TOKYO J. MATH.
VOL. 16, NO. 1, 1993

On the Iwasawa λ-Invariants of Real Quadratic Fields

Hisao TAYA

Waseda University
(Communicated by Y. Shimizu)

Introduction.

Let k be a finite extension of the field of rational numbers $\boldsymbol{Q}$ and p a fixed prime number. A Galois extension K of k is called a $\boldsymbol{Z}_p$-extension if the Galois group $\mathrm{Gal}(K/k)$ is topologically isomorphic to the additive group $\boldsymbol{Z}_p$ of the p-adic integers. Every number field k has at least one $\boldsymbol{Z}_p$-extension, namely the cyclotomic $\boldsymbol{Z}_p$-extension which is contained in the field obtained by adjoining all p-power roots of unity to k.

For a $\boldsymbol{Z}_p$-extension

$$k=k_0\subset k_1\subset k_2\subset\cdots\subset k_n\subset\cdots\subset K=\bigcup_{n=1}^{\infty}k_n$$

with Galois groups $\mathrm{Gal}(k_n/k)\simeq\boldsymbol{Z}/p^n\boldsymbol{Z}$, let h_n be the class number of k_n and p^{e_n} the exact power of p dividing h_n. Then Iwasawa has proved that there exist integers λ, μ and ν, depending only on K/k and p, such that $e_n=\lambda n+\mu p^n+\nu$ for all sufficiently large n. The integers $\lambda=\lambda_p(K/k)$, $\mu=\mu_p(K/k)$ and $\nu=\nu_p(K/k)$ are called the Iwasawa invariants of K/k for p. For convenience, the Iwasawa invariants of the cyclotomic $\boldsymbol{Z}_p$-extension of k for p will be denoted by $\lambda_p(k)$, $\mu_p(k)$ and $\nu_p(k)$.

In [6], Greenberg stated the following conjecture concerning the Iwasawa invariants:

"*If k is totally real, then both $\lambda_p(k)$ and $\mu_p(k)$ vanish.*"

It seems quite difficult to decide whether this conjecture is true, even for real quadratic fields.

Recently in [2], [3], [4] and [5], Fukuda and Komatsu studied Greenberg's conjecture in some real quadratic cases. They defined two invariants n_1 and n_2 in [4] (cf. Section 1), and treated the cases where $2\leq n_1<n_2$ and $n_1=1$ in [3], [4] and the case where $n_1=n_2=2$ in [2], [5] (See Addendum).

Received February 20, 1992
Revised September 16, 1992

In this paper, we shall make further investigation in the real quadratic case, and treat mainly the case $n_2 \geq 2$ (including the case $n_1 = n_2$). Let A_n be the p-primary part of the ideal class group of k_n and D_n the subgroup of A_n consisting of ideal classes which contain products of prime ideals of k_n lying over p. It should be noted that the order of D_n has a close relation to Greenberg's conjecture (see [6]). After recalling the known results in Section 1, we shall give in Section 2 a necessary and sufficient condition for $|D_m| = p|D_n|$ for some $m > n \geq 0$ (Theorem 1), and using this, give in Section 3 a sufficient condition for $\lambda_p(k) = \mu_p(k) = 0$ (Theorem 2). Further, in Appendix we treat the case $n_2 = 1$ and give another proof of a special case of Theorem 1 in [4].

Finally we make the following remark. If k is an arbitrary number field and if p splits completely in k, then $\lambda_p(k) \geq r_2$, where r_2 is the number of complex archimedean primes of k (see [6]). So, for a prime p, we can find k for which $\lambda_p(k)$ is arbitrarily large.

§1. Preliminaries.

Let k be a real quadratic field with class number h, $\varepsilon\,(>1)$ the fundamental unit of k, and p an odd prime number which splits in k, namely $(p) = \mathfrak{p}\mathfrak{p}'$ in k where $\mathfrak{p} \neq \mathfrak{p}'$. Then we can choose $\alpha \in k$ such that $\mathfrak{p}^h = (\alpha)$. Fukuda and Komatsu [4] defined n_1 to be the maximal integer such that $\alpha^{p-1} \equiv 1 (\mathrm{mod}\ p^{n_1} \boldsymbol{Z}_p)$ and n_2 to be the maximal integer such that $\varepsilon^{p-1} \equiv 1 (\mathrm{mod}\ p^{n_2} \boldsymbol{Z}_p)$. Here, n_1 is uniquely determined under the condition $n_1 \leq n_2$.

