TOKYO J. MATH.
VOL. 16, NO. 2, 1993

On The Strong Ergodic Theorems for Commutative Semigroups in Banach Spaces

Hirokazu OKA

Waseda University

(Communicated by S. Suzuki)

1. Introduction.

This paper is concerned with the strong ergodic theorems for commutative semigroups.

Let C be a nonempty closed convex subset of a real Banach space X. A mapping $T: C \to C$ is said to be *asymptotically nonexpansive* if for each $n \geq 1$,

$$\| T^n x - T^n y \| \leq (1+\alpha_n) \| x - y \| \qquad \text{for all } x, y \in C, \tag{1.1}$$

where $\lim_{n\to\infty} \alpha_n = 0$. In particular if $\alpha_n = 0$ for all $n \geq 1$, T is said to be *nonexpansive*. We denote by $F(T)$ the set of fixed points of a mapping T from C into itself. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a family of mappings from C into itself. $\mathscr{T}$ is called an *asymptotically nonexpansive semigroup on* C if $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$, and there exists a function $\alpha(\cdot) : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ with $\lim_{t\to\infty} \alpha(t) = 0$ such that

$$\| T(t)x - T(t)y \| \leq (1+\alpha(t)) \| x - y \| \tag{1.2}$$

for all $x, y \in C$ and $t \geq 0$. In particular, if $\alpha(t) = 0$ for all $t \geq 0$, then $\mathscr{T}$ is called a *nonexpansive semigroup on* C.

Baillon [2] and Bruck [3] proved the strong ergodic theorem for nonexpansive mappings in Hilbert spaces: let T be a nonexpansive mapping from C into itself and let $x \in C$. If $F(T)$ is nonempty and $\lim_{n\to\infty} \| T^n x - T^{n+k} x \|$ exists uniformly in $k = 0, 1, 2, \cdots$, then $\{T^n x : n \geq 1\}$ is strongly almost convergent as $n \to \infty$ to a point of y in $F(T)$, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} T^{i+k} x = y \qquad \text{uniformly in } k = 0, 1, 2, \cdots.$$

The corresponding result for nonexpansive semigroups is the following: let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C. If $\bigcap_{t\geq 0} F(T(t))$ is nonempty and

Received May 22, 1992

$\lim_{t\to\infty} \| T(t)x - T(t+h)x \|$ exists uniformly in $h \geq 0$, then $\{T(t)x : t \geq 0\}$ is strongly almost convergent as $t \to \infty$ to a point of y in $\bigcap_{t\geq 0} F(T(t))$, i.e.,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t T(s+h)x\,ds = y \qquad \text{uniformly in } h \geq 0 .$$

Kobayasi and Miyadera [14] extended these theorems to the case of uniformly convex Banach spaces. Recently, the author [17] obtained the same conclusion for asymptotically nonexpansive mappings. On the other hand, Hirano, Kido, and Takahashi [11] provided nonlinear ergodic theorems for commutative semigroups of nonexpansive mappings in Banach spaces. Recently, the author [18] generalized their results to the case of commutative semigroups of asymptotically nonexpansive mappings. Our results enable us to handle simultaneously ergodic theorems for asymptotically nonexpansive mappings and semigroups, i.e., we can establish the weak almost convergence of $\{T^n x : n \geq 1\}$ $(x \in C)$ and $\{T(t)x : t \geq 0\}$ $(x \in C)$ in a unified way.

Our first purpose is to prove the strong ergodic theorem for commutative semigroups of asymptotically nonexpansive mappings. It seems to be the first time that a unified approach of the strong ergodic theorem for nonlinear semigroups is established.

The second purpose is to provide the ergodic theorem for affine semigroups in general Banach spaces, which is proved by modifying the proof of our first ergodic theorem. See [9] for the characterization of affine semigroups.

This paper consists of five sections. Section 2 is a preliminary part. Section 3 contains some lemmas. In Section 4, we prove the strong ergodic theorem (Theorem 1) for commutative semigroups of asymptotically nonexpansive mappings. Theorem 1 is the first main result which allows us to treat the strong almost convergence for asymptotically nonexpansive mappings and semigroups in a unified way. See Corollaries 2 and 3. In the last section, Section 5, we briefly investigate the ergodic theorem for affine semigroups in general Banach spaces.

