ones of these groups. This is commutative and centralizes* $\mathrm{ad}(P_k)$, $1 \leqq 2k \leqq r$, *but not* $\mathrm{ad}(J_r)$ *or* $\mathrm{ad}(J'_r)$. (ii) *Similarly for* $SO(2r)^{\#}$ *with omission of the outer involutions.*

3.6. REMARK. That $\varepsilon^2 = 1$ is obvious if one knows that $\varepsilon = -1$ in a certain Clifford algebra into which $SO(n)^{\sim}$ is embedded through the spin or half-spin representation. Let Σ_1 denote the subset $\{I \otimes J, J \otimes 1_2, K \otimes J\}$ of SO(4). The matrices in Σ_1 generate the quaternion algebra $\boldsymbol{H}$. The members of $\boldsymbol{H}$ having unit norm form a group which is homeomorphic with the 3-sphere and its Lie algebra is spanned by Σ_1. The group is therefore $Sp(1) = SO(3)^{\sim}$. Next consider another subset $\Sigma_2 = \{J \otimes I, 1_2 \otimes J, J \otimes K\}$ (which is obtained from Σ_1 by the map: $X \otimes Y \mapsto Y \otimes X$). Σ_1 together with Σ_2 spans the Lie algebra of SO(4), which is $Sp(1) \cdot Sp(1) = SO(4)^{\sim}/\{1, \varepsilon\}$, where $2\omega_1 \mapsto \varepsilon = (\delta_2, \delta_2 \varepsilon)$ in $SO(4)^{\sim} = Sp(1) \times Sp(1)$ and $\varepsilon_1 + \varepsilon_2 \mapsto \delta_2$ in the Sp(1) containing Σ_1, while $\varepsilon_1 - \varepsilon_2 \mapsto \delta_2 \varepsilon$ in the other Sp(1) containing Σ_2. Both δ_2 and $\delta_2 \varepsilon$ appear as -1 in SO(4). Thus ε is involutive. The monomorphism f_n: $SO(n)^{\sim} \to SO(n+1)^{\sim}$ carries ε into ε, f_n is the lift of the standard inclusion: $SO(n) \to SO(n+1)$ induced by the one: $\boldsymbol{R}^n \to \boldsymbol{R}^{n+1} = \boldsymbol{R}^n \oplus \boldsymbol{R}$: $x \mapsto x + 0$. In this sense ε is independent of r, as opposed to δ_r.

3.7. REMARK. This is a good place to give more geometric illustrations of the case 2° in Corollary 2.13. One knows $2\omega_1 \mapsto \varepsilon$ in $SO(n)^{\sim}$ (Case 2°). ε is a pole of 1, and the geodesic arc $g[1, \varepsilon\,; 2\omega_1]$ from 1 to ε in the direction of $2\omega_1$ meets the oriented Grassmannian $G_2^o(\boldsymbol{R}^n) = G_2(\boldsymbol{R}^n)^{\sim}$, a component of the centrosome $C(1, \varepsilon)$ (See 1.5b), at the midpoint. In $SO(n)$, one has $\omega_1 \mapsto P_1 \in G_2(\boldsymbol{R}^n)$; $g[1, P_1\,; \omega_1]$ joins 1 to P_1 on the polar $G_2(\boldsymbol{R}^n)$ (Case 1°). Also $\omega_{r-1} + \omega_r \mapsto P_{r-1}$ in $SO(2r)$ (Case 2°); P_{r-1} lies on the polar $G_{2r-2}(\boldsymbol{R}^n)$, $n = 2r$. The projection π of $SO(n)$ onto $SO(n)^*$ carries the two distinct polars $G_2(\boldsymbol{R}^n)$ and $G_{2r-2}(\boldsymbol{R}^n)$ onto a single polar $\pi G_2(\boldsymbol{R}^n)$ just as π does P_1 and P_{r-1} onto a singleton; thus both $\pi g[1, P_1\,; \omega_1]$ and $\pi g[1, P_{r-1}\,; \omega_{r-1} + \omega_r]$ lead to the polar $\pi G_2(\boldsymbol{R}^n)$ but the former is shorter. For the final example, one has $2\omega_r \mapsto -1$ in $SO(2r)$ and $\omega_1 + \omega_r \mapsto$[a point p in the polar $DIII(r)^*$] in $SO(2r)^{\#}$, r even, (both in Case 2°); $g[1, -1\,; 2\omega_r]$ meets a component $DIII(r)$ of $C(1, -1)$ in $SO(2r)$ at the midpoint. In $SO(2r)^*$, r even, the projections of $g[1, J_r\,; \omega_r]$ and $g[1, p\,; \omega_1 + \omega_r]$ lead to a single polar $DIII(r)^*$.

We now work on the other classical groups. It is important to fix the position of the subspace located in the ambient space in the sequel. We place $U(n)$ at $F(\iota, SO(2n))$, where $\iota := \mathrm{ad}(J_n) = \mathrm{ad}(J \otimes 1_n)$; in other words, a unitary matrix $A + iC$, A its real part and C the imaginary one, is identified with $1_2 \otimes A + J \otimes C$ in $SO(2n)$. Then we may choose the maximal torus $A(\mathrm{U}(n))$ of $\mathrm{U}(n)$ to be the same as $A(\mathrm{SO}(2n))$ fixed earlier; and that of the subgroup $\mathrm{SU}(n)$ as its subtorus. Also we use the same orthonormal basis (ε_j), $1 \leqq j \leqq n$, for its Lie algebra of $\mathfrak{a}(\mathrm{U}(n))$ as the one for that of $A(\mathrm{SO}(2n)) = A(\mathrm{U}(n))$. $\mathfrak{a}(\mathrm{SU}(n))$ is given by the equation $\sum p^i = 0$ in $\mathfrak{a}(\mathrm{U}(n))$. The highest root $\alpha^\sim$ for $\mathrm{SU}(n)$ is $\varepsilon_1 - \varepsilon_n = \alpha_1 + \alpha_2 + \cdots + \alpha_r$, $r = n-1$; hence G-Aut$(\mathrm{SU}(n))_{(1)}$ has index 2 in G-Aut$(\mathrm{SU}(n))$, $n > 1$. G-Aut$(\mathrm{SU}(n))$ is the "inner" automorphism group of the normalizer $\{1, K_n\}\mathrm{SU}(n)$ of $\mathrm{SU}(n)$ in $\mathrm{SO}(2n)$, $K_n := K \otimes 1_n$, and it is $F(\iota, \mathrm{SO}(2n)^*)$. Indeed $\mathrm{ad}(K_n)$ acts on $\mathrm{U}(n)$ as the complex conjugation κ. The polar of 1 which contains K_n is called $\mathrm{AI}(n) = \mathrm{SU}(n)/\mathrm{SO}(n)$. If $n = 2m$ is even, there is another polar, $\mathrm{AII}(n) = \mathrm{SU}(n)/\mathrm{Sp}(m)$, which contains $K \otimes J \otimes 1_m$. $\mathrm{SU}(n)$ is 1-connected and the center is $\{\zeta^k \otimes 1_n \mid k \in \boldsymbol{N}, 0 \leqq k < n\}$, where ζ is the 2×2 matrix $\exp(2\pi J/n)$. A shortest geodesic to $\zeta^k 1_n$ is tangent to the k-th fundamental weight $H^k = \omega_k = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k - (k/n)(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)$, $1 \leqq k \leqq r$. One sees $2\omega_k \mapsto \zeta^k 1_n$ in $\mathrm{SU}(n)$ and $\omega_k \mapsto \mathrm{ad}(P_k)$ in $\mathrm{SU}(n)^*$, $1 \leqq k \leqq r$. One has, however, $\mathrm{ad}(P_{n-k}) = \mathrm{ad}(P_k)$. If $\mathrm{SU}(n)$ were thought of as a group of $n \times n$ complex matrices, $\mathrm{ad}(P_k)$ should be written $\mathrm{ad}(I_k)$. $\omega_j + \omega_k$, $j < k$, gives $\mathrm{ad}(P_{k-j})$. Since the center of $\mathrm{SU}(n)$ is cyclic, every involution of $\mathrm{SU}(n)$ can act on every quotient group of $\mathrm{SU}(n)$. We have proved.