For the cyclotomic $\boldsymbol{Z}_p$-extension

$$k = k_0 \subset k_1 \subset k_2 \subset \cdots \subset k_n \subset \cdots \subset k_\infty = \bigcup_{n=1}^{\infty} k_n$$

with Galois group $\mathrm{Gal}(k_\infty/k) = \overline{\langle \sigma \rangle}$, as stated in the introduction, let $A_n\ (= A_n(k))$ be the p-primary part of the ideal class group of k_n and $D_n\ (= D_n(k))$ the subgroup of A_n consisting of ideal classes which contain products of prime ideals of k_n lying over p, and $E_n\ (= E_n(k))$ the unit group of k_n. We also denote by $\mathfrak{p}_n$ (resp. $\mathfrak{p}'_n$) the unique prime ideal of k_n lying above $\mathfrak{p}$ (resp. $\mathfrak{p}'$). In this case we have

$$D_n = \langle Cl(\mathfrak{p}_n) \rangle \cap A_n\,,$$

where $Cl(\mathfrak{p}_n)$ denotes the ideal class represented by $\mathfrak{p}_n$. Let $B_n\ (= B_n(k))$ be the subgroup of A_n consisting of ideal classes which are invariant under the action of $\mathrm{Gal}(k_n/k)$ and $B'_n\ (= B'_n(k))$ the subgroup of A_n consisting of ideal classes which contain ideals invariant under the action of $\mathrm{Gal}(k_n/k)$. For $n \geq r \geq 0$, we put

$$B_n^{(r)} = \{a \in A_n \mid a^{\sigma_r - 1} = 1\}\,,$$

where $\sigma_r = \sigma^{p^r}$. Then we see that

$$D_n \subset B'_n \subset B_n = B_n^{(0)} \subset B_n^{(1)} \subset B_n^{(2)} \subset \cdots \subset B_n^{(n)} = A_n$$

and

$$B'_n = i_{0,n}(A_0)D_n ,$$

where $i_{0,n}$ denotes the induced map by the inclusion of the ideal group of k in the ideal group of k_n. For each $m \geq n \geq 0$, we will let $N_{m,n}$ be the norm map from k_m to k_n and $N_{m,n}$ will also denote the induced maps from A_m to A_n, from E_m to E_n.

The following formulae are well-known and play an important role in the rest of this paper:

(i) $B_n/B'_n \simeq (E_0 \cap N_{n,0}(k_n^\times))/N_{n,0}(E_n)$ for all $n \geq 0$,

(ii) $|B'_n| = |A_0| \dfrac{p^n}{(E_0 : N_{n,0}(E_n))}$ for all $n \geq 0$,

(iii) $|B_n| = |A_0| p^{n_2 - 1}$ for all $n \geq n_2 - 1$.

For details refer to the paper of Fukuda and Komatsu [3], [4], Greenberg [6], and Yokoi [9]. Finally we note that, if k has only one prime lying over p and if A_0 is trivial, then $\lambda_p(k)$, $\mu_p(k)$ and $\nu_p(k)$ are zero (see [7]).

All the notation defined above will be used in the same meaning throughout this paper.

§2. Relation between a new invariant and the order of D_n.

Throughout this section, we assume that A_0 is trivial and that $n_2 \geq 2$. In this case we note that $B'_n = D_n$. We shall give a necessary and sufficient condition for $|D_m| = p|D_n|$ for some $m > n \geq 0$.