2. Preliminaries.

Let C be a nonempty closed convex subset of a real Banach space X and let G be a commutative topological semigroup with the identity. We shall use the same notation as in [11] and [18]. Let X^* be the conjugate space of X. The value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^* \rangle$. For a subset E of X, coE denotes the convex hull of E, $clcoE$ the closed convex hull of E.

Let $m(G)$ be the Banach space of all bounded real valued functions on G with the supremum norm. For each $s \in G$ and $f \in m(G)$, we define an element $r_s f$ in $m(G)$ by $(r_s f)(t) = f(t+s)$ for all $t \in G$. The mapping $r_s : f \mapsto r_s f$ is a continuous linear operator in $m(G)$ for all $s \in G$. Let D be a subspace of $m(G)$ and let μ be an element of D^*, where D^* is the conjugate space of D. Then, we denote by $\mu(f)$ the value of μ at $f \in D$. To specify the variable t, we write the value $\mu(f)$ by $\mu_t\langle f(t) \rangle$ or $\int f(t)\,d\mu(t)$. When D contains a constant function 1, an element μ of D^* is called a *mean on D* if $\|\mu\|=\mu(1)=1$. It is known that $\mu\in D^*$ is a mean on D if and only if

$$\inf\{f(t): t\in G\}\leq\mu(f)\leq\sup\{f(t): t\in G\}$$

for every $f\in D$; see [6]. Further, let D be invariant under r_s for every $s\in G$. Then, a mean on D is said to be *invariant* if $\mu(r_s f)=\mu(f)$ for all $s\in G$ and $f\in D$. For $s\in G$, we can define a point evaluation δ_s by $\delta_s(f)=f(s)$ for every $f\in m(G)$. A convex combination of point evaluations is called a *finite mean on G*.

Let $\mathscr{T}=\{T(t): t\in G\}$ be a commutative semigroup of mappings on C, i.e.,

(a) for each $t\in G$, $T(t)$ is a mapping from C into itself, and

(b) $T(s+t)x=T(s)T(t)x$ for all $s, t\in G$ and $x\in C$.

$F(\mathscr{T})\equiv\bigcap_{s\in G}F(T(s))$ denotes the set of common fixed points of $\{T(s): s\in G\}$. $F(\mathscr{T})$ may be possibly empty. A function $u(\cdot): G\to C$ is said to be an *almost-orbit of* $\mathscr{T}=\{T(t): t\in G\}$ if

$$\lim_{s\in G}\left[\sup_{t\in G}\|u(t+s)-T(t)u(s)\|\right]=0, \tag{2.1}$$

where $\lim_{s\in G}v(s)$ ($v(s)\equiv\sup_{t\in G}\|u(t+s)-T(t)u(s)\|$ for $s\in G$) denotes the limit of a net $v(\cdot)$ on the directed system $(G, \leq)$ and the binary relation $\leq$ on G is defined by $a\leq b$ if and only if there is $c\in G$ with $a+c=b$.

Let a subspace D of $m(G)$ satisfy the following property (*):

(*) D contains a constant function 1 and is invariant under r_s for every $s\in G$.

For each $x^*\in X^*$, a function $h_{x^*}: t\mapsto\langle u(t), x^*\rangle$ is in D.

Suppose that the weak closure of $\{u(t): t\in G\}$ is weakly compact. By [11, Proposition], for any $\mu\in D^*$ there exists a unique element u_μ in X such that

$$\langle u_\mu, x^*\rangle=\int\langle u(t), x^*\rangle d\mu(t) \tag{2.2}$$

for all $x^*\in X^*$. We write u_μ by $\int u(t)d\mu(t)$. If μ is a mean on D, then $\int u(t)d\mu(t)$ is contained in $clco\{u(t): t\in G\}$. Therefore, $\int u(t)d\mu(t)\in C$. Also, if μ is a finite mean on G, say

$$\mu=\sum_{i=1}^n a_i\delta_{t_i}\qquad\left(t_i\in G, a_i\geq 0, i=1, 2, \cdots, n, \sum_{i=1}^n a_i=1\right),$$

then

$$\mu_t\langle u(t)\rangle=\sum_{i=1}^n a_i u(t_i).$$

Let $\mathscr{T}=\{T(t): t\in G\}$ be a commutative semigroup on C. $\mathscr{T}$ is said to be a *commutative* *semigroup of asymptotically nonexpansive mappings on* C if the following condition (c) is satisfied:

(c) For each $t\in G$, there exists $\alpha(t)>0$ such that

$$\| T(t)x-T(t)y \| \leq (1+\alpha(t)) \| x-y \| \qquad \text{for all } x, y\in C$$

with

$$\lim_{t\in G} \alpha(t)=0 . \tag{2.3}$$

If X is uniformly convex and C is bounded, the set $F(\mathscr{T})$ is nonempty bounded closed convex (cf. [22]) and the weak closure of an almost-orbit $\{u(t): t\in G\}$ of $\mathscr{T}$ is weakly compact by the reflexivity of X.