3.8. PROPOSITION (Case of A_r). (i) *In case n is odd,* $\mathrm{ad}(P_k)$, $1 \leqq 2k \leqq r$, *and* $\mathrm{ad}(K_n)$ *form a representative system of the conjugate classes of the group involutions of the covering groups* $\mathrm{SU}(n)/\boldsymbol{Z}_p$ *of* $\mathrm{SU}(n)^*$ *and it is commutative.* (ii) *In case* $n = 2m$ is even, *those involutions together with* $\mathrm{ad}(K \otimes J \otimes 1_m)$ *form a system of the same property.* (iii) *Whether or not n is even,* G-Inv$(\mathrm{SU}(n)/\boldsymbol{Z}_p)$ *is bijective with the restrictions of* $F(\mathrm{ad}(\mathrm{ad}(J_n))$, G-Inv$(\mathrm{SO}(2n)))$ *to* $\mathrm{SU}(n)$; *that is, the restriction of the centralizer of* $\mathrm{ad}(J_n)$ *in* G-Inv$(\mathrm{SO}(2n))$ *gives* G-Inv$(\mathrm{SU}(n)^*)$ *bijectively and every group involution of* $\mathrm{SU}(n)^*$ *lifts to an involution of every covering group.*

We place $\mathrm{Sp}(m)$ at $F(\mathrm{ad}(K \otimes J \otimes 1_m), \mathrm{SU}(2m))$, where $\mathrm{SU}(2m) = F(\mathrm{ad}(J_{2m}), \mathrm{SO}(4m))$. This $\mathrm{Sp}(m)$ may be written $F(\kappa \circ \mathrm{ad}(J \otimes 1_m), \mathrm{SU}(2m))$, where $J \otimes 1_m$ is a member of $\mathrm{SO}(2m) \subset \mathrm{SU}(2m)$. We use $F(\kappa \circ \mathrm{ad}(J_m), A(\mathrm{SU}(2m)))$ as our maximal torus $A(\mathrm{Sp}(m))$. We write ε_k for $\varepsilon_k - \varepsilon_{k+m}$, $1 \leqq k \leqq m$, and call them orthonormal in the Lie algebra $\mathfrak{a}(\mathrm{Sp}(m))$. The highest root $\alpha^\sim$ is $2\varepsilon_1 = 2(\alpha_1 + \alpha_2 + \cdots + \alpha_{r-1}) + \alpha_r$ and $\omega_k = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k$, $1 \leqq k \leqq m$. There is no outer involution. The center of the 1-connected group $\mathrm{Sp}(m)$ is $\{\pm 1\}$. To ω_k there corresponds $\mathrm{ad}(P_k)$ restricted to $\mathrm{Sp}(m)$, $1 \leqq k < m$, where $P_k \in \mathrm{SO}(2m)$. To $H^m = \frac{1}{2}\omega_m$ there corresponds $\mathrm{ad}(J_m)$, where $J_m \in \mathrm{SO}(2m)$; and one has $2H^m \mapsto -1$.

3.9. PROPOSITION (Case of C_r). (i) $\mathrm{ad}(P_k)$, $1 \leqq k < r$, *and* $\mathrm{ad}(J_m)$ *form a representative system of the group involutions of conjugate classes of* $\mathrm{Sp}(m)$ *and* $\mathrm{Sp}(m)^*$ *and it is commutative.* (ii) G-Inv$(\mathrm{Sp}(m))$ *is bijective with the restrictions of* $F(\mathrm{ad}(\mathrm{ad}(\mathrm{ad}(K \otimes J_m))))$, G-Inv$(\mathrm{SU}(2m)))$ *to* $\mathrm{Sp}(m)$.

Now we turn to the exceptional groups. We assume some knowledge of E_8 including its existence, but not of E_6, E_7 or F_4. We denote by these both the compact 1-connected groups and their root systems. E_8 contains $SO(16)^\#$ and we will show that all the group involutions of E_8 are conjugate to those of $SO(16)^\#$ extended to E_8.

3.10. PROPOSITION (Case of E_8). *E_8 has two involutions, denoted by σ^{VIII} and σ^{IX} corresponding to the two polars; $\sigma^{VIII}:=\mathrm{ad}(\varepsilon)$ and $\sigma^{IX}:=\mathrm{ad}(\varepsilon P_2)$ in terms of members of the subgroup $SO(16)^\#=F(\mathrm{ad}(\varepsilon), E_8)$ (See the proof). σ^{VIII} and σ^{IX} make a commutative representative system of the conjugate classes of the group involutions of E_8.*

PROOF. We write $\alpha(q^1, \cdots, q^r)$ for a linear combination $q^1\alpha_1+\cdots+q^r\alpha_r$ of the simple roots and, similarly, $\varepsilon(p^1, \cdots, p^r)$ for $p^1\varepsilon_1+\cdots+p^r\varepsilon_r$. Then the highest root $\alpha^\sim=\alpha^\sim(E_8)$ is α (2, 3, 4, 6, 5, 4, 3, 2). There is no outer involution. By Theorem 2.11, there is a meridian $M^-(p)$ for the polar $M^+(p)$ of the unit element 1 that corresponds to the first number 2 in the above expression of $\alpha^\sim$; $M^-(p)=F(\mathrm{ad}(p), E_8)$, the centralizer of p. Its Lie algebra is D_8. We choose a common Cartan subalgebra $\mathfrak{a}(D_8)=\mathfrak{a}(E_8)$. Besides we make $\varepsilon_1, \cdots, \varepsilon_j, \cdots, \varepsilon_8$, the orthonormal basis vectors for $\mathfrak{a}(E_8)$, correspond to the vectors $\varepsilon_8, \varepsilon_7, \cdots, \varepsilon_k, \cdots, \varepsilon_2, -\varepsilon_1$, considered earlier for $D_8, j+k=9$ for $1\leqq j\leqq 7$, notationwise, referring to the tables at the end of Chap. 6 of [B]; thus the root system $R(E_8)$ is the union of $R(D_8)$ and the weights of the half-spin representation $\omega_7=\omega_7(D_8)$ of D_8. One sees that $M^-(p)$ is the semi-spinor group $SO(16)^\sim/\{1, \delta\varepsilon\}$ which we denote by $SO(16)^\#$. It meets the polar $M^+(p)$ at ε, the only pole of 1 in $SO(16)^\#$; hence $p=\varepsilon$. The polar is called EVIII. σ^{VIII} is $\mathrm{ad}(\varepsilon)$ acting on E_8 by definition. $\omega_7(D_8)$, etc. being abbreviated to ω_7, etc. one sees $\omega_1(E_8)=-2\omega_1$, $\omega_2(E_8)=\omega_8-3\omega_1$, $\omega_3(E_8)=\omega_7-4\omega_1$, $\omega_4(E_8)=\omega_6-6\omega_1$, $\omega_5(E_8)=\omega_5-5\omega_1$, $\omega_6(E_8)=\omega_4-4\omega_1$, $\omega_7(E_8)=\omega_3-3\omega_1$, and $\omega_8(E_8)=\omega_2-2\omega_1$. Next from the second coefficient 2 in $\alpha^\sim$ and Theorem 2.11, one obtains another meridian $M^-(q)$, where the point q lies in the direction of $\omega_8(E_8)=\omega_2-2\omega_1$, a root of D_8. Again by Theorem 2.11, $M^-(q)$ is a dot product of $Sp(1)$ and a simple group with the root system E_7. q is the common pole of 1 in $Sp(1)$ and the group E_7; one sees q equals $\varepsilon 1_2\otimes I_2=\varepsilon P_2$. σ^{IX} is $\mathrm{ad}(\varepsilon P_2)$ by definition. The polar $M^+(q)$ is called EIX. σ^{VIII} commutes with σ^{IX}, since they have been defined by two members of a maximal torus. They form a representative system of the congruence classes of the group involutions of E_8 by Theorem 2.11. QED