Now fix $r \geq 0$ for a while and put $|D_r| = p^j$. Assume that $0 \leq j \leq n_2 - 2$. (If $j = n_2 - 1$, then $B_n = D_n$ for large $n \geq 0$ from (iii), hence Greenberg's criterion implies that $\lambda_p(k) = \mu_p(k) = 0$ (cf. Appendix).) Then we can choose $\alpha_r \in k_r$ such that $\mathfrak{p}_r'^{hp^j} = (\alpha_r)$. We define a new invariant $m_r \in N$ for k_r/k and p, by

$$\mathfrak{p}^{m_r} \,\|\, (N_{r,0}(\alpha_r)^{p-1} - 1) \qquad \text{in } k .$$

Since $N_{r,0}(\alpha_r)^{p-1} \in (1 + pZ_p)^{p^r}$, we have $r + 1 \leq m_r$. On the other hand, it follows from (ii) that $N_{r,0}(E_r) = E_0^{p^{r-j}}$. Thus there exists $\varepsilon_r \in E_r$ such that $\mathfrak{p}^{r+n_2-j} \,\|\, (N_{r,0}(\varepsilon_r)^{p-1} - 1)$, therefore we can choose $\alpha_r \in k_r$ such that $m_r \leq r + n_2 - j$. Hence we see that m_r is uniquely determined under the condition $r + 1 \leq m_r \leq r + n_2 - j$.

Here we should mention that if we put $n_2 = 2$ and $j = 0$, then m_r is equal to $n_1^{(r)}$ which was defined by Fukuda [2], and that if we put $r = 0$, then $j = 0$, so m_0 is equal to n_1. Now we prove the following theorem.

THEOREM 1. *Let k be a real quadratic field and p an odd prime number which splits in k. Assume that A_n is cyclic for each $n \geq 0$, that A_0 is trivial, and that* $|D_r| = p^j$

for some $r\geq 0$. *Then*

$$m_r=r+s \Leftrightarrow |D_{r+t}|=\begin{cases} p^{j+1} & \text{if } t=s\,, \\ p^j & \text{if } 0<t<s\,, \end{cases}$$

for $1\leq s\leq n_2-1-j$.

Before proceeding to the proof, we prepare two lemmas. Let $k_{\mathfrak{p}_n}$ be the completion of k_n at $\mathfrak{p}_n$ and $E_{\mathfrak{p}_n}$ the unit group of $k_{\mathfrak{p}_n}$. We put

$$U_n=\{u\in E_{\mathfrak{p}_n} \mid u\equiv 1 \pmod{\mathfrak{p}_n}\}$$

and

$$U_n^{(r)}=\{u\in U_n \mid N_{n,0}(u)\equiv 1 \pmod{p^{n+r+1}}\}$$

for $0\leq r\leq n$, respectively.

Lemma 1. *Let k and p be as in Theorem* 1. *Then we have* $N_{n+j,n}(U_{n+j})=U_n^{(j)}$ *for all* $n\geq j$.

Proof. Let $\varepsilon_n=N_{n+j,n}(\varepsilon_{n+j})\in N_{n+j,n}(U_{n+j})$. Then we see that

$$N_{n,0}(\varepsilon_n)=N_{n+j,0}(\varepsilon_{n+j})\equiv 1 \pmod{p^{n+j+1}}\,.$$

Hence we have $\varepsilon_n\in U_n^{(j)}$, so $N_{n+j,n}(U_{n+j})\subset U_n^{(j)}$ for all $n\geq j$.

We now consider the composite map φ of

$$N_{n,0}: U_n \to 1+p^{n+1}\boldsymbol{Z}_p \quad \text{and} \quad 1+p^{n+1}\boldsymbol{Z}_p \to (1+p^{n+1}\boldsymbol{Z}_p)/(1+p^{n+j+1}\boldsymbol{Z}_p)\,.$$

It is easy to see that φ is surjective and its kernel is $U_n^{(j)}$. Therefore we obtain

$$U_n/U_n^{(j)}\simeq(1+p^{n+1}\boldsymbol{Z}_p)/(1+p^{n+j+1}\boldsymbol{Z}_p)\simeq\boldsymbol{Z}/p^j\boldsymbol{Z}\,.$$

On the other hand, since k_{n+j}/k_n is totally ramified at $\mathfrak{p}_n$, we obtain, by local class field theory,

$$U_n/N_{n+j,n}(U_{n+j})\simeq \mathrm{Gal}(k_{\mathfrak{p}_{n+j}}/k_{\mathfrak{p}_n})\simeq\boldsymbol{Z}/p^j\boldsymbol{Z}\,.$$

It follows that $N_{n+j,n}(U_{n+j})=U_n^{(j)}$. This completes the proof of Lemma 1. □

Lemma 2. *Let k and p be as in Theorem* 1. *Assume that* A_n *is cyclic for each* $n\geq 0$ *and* A_0 *is trivial. If* $|D_r|=p^j$ *for some* $r\geq 0$, *then we have* $A_{r+t}=B_{r+t}^{(r)}$ *for* $0\leq t\leq n_2-1-j$.

Proof. First, we consider the case $t=n_2-1-j$. We have to show that $A_{r+n_2-1-j}=B_{r+n_2-1-j}^{(r)}$. Let $\varepsilon_r\in E_r$. Since $|D_r|=p^j$, it follows from (ii) that $N_{r,0}(E_r)=E_0^{p^{r-j}}$. Thus we have $N_{r,0}(\varepsilon_r^{p-1})\equiv 1 \pmod{p^{r+n_2-j}}$, hence

$$\varepsilon_r^{p-1}\in U_r^{(n_2-1-j)}=N_{r+n_2-1-j,r}(U_{r+n_2-1-j})$$

from Lemma 1. It follows that ε_r is a local norm from k_{r+n_2-1-j} at $\mathfrak{p}_r$. Since any place which does not lie above p is unramified in k_{r+n_2-1-j}/k_r, the product formula for the norm residue symbol and Hasse's norm theorem imply that ε_r is a global norm from k_{r+n_2-1-j}, so that

$$E_r \subset N_{r+n_2-1-j,r}(k_{r+n_2-1-j}^{\times}) .$$

Then by the genus formula for k_{r+n_2-1-j}/k_r, we obtain

$$|B_{r+n_2-1-j}^{(r)}| = \frac{|A_r|p^{n_2-1-j}p^{n_2-1-j}}{p^{n_2-1-j}(E_r : E_r \cap N_{r+n_2-1-j,r}(k_{r+n_2-1-j}^{\times}))} = |A_r|p^{n_2-1-j} .$$

Now we assume that $B_{r+n_2-1-j}^{(r)} \subsetneqq A_{r+n_2-1-j}$. Then there exists $a \in A_{r+n_2-1-j}$ such that $a^{\sigma_r-1} \neq 1$ and $a^{(\sigma_r-1)^2}=1$. It follows from the remark mentioned below that there exist $u \in \boldsymbol{Z}_p[\mathrm{Gal}(k_{r+n_2-1-j}/k_r)]^{\times}$ and $v \in \boldsymbol{Z}_p[\mathrm{Gal}(k_{r+n_2-1-j}/k_r)]$ such that

$$1+\sigma_r+\cdots+\sigma_r^{p^{n_2-1-j}-1}=(\sigma_r-1)^2 v+p^{n_2-1-j}u .$$

Hence

$$a^{|A_r|(1+\sigma_r+\cdots+\sigma_r^{p^{n_2-1-j}-1})}=a^{|A_r|(\sigma_r-1)^2 v+|A_r|p^{n_2-1-j}u} .$$

Therefore we have

$$a^{|A_r|p^{n_2-1-j}}=1 .$$

But A_{r+n_2-1-j} is cyclic, so it follows that $a \in B_{r+n_2-1-j}^{(r)}$, which is a contradiction.

Next, we assume that $0 \le t \le n_2-2-j$. Since

$$E_r \subset N_{r+n_2-1-j,r}(k_{r+n_2-1-j}^{\times}) \subset N_{r+t,r}(k_{r+t}^{\times}) ,$$

the genus formula for k_{r+t}/k_r implies that $|B_{r+t}^{(r)}|=|A_r|p^t$. Therefore we can show that $A_{r+t}=B_{r+t}^{(r)}$ by the above argument. This completes the proof of Lemma 2. □

REMARK. Let G be a cyclic group with generator ρ, and g the order of G. It is easy to see that, for each positive integer N,

$$1+\rho+\rho^2+\cdots+\rho^N=(\rho-1)^2 v+\frac{1}{2}(N+1)(N\rho+2-N) ,$$

where

$$v=\sum_{i=0}^{N-2}(N-1-i)(\rho^i+\rho^{i-1}+\cdots+\rho+1) .$$

In particular, if we put $N=p^r-1$ and $g=p^r$, then we have

$$1+\rho+\rho^2+\cdots+\rho^{p^r-1}=(\rho-1)^2 v+p^r\left(\frac{p^r-1}{2}\rho+\frac{3-p^r}{2}\right).$$