3.11. REMARK. Clearly εP_2 is conjugate with P_2, εI_4 and I_4 in $SO(16)^\#$; indeed all these lie in the polar $G_4^o(\boldsymbol{R}^{16})$ of 1 in $SO(16)^\#$. In particular, σ^{VIII} is conjugate with $\mathrm{ad}(P_2)$ even in $SO(16)^\#$. We record $SO(16)^\#=F(\sigma^{VIII}, E_8)$ and $Sp(1)\cdot E_7=F(\sigma^{IX}, E_8)$. These are the meridians for the polars EVIII and EIX respectively.

3.12. REMARK. There are many ways of checking results because of abundant interrelationship between the spaces. Here is a simple example. It is not hard to see that, among the polars $\{\varepsilon\}$, 2 copies of DIII(8)*, $G_4^o(\boldsymbol{R}^{16})$ and $G_8(\boldsymbol{R}^{16})^\#$ of $SO(16)^\#$, one of DIII(8)* and $G_4^o(\boldsymbol{R}^{16})$ lie in EIX; indeed $\omega_7(D_8)$ and $\alpha_1(D_8)$ are conjugate in E_8 (or with respect to its Weyl group $W(E_8)$) so that this DIII(8)* in EIX contains $J\otimes I_1$ (on the 1-parameter subgroup with the initial tangent $\omega_7(\mathrm{D}_8)$) and the other DIII(8)* contains $\varepsilon J\otimes I_1$. Assuming the knowledge of the Euler numbers of the polars of $\mathrm{SO}(16)^{\#}$, one finds that the Euler number χEIX is $120=56+64$ and $\chi\mathrm{EVIII}=1+64+70=135$; thus $1+\chi\mathrm{EVIII}+\chi\mathrm{EIX}$ equals 2^8, as it should.

3.13. PROPOSITION (Case of E_7). *E_7 has three group involutions, σ^{V}, σ^{VI} and σ^{VII} corresponding to the 3 polars in $\mathrm{E}_7^{\#}=\mathrm{E}_7/\boldsymbol{Z}_2$. Here $\sigma^{\mathrm{V}}:=\mathrm{ad}(K\otimes 1_8)|\mathrm{E}_7$; σ^{VI} is $\mathrm{ad}(\varepsilon)=\sigma^{\mathrm{VIII}}$ restricted to E_7; and σ^{VII} is $\mathrm{ad}(J_2P_1)|\mathrm{E}_7$, where J_2 is a member of $\mathrm{Sp}(1)$ (that is, the member $J\otimes 1_2$ of Σ_1 in Remark 3.6) and P_1 is that of $\mathrm{SO}(12)^{\sim}$, both of which are subgroups of $\mathrm{Sp}(1)\cdot\mathrm{SO}(12)^{\sim}\subset\mathrm{E}_7\cap\mathrm{SO}(4)^{\sim}\cdot\mathrm{SO}(12)^{\sim}\subset\mathrm{SO}(16)^{\#}$, if E_7 is located in $F(\sigma^{\mathrm{IX}},\mathrm{E}_8)$. σ^{V}, σ^{VI} and σ^{VII} make a commutative representative system of the conjugate classes of the group involutions of E_7 and of $\mathrm{E}_7^{\#}$.*

PROOF. We place E_7 at the position in E_8 specified in the above. Thus a Cartan subalgebra $\mathfrak{a}(\mathrm{E}_7)$ of E_7 is given by the equation $p^1=p^2$, i.e. one has $\mathfrak{a}(\mathrm{E}_7)=\{\varepsilon(p^1,\cdots,p^8)\in\mathfrak{a}(\mathrm{E}_8)\mid p^1=p^2\}$. We choose a system of simple roots $\alpha_1,\cdots,\alpha_7$, just omitting α_8 from those in $R(\mathrm{E}_8)$. One has the fundamental weights $\omega_1(\mathrm{E}_7)=-\omega_2=-\omega_2(\mathrm{D}_8)$, $\omega_2(\mathrm{E}_7)=\omega_8-(3/2)\omega_2$, $\omega_3(\mathrm{E}_7)=\omega_7-2\omega_2$, $\omega_4(\mathrm{E}_7)=\omega_6-3\omega_2$, $\omega_5(\mathrm{E}_7)=\omega_5-(5/2)\omega_2$, $\omega_6(\mathrm{E}_7)=\omega_4-2\omega_2$, and $\omega_7(\mathrm{E}_7)=\omega_3-(3/2)\omega_2$, where the right hand sides involve fundamental weights of D_8 only. Now that the location of E_7 has been fixed in E_8 together with its Cartan subalgebra, we are ready to determine the polars of $\mathrm{E}_7^{\#}$ (which are all contained in those of E_8 somehow) and the involutions of $\mathrm{E}_7^{\#}$ by heavy use of Theorem 2.11 as in the foregoing proof. The highest root $\alpha^{\sim}(\mathrm{E}_7)$ is $\omega_1(\mathrm{E}_7)=\alpha(2,2,3,4,3,2,1)$. There is no outer involution; in particular every involution is the restriction of that of E_8 to E_7.

The first coefficient 2 gives the meridian $\mathrm{Sp}(1)\cdot\mathrm{SO}(12)^{\sim}=\mathrm{E}_7\cap\mathrm{SO}(16)^{\#}$. This meets the polar, called EVI, at P_2, since $\omega_1(\mathrm{E}_7)=-\omega_2\mapsto P_2$. It is more convenient to denote this P_2 by δ_2, which is the pole of 1 in $\mathrm{Sp}(1)$. One has $\mathrm{Sp}(1)\cdot\mathrm{SO}(12)^{\sim}=(\mathrm{Sp}(1)\times\mathrm{SO}(12)^{\sim})/\{1,(\delta_2,\delta_6\varepsilon)\}$, as one knows $\delta_2=\delta_6\varepsilon$ in $\mathrm{SO}(16)^{\#}$. The Cartan subalgebra $\mathfrak{a}(\mathrm{D}_6)$ of $\mathrm{SO}(12)^{\sim}$ is spanned by ε_3, ε_4, ε_5, ε_6, ε_7 and ε_8, that is, $\mathfrak{a}(\mathrm{D}_6)\perp\{\varepsilon_1,\varepsilon_2\}$. The poles of 1 in $\mathrm{Sp}(1)\cdot\mathrm{SO}(12)^{\sim}$ are identified with ε, δ_6 and $\delta_6\varepsilon$ in $\mathrm{SO}(12)^{\sim}$. One has $2\omega_6(\mathrm{D}_6)=\varepsilon_3+\cdots+\varepsilon_8\mapsto\delta_6$ in E_7, while $\omega_1(\mathrm{E}_7)\mapsto\delta_2=\delta_6\varepsilon$. Since $\omega_8(\mathrm{E}_8)\mapsto\delta_6$, δ_6 is the pole in E_7 by the definition of E_7; the center of E_7 is $\{1,\delta_6\}$.

The second coefficient 2, thus, gives the meridian with the root system A_7 in the bottom space $\mathrm{E}_7^{\#}$ but not in E_7, since $2\omega_2(\mathrm{E}_7)\mapsto\delta_6$. In E_7 the centrosome has the component, called EV, in correspondence with $\omega_2(\mathrm{E}_7)$. The meridian in $\mathrm{E}_7^{\#}$ is $\mathrm{SU}(8)/\boldsymbol{Z}_4$ because of the relative position of the root $\alpha_2(\mathrm{E}_7)$ to the Dynkin diagram of A_7 within the extended Dynkin diagram of E_7; indeed $\alpha_2(\mathrm{E}_7)$ indicates the isotropy representation $\omega_4(\mathrm{A}_7)$ for EV, or $\alpha_2(\mathrm{E}_7)=-\omega_4(\mathrm{A}_7)$ with respect to an appropriate basis for $\mathfrak{a}(\mathrm{E}_7)=\mathfrak{a}(\mathrm{A}_7)$.

The 6-th coefficient 2 in $\alpha^{\sim}(\mathrm{E}_7)$ gives the other polar EVI in E_7, which can share the meridian $\mathrm{Sp}(1)\cdot\mathrm{SO}(12)^{\sim}$ with the first EVI because of the other pole ε of 1 in

$\mathrm{Sp}(1)\cdot\mathrm{SO}(12)^\sim$. This EVI is therefore a subspace of EVIII, while the first EVI $\ni \delta_2$ (which is P_2 in $\mathrm{SO}(16)^\#$) is contained in EIX. The pole δ_6 of 1 in E_7 gives rise to the covering transformation for the double covering map; $\mathrm{E}_7 \to \mathrm{E}_7^*$, and the translation δ_6 permutes the two polars, since it carries $\delta_2 = \delta_6 \varepsilon$ into ε. Hence the covering map projects the two polars onto a single polar EVI in E_7^*. We choose $\sigma^{\mathrm{VI}} := \mathrm{ad}(\varepsilon)|\mathrm{E}_7$.

The 7-th coefficient 1 in $\alpha^\sim(\mathrm{E}_7)$ gives the subgroup $T\cdot\mathrm{E}_6$, $T \cong \mathrm{U}(1)$, and the other component of the centrosome $C(1, \delta_6)$, denoted by EVII, which project respectively to a meridian and the corresponding polar in E_7^*. Here E_6 is of course the 1-connected group of the root system E_6. $T\cdot\mathrm{E}_6$ denotes $(T\times\mathrm{E}_6)/\boldsymbol{Z}_3$; indeed $\omega_7(\mathrm{E}_7)$ is an initial tangent to T and orthogonal to $\mathfrak{a}(\mathrm{E}_6)$ in $\mathfrak{a}(\mathrm{E}_7)$, whence the orthogonal projection of $\alpha^\sim(\mathrm{E}_7)$ into $\mathfrak{a}(\mathrm{E}_6)$ is $-(2/3)\omega_3 = \omega_1(\mathrm{E}_6)$, the denominator suggesting the center $\boldsymbol{Z}_3$ of E_6; also $\mathfrak{a}(\mathrm{E}_6)$ is defined by the equation $p^2 = p^3$ in $\mathfrak{a}(\mathrm{E}_7)$. We add that σ^{VII} is conjugate with $\mathrm{ad}(J_6)$, $J_6 \in \mathrm{SO}(12)^\sim$, since the vector $\omega_7(\mathrm{E}_7)$ in $\mathfrak{a}(\mathrm{E}_7)$ is carried into $\omega_6(\mathrm{D}_6)$ by the reflection with respect to the root $\frac{1}{2}\varepsilon(1, 1, -1, 1, 1, 1, 1)$ of E_7. For another consequence, J_6 lies on DIII(6), a component of $\mathrm{Sp}(1)\cdot\mathrm{SO}(12)^\sim \cap$ EVII, and J_2P_1 does on $S^2\cdot G_2^\circ(\boldsymbol{R}^{12})$, the other component.

Now $K_8 := K\otimes 1_8$ is an involutive member of $\mathrm{SO}(16)^\#$. $\mathrm{ad}(K_8)$ stabilizes our maximal torus $A = A(\mathrm{E}_8)$ and acts on it as the symmetry s_1. Hence $\mathrm{ad}(K_8)$ fixes every involutive member of A. In particular it stabilizes E_7; K_8 is a member of $\mathrm{Sp}(1)\cdot\mathrm{E}_7$ and not of E_7. Also it follows that the subspace $F(s_1\circ\mathrm{ad}(K_8), G)_{(1)}$ of G has equal rank to that of G for any compact $\mathrm{ad}(K_8)$-invariant subgroup G of E_8 with a maximal torus contained in A. This applies to E_7 as well as any group $F(\mathrm{S}, \mathrm{E}_8)$ defined for a finite set S of $\mathrm{ad}(t)$, t involutive members of A. Therefore the subspace $F(s_1\circ\sigma^{\mathrm{V}}, \mathrm{E}_7)_{(1)} =:$ EV of E_7 has rank $=7$ and shares the root system with E_7.

Finally, we come to the statement on the representative system. In view of Theorem 2.11, there are involutive members of E_7^* in the directions of $\omega_1(\mathrm{E}_7)$, $\omega_2(\mathrm{E}_7)$, $\omega_6(\mathrm{E}_7)$, $\omega_7(\mathrm{E}_7)$ and nothing more up to conjugacy. $\omega_1(\mathrm{E}_7)$ corresponds to σ^{VI}, since $\omega_1(\mathrm{E}_7)\mapsto\delta_2$ which projects to the same element of E_7^* as ε does, that is, one has $\omega_1(\mathrm{E}_7)\mapsto\varepsilon$ in E_7^*. Since $\omega_6(\mathrm{E}_7)\mapsto\delta_2P_2 = \delta_6\varepsilon P_2$ in E_7 hence $\omega_6(\mathrm{E}_7)\mapsto\varepsilon P_2$ in E_7^*, we assert that ω_6 gives ε; in fact $2\varepsilon_3\mapsto\varepsilon$ and $2\varepsilon_3$ is conjugate with ω_6 by the Weyl group of E_7, as one easily sees. Thus both $\omega_1(\mathrm{E}_7)$ and $\omega_6(\mathrm{E}_7)$ correspond to the class of σ^{VI}. One sees that $\omega_7(\mathrm{E}_7)$ gives σ^{VII}, since $\omega_7(\mathrm{E}_7) = \omega_3 - (3/2)\omega_2 = -\frac{1}{2}\omega_2 + \varepsilon_3 = -\frac{1}{2}\omega_2 + \omega_1(\mathrm{D}_6)\mapsto J_2P_1$. Therefore $\omega_2(\mathrm{E}_7)$ must give σ^{V}; this can be checked directly with computations. QED