We let

$$\alpha=\frac{p^r-3}{2}\,,\quad \beta=\frac{p^r-1}{2}\quad \text{and}\quad u=\beta\rho-\alpha\,.$$

Then it follows that

$$(\beta\rho-\alpha)\{(\beta\rho)^{p^r-1}+(\beta\rho)^{p^r-2}\alpha+\cdots+\alpha^{p^r-1}\}=\beta^{p^r}-\alpha^{p^r}\equiv\beta-\alpha\equiv 1 \qquad (\mathrm{mod}\ p)\,.$$

Hence we have $\beta^{p^r}-\alpha^{p^r}\in \boldsymbol{Z}_p^{\times}$, so $u\in \boldsymbol{Z}_p[G]^{\times}$. Consequently there exist $u\in \boldsymbol{Z}_p[G]^{\times}$ and $v\in \boldsymbol{Z}_p[G]$ such that

$$1+\rho+\rho^2+\cdots+\rho^{p^r-1}=(\rho-1)^2v+p^r u\,.$$

In particular, for each $a\in A_n$, $a^{p^i u}=1$ implies that $a^{p^i}=1$.

Proof of Theorem 1. If $|D_{n+1}|\neq|D_n|$ for some $n\geq 0$, then $|D_{n+1}|=p|D_n|$. Therefore it is sufficient to prove that

$$m_r\geq r+t+1 \quad \text{if and only if} \quad |D_{r+t}|=|D_r|$$

for $1\leq t\leq n_2-1-j$.

Assume now that $|D_{r+t}|=|D_r|=p^j$ where $1\leq t\leq n_2-1-j$. Then we have $\mathfrak{p}'^{hp^j}_{r+t}=(\alpha_{r+t})$ for some $\alpha_{r+t}\in k_{r+t}$. Let $\alpha_r=N_{r+t,r}(\alpha_{r+t})$, so that $\mathfrak{p}'^{hp^j}_r=(\alpha_r)$. Thus we obtain

$$N_{r,0}(\alpha_r^{p-1})=N_{r+t,0}(\alpha_{r+t}^{p-1})\in 1+p^{r+t+1}\boldsymbol{Z}_p\,.$$

Hence $m_r\geq r+t+1$.

Conversely, we assume that $m_r\geq r+t+1$ where $1\leq t\leq n_2-1-j$. Let α_r be an element of k_r such that $\mathfrak{p}'^{hp^j}_r=(\alpha_r)$. We then have $N_{r,0}(\alpha_r)^{p-1}\in 1+p^{r+t+1}\boldsymbol{Z}_p$, hence

$$\alpha_r^{p-1}\in U_r^{(t)}=N_{r+t,r}(U_{r+t})$$

from Lemma 1. Therefore it follows that there exists $\alpha_{r+t}\in k_{r+t}$ such that $\alpha_r^{p-1}=N_{r+t,r}(\alpha_{r+t})$ from Hasse's norm theorem and the product formula. Since

$$N_{r+t,r}(\mathfrak{p}'^{(p-1)hp^j}_{r+t}(\alpha_{r+t}^{-1}))=\mathfrak{p}'^{(p-1)hp^j}_r(\alpha_r^{-1})^{(p-1)}=(1)\,,$$

we see that

$$\mathfrak{p}'^{(p-1)hp^j}_{r+t}(\alpha_{r+t}^{-1})=\mathfrak{a}_{r+t}^{\sigma_r-1}$$

for some ideal $\mathfrak{a}_{r+t}$ of k_{r+t}. This implies that $D_{r+t}^{p^j}\subset A_{r+t}^{\sigma_r-1}$. Hence, by Lemma 2

$$D_{r+t}^{p^j}\subset A_{r+t}^{\sigma_r-1}=B_{r+t}^{(r)\,\sigma_r-1}=1\,.$$

Since D_m has a subgroup which is isomorphic to D_n for $m\geq n\geq 0$, it follows that $|D_{r+t}|=p^j=|D_r|$. This completes the proof of Theorem 1. □

§3. Application to λ-invariants of real quadratic fields.