3.14. Remark. We record $F(\sigma^{\mathrm{VI}}, \mathrm{E}_7) = \mathrm{Sp}(1)\cdot\mathrm{SO}(12)^\sim$. One has $\mathrm{U}(1)\cdot\mathrm{E}_6 = F(\sigma^{\mathrm{VII}}, \mathrm{E}_7)$. σ^{VII} is conjugate with $\mathrm{ad}(J_6)$, $J_6\in\mathrm{SO}(12)^\sim$. We add some explanations without proof. The involution $\mathrm{ad}(K_8)$ on E_7^* equals $\mathrm{ad}(b)$ for a member b in $S^2\cdot G_6(\boldsymbol{R}^{12})^\# \subset \mathrm{EV}^* \subset \mathrm{E}_7^*$. The involutive member c of E_7^* in the direction of $\omega_2(\mathrm{E}_7)$ lies in $\mathrm{DIII}(6)^* \subset \mathrm{EV}^*$. The intersection $\mathrm{SO}(4)^\sim\cdot\mathrm{SO}(12)^\# \cap \mathrm{EV}^*$ is the disjoint union of these two subspaces; in particular the Euler number $\chi\mathrm{EV}^* = \chi S^2\cdot G_6(\boldsymbol{R}^{12})^\# + \chi$ $\mathrm{DIII}(6)^* = 20 + 16 = 36$. Similarly $\mathrm{SO}(4)^\sim\cdot\mathrm{SO}(12)^\#$ meets EVII^* in the disjoint union of

$S^2 \cdot G_2^\circ(\boldsymbol{R}^{12})$ and DIII(6)*; thus χEVII* $=12+16=28$. The same group $\mathrm{SO}(4)^\sim \cdot \mathrm{SO}(12)^\#$ meets the other polar EVI in the disjoint union of $S^2 \cdot$DIII(6), $G_4^\circ(\boldsymbol{R}^{12})$ and a single point; χEVI $=32+30+1=63$. Finally the sum χEV* $+\chi$EVII* $+\chi$EVI $+\chi\{1\}=2^7$, cross-checking our result partly.

3.15. REMARK. We describe the polars of EV* for a later use. Their meridians have the same root systems with multiplicity one and the same fundamental groups as those of the meridians in E_7^*, hence they are AI(8)$/\boldsymbol{Z}_4$, $S^2 \cdot G_6(\boldsymbol{R}^{12})^\#$ and $T\cdot$EI in correspondence with $\mathrm{SU}(8)/\boldsymbol{Z}_4$, $\mathrm{Sp}(1)\cdot\mathrm{SO}(12)^\#$ and $T\cdot\mathrm{E}^6$ respectively in E_7^*, where EI is the space with the root system of E_6 with multiplicity 1. The c-orthogonal to AI(8)$/\boldsymbol{Z}_4$ in EV* is isomorphic with itself by the explained property of $\mathrm{ad}(K_8)$. The isotropy subgroup for U(1)$\cdot$EI (and for EI) is Sp(4)*, which, placed as the c-orthogonal to EI at 1 in E^6, meets the polars, EII and EIII, of 1 in E^6 at polars CI(4)* $\amalg$ $G_1(\boldsymbol{H}^4)$ and the polar $G_2(\boldsymbol{H}^4)^*$ in Sp(4)*; CI(4)* and $G_2(\boldsymbol{H}^4)^*$ are the polars in EI.

3.15a. REMARK. The fixed point set $F(\sigma^{\mathrm{V}}, \mathrm{E}_7^*)=2\times\mathrm{SU}(8)/\boldsymbol{Z}_4$ (2 copies). It cannot be connected, since $F(\sigma^{\mathrm{V}}, \mathrm{E}_7^*)_{(1)}=\mathrm{SU}(8)/\boldsymbol{Z}_4$ is the meridian to EV* which has the polars $G_4(\boldsymbol{C}^8)^*$, AI(8)$/\boldsymbol{Z}_4$ and AII(4)*, but the last two cannot be polars of $\mathrm{SU}(8)/\boldsymbol{Z}_4$.

3.15b. REMARK. We have shown that the centrosome $C(1, \delta_6)$ in E_7 is the disjoint union of EV and EVII. Since these are not isomorphic with each other, the projection π of E_7 onto E_7^* restricts to double covering maps of EV and EVII. Therefore the polars in E_7^* are π(EV), EVI and π(EVII). Moreover π(EV) and π(EVII) are the bottoms EV* and EVII* by 1.9.

3.16. REMARK. The polars EVI and $S^2\cdot$EV of 1 in Sp(1)$\cdot\mathrm{E}_7$ are contained in the polar EVIII in E_8, while the polars EVI and $S^2\cdot$EVII are in EIX. The ratio $d(1, \mathrm{EVIII})^2 : d(1, \mathrm{EIX})^2$ of the distance squared from the unit element to the polars is $2:1$, by 2.14 and $\|\omega_1(\mathrm{E}_8)\|^2 : \|\omega_8(\mathrm{E}_8)\|^2=2:1$.

3.17. PROPOSITION (Case of E_6). E_6 *admits four group-involutions* σ^{I}, σ^{II}, σ^{III} *and* σ^{IV} *which constitute a complete system of the conjugate classes. Here* $\sigma^{\mathrm{I}}:=\mathrm{ad}(K_8)|\mathrm{E}_6=\sigma^{\mathrm{V}}|\mathrm{E}_6$; $\sigma^{\mathrm{II}}:=\mathrm{ad}(P_2)|\mathrm{E}_6$; σ^{III} *is the restriction* $\mathrm{ad}(\varepsilon)|\mathrm{E}_6$; *and* $\sigma^{\mathrm{IV}}:=\mathrm{ad}(Q_2)|\mathrm{E}_6$, Q_2 *denoting the member of* $\mathrm{SO}(16)^\#$ *which corresponds to* $K\otimes(1_4\oplus 0_4)+1_2\otimes(0_4\oplus 1_4)$ *in* SO(16) *if* E_6 *is located in* $F(\sigma^{\mathrm{VII}}, \mathrm{E}_7)$. *The inner ones among these are* σ^{II} *and* σ^{III}. *These* σ^{I}, σ^{II}, σ^{III} *and* σ^{IV} *commute with each other.* Q_2 is conjugate with P_2 and I_4 in $\mathrm{SO}(16)^\#$.