In this section, we shall apply our result of the previous section to the Iwasawa λ-invariant of k. We first prove the following lemma.

LEMMA 3. *Let k and p be as in Theorem 1. If A_n is cyclic for all $n \geq 0$ and D_r is non-trivial for some $r \geq 0$, then $\lambda_p(k) = \mu_p(k) = 0$.*

PROOF. Since $|D_n|$ remains bounded as $n \to \infty$ from (iii), it suffices to prove that if $|D_n| = |D_{n+1}|$, then $|A_n| = |A_{n+1}|$ for all sufficiently large n. We now assume that $|A_n| < |A_{n+1}|$ for all sufficiently large n. Since k_{n+1}/k_n is totally ramified at $\mathfrak{p}_n$, $N_{n+1,n}$: $A_{n+1} \to A_n$ is surjective. Thus $\mathrm{Ker}(N_{n+1,n})$ is non-trivial. Since A_n is cyclic and D_n is non-trivial, this implies that $\mathrm{Ker}(N_{n+1,n}) \cap D_{n+1}$ is non-trivial. Therefore we have $|D_n| < |D_{n+1}|$, because $N_{n+1,n} : D_{n+1} \to D_n$ is surjective. This completes the proof of Lemma 3. □

REMARK. Let K be a finite totally real extension of $\boldsymbol{Q}$ and p an odd prime number which is totally ramified in K_∞/K. If Leopoldt's conjecture is valid for K, then $|B_n(K)|$ remains bounded as $n \to \infty$ (see [6]). Hence, in general, it follows from the above proof that Lemma 3 holds for such a field K under the same assumptions.

From Lemma 3, we have only to consider the case $|D_r| = 1$ for some $r \geq 0$. Now we prove the following theorem, which gives a sufficient condition for the Iwasawa invariants $\lambda_p(k)$ and $\mu_p(k)$ to vanish in the case $n_2 \geq 2$.

THEOREM 2. *Let k and p be as in Theorem 1, and $k^* = k(\zeta_p)$ where ζ_p is a primitive p-th root of unity. Put $\lambda_p^-(k^*) = \lambda_p(k^*) - \lambda_p((k^*)^+)$ where $(k^*)^+$ is the maximal real subfield of k^*. Assume that*

(1) $n_2 \geq 2$,
(2) $A_0 = 1$,
(3) $\lambda_p^-(k^*) = 1$,
(4) $|D_r| = 1$ *for some* $r \geq 0$.

Then $m_r = r + s$ if and only if $|D_{r+s}| = p$ and $|D_{r+s-1}| = 1$ for $1 \leq s \leq n_2 - 1$. In particular, if $m_r \neq r + n_2$, then $\lambda_p(k) = \mu_p(k) = 0$.

REMARK. In [1], Ferrero and Washington proved that $\mu_p(K)$ always vanishes when K is abelian over $\boldsymbol{Q}$.

PROOF OF THEOREM 2. Let k_n^* be the n-th layer of the cyclotomic $\boldsymbol{Z}_p$-extension k_∞^*/k^* and $A_n^* = A_n(k^*)$ as defined in Section 1. Then k_n^* is a CM-field, so we can define $(A_n^*)^+$ by the p-primary part of the ideal class group of its maximal real subfield and $(A_n^*)^-$ by the kernel of the norm map from A_n^* to $(A_n^*)^+$. Since $\mu_p(k^*)$ vanishes, the assumption (3) implies that $(A_n^*)^-$ is cyclic for $n \geq 0$ (cf. Cor. 13.29 in [10]). It follows from the reflection theorem that $(A_n^*)^+$ is cyclic, hence so is A_n for $n \geq 0$. Therefore

Theorem 1 says that

$$m_r=r+s \Leftrightarrow |D_{r+t}|=\begin{cases} p & \text{if } t=s\,, \\ 1 & \text{if } 0<t<s\,, \end{cases}$$

for $1\leq s\leq n_2-1$. We have finished the proof of our theorem. □

REMARK. When $n_2\geq 3$, we can replace the assumption (3) of Theorem 2 by the following:

(3′) A_0^* is an elementary abelian p-group.