PROOF. We place E_6 at the position in E_7 as stated; thus our Cartan subalgebra $\mathfrak{a}(\mathrm{E}_6)$ of E_6 is given by the equation $p^2=p^3$ in $\mathfrak{a}(\mathrm{E}_7)$, that is, $\mathfrak{a}(\mathrm{E}_6)=\{p^1\varepsilon_1+\cdots+p^8\varepsilon_8 \mid p^1=p^2=p^3\}$. We choose a system of simple roots $\alpha_1, \cdots, \alpha_6$, just omitting α_7 from those in $R(\mathrm{E}_7)$. One has the fundamental weights $\omega_1(\mathrm{E}_6)=-(2/3)\omega_3$, $\omega_2(\mathrm{E}_6)=\omega_8-\omega_3$, $\omega_3(\mathrm{E}_6)=\omega_7-(4/3)\omega_3$, $\omega_4(\mathrm{E}_6)=\omega_6-2\omega_3$, $\omega_5(\mathrm{E}_6)=\omega_5-(5/3)\omega_3$ and $\omega_6(\mathrm{E}_6)=\omega_4-(4/3)\omega_3$, where the right hand sides involve fundamental weights of D_8

only. The highest root $\alpha^{\sim}(E_6)$ is $\omega_2(E_6)=\alpha(1, 2, 2, 3, 2, 1)$. G-Aut$(E_6)$ is the extension of G-Aut$(E_6)_{(1)}$ by $\{1, \sigma^1\}$; in fact the composite $\sigma^1 \circ w_o$ realizes the symmetry of the Dynkin diagram, where w_o denotes the member of the Weyl group that has the greatest length [B]. This outer involution makes $\omega_1(E_6)$ conjugate with $\omega_6(E_6)$ and $\omega_3(E_6)$ with $\omega_5(E_6)$. One sees that the center of E_6 has order 3 and is generated by a member in the direction of $2\omega_1(E_6)$; E_6 has no pole.

The first coefficient 1 in $\alpha^{\sim}(E_6)$ indicates the meridian $U(1)\cdot SO(10)^{\sim}$ and the corresponding involutive member b lies in $U(1)$ which has a tangent vector $\omega_1(E_6)$. The subgroup $U(1)\cdot SO(10)^{\sim}$ of E_6 is contained in $SO(6)^{\sim}\cdot SO(10)^{\sim}$, that of $SO(16)^{\#}$. Hence one finds $b=\varepsilon$; thus $3\omega_1(E_6)\mapsto\varepsilon$. We might add that the subgroup $U(1)\cdot SO(10)^{\sim}$ projects onto the isomorphic group $(U(1)/\boldsymbol{Z}_3)\cdot SO(10)^{\sim}$ of E_6^*. The polar is denoted by EIII, which is a subspace of EVIII.

The second coefficient 2 in $\alpha^{\sim}(E_6)$ indicates the meridian $Sp(1)\cdot SU(6)$. $\omega_2(E_6)$ is tangent to $Sp(1)$. The polar which $\omega_2(E_6)=\alpha^{\sim}(E_6)$ defines is denoted by EII ; EII is a subspace of EIX.

The third coefficient 2 gives the same polar EII. In fact $3\omega_3(E_6)$ is congruent with the root $\alpha_1(E_6)$ modulo the unit lattice of E_6. Similarly for $\omega_5(E_6)$; $3\omega_5(E_6)$ is congruent with the root $\alpha_6(E_6)$. Also $\omega_6(E_6)$ is congruent with $\alpha_2(E_6)+\alpha_5(E_6)$; hence it leads to the polar $G_8^{\circ}(\boldsymbol{R}^{10})$ in $SO(10)^{\sim}$. $G_8^{\circ}(\boldsymbol{R}^{10})$ is contained in EIII; in fact w_o carries $\omega_1(E_6)$ to $-\omega_6(E_6)$.

Therefore E_6 and E_6^* have exactly two polars $(\neq\{1\})$ EII and EIII; EII corresponds to $\omega_2(E_6)$, $\omega_3(E_6)$ and $\omega_5(E_6)$, while EIII corresponds to $\omega_1(E_6)$ and $\omega_6(E_6)$. (As to the Euler numbers, one has $\chi\mathrm{EII}+\chi\mathrm{EIII}+1=36+27+1=2^6$.) We have classified the inner involutions of E_6.

We turn to the outer involutions. We have to determine the polars of G-Aut(E_6). To do it, we look at $F:=F(\sigma^{\mathrm{VII}}, E_7^*)$. This is not connected, since, on one hand, $F(\sigma^{\mathrm{VII}}, E_7)$ is $T\cdot E_6$, where T is the circle $U(1)$ with the initial tangent $\omega_7(E_7)$ and, on the other hand, ad(K_8) acts on T as the point symmetry s_1; thus the centrosome $C(1, \delta_6)$ in T projects into F, giving another component. For another proof, the group $T\cdot E_6$ in E_7^* meets the polar EVII* at a point p and EIII, while the space EVII* having the root system C_3 must have another polar of p, which is $T\cdot$EIV by definition. We have also proved that F acts on E_6 through the adjoint action as G-Aut(E_6); recall G-Aut(E_6) has two components. The component $F_{(q)}$ has to meet the polar EV* of the same root system as E_7 exactly for the reason why F is disconnected; one has $F_{(1)}\cap \mathrm{EV}^*=\mathrm{EII}$ and $F_{(q)}\cap \mathrm{EV}^*=T\cdot\mathrm{EI}$. Now the picture is complete for our purpose; G-Aut(E_6) contains exactly two conjugate classes of outer involutions. We will show that these are represented respectively by σ^1 and $\sigma^{\mathrm{IV}}=\mathrm{ad}(Q_2)$ as defined in the proposition, after some study of Q_2.

Q_2 is conjugate with $P_2\in G_4^{\circ}(\boldsymbol{R}^{16})\subset$ EIX in $SO(16)^{\#}$. ad(Q_2) acts on $\mathfrak{a}(E_8)$ carrying ε_j into $-\varepsilon_j$, $j\leqq 4$, and fixing ε_k, $k>4$. Hence a straightforward calculation shows that EIX has the root system $F_4=F_4(8, 1)$, which means the shorter roots have multiplicity 8 and the longer ones have multiplicity 1. In particular the space EIX has rank 4.

Therefore the space EIV $\subset$EVII$\subset$EIX has rank $\leqq 4$ ($=2$ actually). On the other hand EI corresponding to σ^1 clearly has rank $6=\mathrm{rank}(\mathrm{E}_6)$.

Finally, Q_2 commutes with K_8, both of which commute with every involutive member of our maximal torus $A(\mathrm{E}_8)$. QED

3.18. Remark. It does not take too much work to see that $F(\sigma^1, \mathrm{E}_6)=\mathrm{Sp}(4)^*$, $F(\sigma^{\mathrm{II}}, \mathrm{E}_6)=\mathrm{Sp}(1)\cdot\mathrm{SU}(6)$, $F(\sigma^{\mathrm{III}}, \mathrm{E}_6)=\mathrm{U}(1)\cdot\mathrm{SO}(10)^\sim$ and $F(\sigma^{\mathrm{IV}}, \mathrm{E}_6)=\mathrm{F}_4$ (See the next proof for F_4). Their c-orthogonal spaces are EI, EII, EIII and EIV respectively.