Indeed, under the assumption (2) of Theorem 2, this assumption (3′) implies that A_n is cyclic for $n\geq 0$ (see [4] or [6]). Hence, applying Theorem 1, we obtain the above result.

By the way, in the proof of Theorem 2, we used the Ferrero-Washington theorem. But if we replace (3) by (3′), then it follows immediately that $\lambda_p(k)=\mu_p(k)=0$ without using the Ferrero-Washington theorem.

In the above theorem, the assumption (2) implies that $|D_0|=1$, so we can put $r=0$. Therefore we obtain the next corollary.

COROLLARY (Fukuda-Komatsu [4]). *Let k, k^* and p be as in Theorem 2. Assume that*

(1) $n_1\neq n_2$ (*i.e.*, $1\leq n_1<n_2$),

(2) $A_0=1$,

(3) $\lambda_p^-(k^*)=1$.

Then we have $\lambda_p(k)=\mu_p(k)=0$.

REMARK. By this corollary, we know that we need not define m_r when $n_1\neq n_2$. However, the invariant m_r plays an important role in Theorem 1, and also in Theorem 2 when $n_1=n_2\geq 2$.

Appendix. Another proof of a special case of Theorem 1 in [4].

In this appendix we treat the case $n_2=1$ and give another proof of a special case of Theorem 1 in [4].

Let K be a finite totally real extension of $\boldsymbol{Q}$ and p an odd prime number which splits completely in K. We will denote by K_n the n-th layer of the cyclotomic $\boldsymbol{Z}_p$-extension K_∞/K, and $A_n(K)$, $E_n(K)$ etc., will be as defined in Section 1. When Leopoldt's conjecture is valid for K, Greenberg [6] proved the following results:

(iv) $B_n(K)=D_n(K)$ for all sufficiently large $n \Leftrightarrow \lambda_p(K)=\mu_p(K)=0$,

(v) $|B_n(K)|$ remains bounded as $n\to\infty$.

First we consider the case where K has the following property:

(*) *For all $n\geq 0$, every unit of K, which is a $\mathfrak{p}$-adic p^n-th power for every prime ideal $\mathfrak{p}$ of K lying over p, is actually a p^n-th power in K.*

Here we note that Leopoldt's conjecture is valid for K which has the property (*).

In fact, Leopoldt's conjecture is equivalent to the following statement: For each positive integer s, there exists a positive integer t such that if a unit ε of K is a $\mathfrak{p}$-adic p^t-th power for all prime ideals $\mathfrak{p}$ of K lying over p, then ε is a p^s-th power in K. We first prepare the following lemma.

LEMMA 4. *Let K and p be as above. Assume that K has the property* (*) *and* $A_0(K)=D_0(K)$. *Then we have* $\lambda_p(K)=\mu_p(K)=0$.

PROOF. Let $c\in B_n(K)$ and $\mathfrak{a}$ an ideal of K_n such that $\mathfrak{a}\in c$. Then we have $\mathfrak{a}^{\rho-1}=(\alpha)$ for some $\alpha\in K_n$, where ρ denotes a generator of $\mathrm{Gal}(K_n/K)$. Let $\varepsilon=N_{n,0}(\alpha)$, then clearly $\varepsilon\in E_0(K)$. Thus ε is a $\mathfrak{p}$-adic p^n-th power for every prime ideal $\mathfrak{p}$ of K lying over p by local class field theory. It follows from the property (*) that ε is actually a p^n-th power in K, namely $\varepsilon=\varepsilon_0^{p^n}$ for some $\varepsilon_0\in E_0(K)$. Therefore we have $N_{n,0}(\alpha)=N_{n,0}(\varepsilon_0)$, so $N_{n,0}(\alpha\varepsilon_0{}^{-1})=1$. Hilbert's Theorem 90 implies that

$$\alpha\varepsilon_0{}^{-1}=\beta^{\rho-1} \qquad \text{for some } \beta\in K_n^\times .$$