3.19. Proposition (Case of F_4). *Two involutions $\sigma_{\mathrm{F}}^{\mathrm{I}}$ and $\sigma_{\mathrm{F}}^{\mathrm{II}}$ make a representative system of the conjugacy classes of the group involutions of the group* F_4; *here* $\sigma_{\mathrm{F}}^{\mathrm{I}}:=\mathrm{ad}(P_2)$, $P_2\in\mathrm{SO}(8)^\sim\subset\mathrm{F}_4$, *and* $\sigma_{\mathrm{F}}^{\mathrm{II}}:=\mathrm{ad}(\varepsilon)|\mathrm{F}_4$ *if* F_4 *is positioned at* $F(\sigma^{\mathrm{IV}}, \mathrm{E}_6)$ *in* E_6. *The system* $\{\sigma_{\mathrm{F}}^{\mathrm{I}}, \sigma_{\mathrm{F}}^{\mathrm{II}}\}$ *is commutative.*

Proof. Recall that $\mathrm{ad}(Q_2)$ acts on $\mathfrak{a}(\mathrm{D}_8)$ with the fixed point set $F(\mathrm{ad}(Q_2), \mathfrak{a}(\mathrm{D}_8))$ spanned by ε_k, $5\leqq k\leqq 8$. This is also spanned by the roots $\alpha_5=\varepsilon_5-\varepsilon_6$, $\alpha_6=\varepsilon_6-\varepsilon_7$, $\alpha_7=\varepsilon_7-\varepsilon_8$, $\alpha_8=\varepsilon_7+\varepsilon_8$ of $\mathrm{SO}(16)^\#$, which form a system of simple roots of the Lie algebra D_4 of a subgroup $\mathrm{SO}(8)^\sim$. One has $F(\mathrm{ad}(Q_2), R(\mathrm{E}_6))=R(\mathrm{D}_4)$. We write $\mathfrak{a}(\mathrm{F}_4)$ for $F(\mathrm{ad}(Q_2), \mathfrak{a}(\mathrm{D}_8))$; in fact, the projection of $R(\mathrm{E}_6)$ onto $\mathfrak{a}(\mathrm{F}_4)$ is $R(\mathrm{F}_4)$, that is, $F(\mathrm{ad}(Q_2), R(\mathrm{E}_6))=\mathrm{F}_4$. Thus $R(\mathrm{F}_4)$ consists of $R(\mathrm{D}_4)$, $\pm\varepsilon_k$, $5\leqq k\leqq 8$, and $\frac{1}{2}\varepsilon(0, 0, 0, 0, \pm 1, \pm 1, \pm 1, \pm 1)$ in our setting. The highest root is $\alpha(2, 3, 4, 2)$ (in the numbering of [B]). Hence F_4 has two polars, FI and FII; the meridians are isomorphic with $\mathrm{Sp}(1)\cdot\mathrm{SP}(3)$ and $\mathrm{SO}(9)^\sim$ respectively. Now the proposition is obvious. QED

3.20. Remark. We explain more about the polars of F_4. FII has rank 1, since none of the roots $\frac{1}{2}\varepsilon(0, 0, 0, 0, \pm 1, \pm 1, \pm 1, \pm 1)$ (which make $R(\mathrm{F}_4)-R(\mathrm{B}_4)$) is strongly orthogonal to any one among themselves. On the other hand, the polars of $\mathrm{SO}(9)^\sim$ are $G_4^\circ(\boldsymbol{R}^9)$, $G_8^\circ(\boldsymbol{R}^9)=S^8$ and the pole ε. Thus S^8 is the only polar in FII, the Cayley projective plane, and $G_4^\circ(\boldsymbol{R}^9)$ is a subspace of FI. Hence the Euler number $\chi\mathrm{FI}=\chi G_4^\circ(\boldsymbol{R}^9)=12$, while $\chi\mathrm{FII}=1+\chi S^8=3$ obviously. The polars of P_2 in FI are $S^2\cdot\mathrm{CI}(3)$ and $G_2(\boldsymbol{H}^3)$, which make those of 1 in $\mathrm{Sp}(1)\cdot\mathrm{Sp}(3)$ together with P_2 and $G_1(\boldsymbol{H}^3)\subset\mathrm{FII}$.

Now we come to the last exceptional group G_2. The subgroup $\mathrm{SO}(8)^\sim$ of F_4 admits a group automorphism of $\mathrm{SO}(8)^\sim$, T, such that (1) $\mathrm{T}^3=1$ and (2) T cyclically permutes the three poles of 1 in $\mathrm{SO}(8)^\sim$. One has $\mathrm{G}_2=F(\mathrm{T}, \mathrm{SO}(8)^\sim)$ by definition. T is outer; in fact T acts on the fundamental group of $\mathrm{SO}(8)^*$ nontrivially. One may assume that T stabilizes $\mathfrak{a}(\mathrm{D}_4)=\mathfrak{a}(\mathrm{F}_4)$. By (2), T cyclically permutes the three simple roots α_5, α_7 and α_8 of D_4 and fixes α_6 (See the proof of 3.19 for these roots). It follows that the group G_2 does have the root system G_2; $\beta_1:=\alpha_6$ and $\beta_2:=\alpha_5+\alpha_7+\alpha_8$ make a system of simple roots. The highest root $\alpha^\sim(\mathrm{G}_2)$ is $3\beta_1+2\beta_2$.

3.21. Proposition (Case of G_2). *Every group involution* $\neq 1$ *of* G_2 *is conjugate with* $\mathrm{ad}(P_2)$ *of* $\mathrm{SO}(8)^*$ *restricted to* G_2.

PROOF. The automorphism T stabilizes the only remaining polar $G_4^{\circ}(\boldsymbol{R}^8)$ in $\mathrm{SO}(8)^{\sim}$, which has the root system of $\mathrm{SO}(8)^{\sim}$. Therefore the only polar in G_2 is $\mathrm{GI} := F(\mathrm{T}, G_4^{\circ}(\boldsymbol{R}^8))$. The meridian is $\mathrm{SO}(4) \cong F(\mathrm{ad}(P_2), G_2) \subset \mathrm{SO}(4)^{\sim} \cdot \mathrm{SO}(4)^{\sim}$. QED

We summarize the above discussions on the involutions of exceptional spaces from a different angle; we describe the effect of each of the involutions given by the involutive members ε, P_2, K_8 and Q_2 of $\mathrm{SO}(16)^{\#}$ or its appropriate subgroups. This will elucidate the interrelationship between exceptional spaces and the effects of involutions. Some inclusions in the diagrams below may be different from those defined earlier.

3.22. PROPOSITION (Effect of ε). *One has the commutative diagram of monomorphisms*:

$$\begin{tikzcd}
\mathrm{SO}(9)^{\sim} \arrow[r] \arrow[d] & \mathrm{SO}(10)^{\sim} \arrow[r] \arrow[d] & \mathrm{SO}(12)^{\sim} \arrow[r] \arrow[d] & \mathrm{SO}(16)^{\#} \arrow[d] \\
\mathrm{SO}(9)^{\sim} \arrow[d] & \mathrm{U}(1)\cdot\mathrm{SO}(10)^{\sim} \arrow[d] & \mathrm{Sp}(1)\cdot\mathrm{SO}(12)^{\sim} \arrow[d] & \mathrm{SO}(16)^{\#} \arrow[d] \\
\mathrm{F}_4 \arrow[r] & \mathrm{E}_6 \arrow[r] & \mathrm{E}_7 \arrow[r] & \mathrm{E}_8 \\
\mathrm{FII} \arrow[r] \arrow[u] & \mathrm{EIII} \arrow[r] \arrow[u] & \mathrm{EVI} \arrow[r] \arrow[u] & \mathrm{EVIII}, \arrow[u]
\end{tikzcd}$$

in which $\mathrm{ad}(\varepsilon)$, $\varepsilon \in \mathrm{SO}(9)^{\sim}$, *stabilizes and so acts on all the spaces above equivariantly. On each column, the second space B is the fixed point set* $F(\mathrm{ad}(\varepsilon), C)$ *of* $\mathrm{ad}(\varepsilon)$ *acting on the third space C which is a group. And B is c-orthogonal to* $D = C/B$ *at* 1 *in C.*