It is easily shown that $\mathfrak{a}(\beta^{-1})$ is a $\mathrm{Gal}(K_n/K)$-invariant ideal and $\mathfrak{a}(\beta^{-1})$ is contained in c. Thus we have $c\in B'_n(K)$, which implies that

$$B_n(K)=B'_n(K) .$$

On the other hand, since $i_{0,n}(D_0(K))\subset D_n(K)$, it follows that

$$B'_n(K)=i_{0,n}(A_0(K))D_n(K)=i_{0,n}(D_0(K))D_n(K)=D_n(K)$$

for all $n\geq 0$. Therefore

$$B_n(K)=D_n(K) .$$

Now Lemma 4 follows immediately from (iv). □

Applying the above lemma to real quadratic fields, we obtain the following special case of Theorem 1 in [4].

PROPOSITION (Special case of Theorem 1 [4]). *Let k be a real quadratic field and p an odd prime number which splits in k. Assume that*

(1) $n_2=1$,

(2) $A_0=D_0$.

Then we have $\lambda_p(k)=\mu_p(k)=0$.

PROOF. By the assumption (1), we have $\varepsilon^{p-1}\not\equiv 1 \pmod{p^2\boldsymbol{Z}_p}$. It follows from local class field theory that ε^{p-1} is not a $\mathfrak{p}$-adic (resp. $\mathfrak{p}'$-adic) p-th power, hence ε is not also a $\mathfrak{p}$-adic (resp. $\mathfrak{p}'$-adic) p-th power. This shows that k has the property (*). Lemma 4 then implies that $\lambda_p(k)=\mu_p(k)=0$, finishing the proof of our proposition. □

ACKNOWLEDGEMENT. The author wishes to thank Professor N. Adachi, Professor K. Komatsu and Dr. T. Fukuda for their helpful suggestions and warm encouragements.

ADDENDUM. We found out a computational error in Theorem of [5]. For $k=\boldsymbol{Q}(\sqrt{727})$, we calculated n_1, n_2 and obtained $n_1=2$, $n_2=3$. Hence the lemma in [5] can not be applied to $k=\boldsymbol{Q}(\sqrt{727})$, so we do not know whether $\lambda_3(k)=0$ or not. Dr. T. Fukuda told the author that $E_{1,0}(E_1)=E_0$, which is one of assumptions of the lemma in [5], is sure to hold.

References

[1] B. FERRERO and L. C. WASHINGTON, The Iwasawa invariant μ_p vanishes for abelian number fields, Ann. of Math., **109** (1979), 377–395.

[2] T. FUKUDA, Iwasawa λ-invariants of certain real quadratic fields, Proc. Japan Acad. Sect. A, **65** (1989), 260–262.

[3] T. FUKUDA and K. KOMATSU, On the λ invariants of $\boldsymbol{Z}_p$-extensions of real quadratic fields, J. Number Theory, **23** (1986), 238–242.

[4] T. FUKUDA and K. KOMATSU, On $\boldsymbol{Z}_p$-extensions of real quadratic fields, J. Math. Soc. Japan, **38** (1986), 95–102.

[5] T. FUKUDA, K. KOMATSU and H. WADA, A remark on the λ-invariants of real quadratic fields, Proc. Japan Acad. Sect. A, **62** (1986), 318–319.

[6] R. GREENBERG, On the Iwasawa invariants of totally real fields, Amer. J. Math., **98** (1976), 263–284.

[7] K. IWASAWA, A note on class numbers of algebraic number fields, Abh. Math. Sem. Univ. Hamburg, **20** (1956), 257–258.

[8] K. IWASAWA, On $\boldsymbol{Z}_p$-extensions of algebraic number fields, Ann. of Math., **98** (1973), 246–326.

[9] H. YOKOI, On the class number of a relatively cyclic number field, Nagoya Math. J., **29** (1967), 31–44.

[10] L. C. WASHINGTON, *Introduction to Cyclotomic Fields*, Graduate Text in Math., **83** (1982), Springer-Verlag.

Present Address:
DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE AND ENGINEERING, WASEDA UNIVERSITY
OKUBO, SHINJUKU-KU, TOKYO 169-50, JAPAN
e-mail: taya@math.waseda.ac.jp