3.23. COROLLARY.′ *The isotropy representations for* FII, EIII, EVI *and* EVIII *are the restrictions of the half-spin representation of* $\mathrm{SO}(16)^{\#}$ *to the corresponding subgroups in the second row; in particular these spaces have dimensions* $2^4, 2^5, 2^6$ *and* 2^7 *respectively.*

3.24. PROPOSITION (Effect of P_2). *One has the commutative diagram of monomorphisms*:

$$\begin{tikzcd}
\mathrm{SO}(4) \arrow[d] & \mathrm{Sp}(1)\cdot\mathrm{Sp}(3) \arrow[r] \arrow[d] & \mathrm{Sp}(1)\cdot\mathrm{SU}(6) \arrow[r] \arrow[d] & \mathrm{Sp}(1)\cdot\mathrm{SO}(12)^{\sim} \arrow[r] \arrow[d] & \mathrm{Sp}(1)\cdot\mathrm{E}_7 \arrow[d] \\
\mathrm{G}_2 \arrow[r] & \mathrm{F}_4 \arrow[r] & \mathrm{E}_6 \arrow[r] & \mathrm{E}_7 \arrow[r] & \mathrm{E}_8 \\
\mathrm{GI} \arrow[r] \arrow[u] & \mathrm{FI} \arrow[r] \arrow[u] & \mathrm{EII} \arrow[r] \arrow[u] & \mathrm{EVI} \arrow[r] \arrow[u] & \mathrm{EIX} \arrow[u] \\
 & G_4^{\circ}(\boldsymbol{R}^9) \arrow[r] \arrow[u] & G_4^{\circ}(\boldsymbol{R}^{10}) \arrow[r] \arrow[u] & G_4^{\circ}(\boldsymbol{R}^{12}) \arrow[r] \arrow[u] & G_4^{\circ}(\boldsymbol{R}^{16}), \arrow[u]
\end{tikzcd}$$

in which $\mathrm{ad}(P_2)$, $P_2 \in F(\mathrm{T}, \mathrm{SO}(8)^{\sim}) \subset \mathrm{SO}(8)^{\sim}$, *acts on all the spaces equivariantly. On* *each column, the groups A above B in the second row are the centralizers* $F(\mathrm{ad}(P_2), B)$. *The orthogonal space* B/A *is the space* C *in the third row. Take a member* q *of* F_4 *which is conjugate with* P_2 *and commutes with* P_2. *Then the connected fixed point set* $F(\mathrm{ad}(q), C)_{(1)}$ *is the space* $D=G_4^{\circ}(\boldsymbol{R}^n)$ *below* C *in the diagram.*

3.25. REMARK. The spaces C in the third row with GI replaced by F_4 are all of the spaces that have the root system F_4. The spaces C in the third row are all polars in the groups in the second row. Except for G_2, those 1-connected groups in the second row have exactly two polars $\neq 1$ of 1 in them; the other polars appear in the fourth row of the diagram in the preceding proposition 3.22. The spaces $G_4^{\circ}(\boldsymbol{R}^n)$ in the fourth row have the root system B_4.

3.26. PROPOSITION (Effect of K_8). $\mathrm{ad}(K_8)$ *stabilizes all the spaces in the commutative diagram below and commutes with the monomorphisms (indicated by the arrows) in it*:

$$\begin{array}{ccccccc}
\mathrm{Sp}(1)\cdot\mathrm{Sp}(3) & \rightarrow & \mathrm{Sp}(4)^* & \rightarrow & \mathrm{SU}(8)/\boldsymbol{Z}_2 & \rightarrow & \mathrm{SO}(16)^{\#} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{F}_4 & \rightarrow & \mathrm{E}_6 & \rightarrow & \mathrm{E}_7 & \rightarrow & \mathrm{E}_8 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{FI} & \rightarrow & \mathrm{EI} & \rightarrow & \mathrm{EV} & \rightarrow & \mathrm{EVIII}\,.
\end{array}$$

On each column, the first space A is the fixed point set $F(\mathrm{ad}(K_8), B)$ *in the second space B which is a group. The third space C is the component* $F(s_1 \circ \mathrm{ad}(K_8), B)_{(1)}$ *through* 1 *of the fixed point set of the space involution acting on B. A is orthogonal to C at* 1. *And C has the isomorphic root system with that of B, with multiplicity* 1. (*Inclusions such as* $\mathrm{SO}(16)^{\#}\rightarrow\mathrm{E}_8$ *in the diagram are different from those defined earlier.*)

3.27. PROPOSITION (Effect of Q_2). $\mathrm{ad}(Q_2)$, $Q_2=K\otimes(1_4\oplus 0_4)+1_2\otimes(0_4\oplus 1_4)\in \mathrm{SO}(16)^{\#}$, *acts on the spaces in this commutative diagram of monomorphisms*:

$$\begin{array}{ccccccccc}
\mathrm{F}_4 & & \mathrm{F}_4 & & T\cdot\mathrm{E}_6 & & T\cdot\mathrm{E}_6 & & S^3\cdot\mathrm{E}_7 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{E}_6 & \rightarrow & T\cdot\mathrm{E}_6 & \rightarrow & \mathrm{E}_7 & \rightarrow & S^3\cdot\mathrm{E}_7 & \rightarrow & \mathrm{E}_8 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{EIV} & \rightarrow & T\cdot\mathrm{EIV} & \rightarrow & \mathrm{EVII} & \rightarrow & S^2\cdot\mathrm{EVII} & \rightarrow & \mathrm{EIX}\,,
\end{array}$$

in which the groups in the first row are the fixed point sets of $\mathrm{ad}(Q_2)$ *acting on the groups below them, the second row shows monomorphisms as the fixed point sets of single involutions and the spaces in the third row are the completely orthogonal spaces to those in the first row at* 1.

3.28. REMARK. In the Satake diagrams of EIV, EVII and EIX, the black vertices are α_5, α_6, α_7 and α_8 (of D_8) and there are no arrows.

3.29. Remark. The diagram below exhibits all the 1-connected spaces G/K whose automorphism groups G are simple and exceptional (and which are not groups) along with monomorphisms between them. (FII is the unique polar in EIV.)

$$\begin{tikzcd}
 & & & \mathrm{EIV} \arrow[drr] & & & & & \\
G_1(H^3) \arrow[r] \arrow[dr] & \mathrm{FII} \arrow[rr] \arrow[urr] & & \mathrm{EIII} \arrow[rr] \arrow[drr] & & \mathrm{EVII} \arrow[drr] & & & \\
\mathrm{GI} \arrow[r] & \mathrm{FI} \arrow[rr] \arrow[dr] & & \mathrm{EII} \arrow[rr] \arrow[dr] & & \mathrm{EVI} \arrow[rr] \arrow[dr] & & \mathrm{EIX} \arrow[dr] & \\
 & & \mathrm{EI} \arrow[rr] & & \mathrm{EV} \arrow[rr] & & \mathrm{EVIII} \arrow[rr] & & \mathrm{E}_8
\end{tikzcd}$$

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