3. Lemmas.

Throughout this section, X is a uniformly convex real Banach space, C a nonempty bounded closed convex subset of X, G a commutative topological semigroup with the identity, $\mathscr{T}=\{T(t): t\in G\}$ a commutative semigroup of asymptotically nonexpansive mappings on C satisfying (c) in Section 2, and $\{u(t): t\in G\}$ an almost-orbit of $\mathscr{T}$. Moreover assume that a subspace D of $m(G)$ satisfies the property (*) in Section 2 and G is totally ordered, i.e., for all $a, b\in G$, either $a\geq b$ or $b\geq a$ holds.

In what follows, we suppose that an almost-orbit $\{u(t): t\in G\}$ satisfies the following:

$$\lim_{t\in G} \| u(t+h)-u(t) \| \text{ exists uniformly in } h\in G. \tag{3.1}$$

Then, $u(\cdot)$ is called *asymptotically isometric* (cf. [3]). Since $\lim_{t\in G} \| u(t+h)-u(t) \|$ exists for each $h\in G$ (see [18, Lemma 2]), in (3.1) we require the uniformity of the limit in $h\in G$. Put $d=\sup_{x\in C} \| x \|$ and $M=\sup_{t\in G}(1+\alpha(t))$.

LEMMA 1. *For any* $\varepsilon>0$ *there exist* $r_0(\varepsilon)\in G$ *and* $s_0(\varepsilon)\in G$ *such that*

$$\left\| T(r)\left(\sum_{i=1}^{n} a_i u(s_i)\right)-\sum_{i=1}^{n} a_i T(r)u(s_i)\right\| <\varepsilon$$

for any $r\geq r_0(\varepsilon)$, $n\geq 1$, $a_1, \cdots, a_n\geq 0$ *with* $\sum_{i=1}^{n} a_i=1$, *and* $s_1, \cdots, s_n\geq s_0(\varepsilon)$.

FROOF. Since X is uniformly convex and C is bounded, by Bruck [4, Theorem 2.1] there exists a strictly increasing, continuous, and convex function $\gamma: \boldsymbol{R}^+\to \boldsymbol{R}^+$ with $\gamma(0)=0$ such that

$$\left\| T(t)\left(\sum_{i=1}^{n} a_i x_i\right)-\sum_{i=1}^{n} a_i T(t)x_i\right\| \tag{3.2}$$

$$\leq (1+\alpha(t))\gamma^{-1}\left(\max_{1\leq i,j\leq n}\left[\| x_i - x_j \| - \frac{1}{1+\alpha(t)}\| T(t)x_i - T(t)x_j \|\right]\right)$$

for any $t \in G, n \geq 1, a_1, \cdots, a_n \geq 0$ with $\sum_{i=1}^n a_i = 1$, and $x_1, \cdots, x_n \in C$.

For any $\varepsilon > 0$, we can choose $\delta > 0$ such that

$$\gamma^{-1}(\delta) < \varepsilon/M$$

because γ^{-1} is continuous and $\gamma^{-1}(0) = 0$. Then, by (2.3) there exists $r_0(\varepsilon) \in G$ such that

$$\alpha(r) < \delta/8d$$

for all $r \geq r_0(\varepsilon)$. Also, from (2.1) and (3.1), there is $s_0(\varepsilon) \in G$ such that

$$\sup_{h\in G} \left| \| u(s) - u(s+h) \| - \| u(s') - u(s'+h) \| \right| < \delta/4$$

and

$$\sup_{t \in G} \| u(t+s) - T(t)u(s) \| < \delta/4$$

for any $s, s' \geq s_0(\varepsilon)$. Let $n \geq 1$ and $s_1, \cdots, s_n \geq s_0(\varepsilon)$. If $r \geq r_0(\varepsilon)$, then

$$\begin{aligned}
&\| u(s_i) - u(s_j) \| - \frac{1}{1+\alpha(r)} \| T(r)u(s_i) - T(r)u(s_j) \| \\
&\quad \leq \| u(s_i) - u(s_j) \| - \| u(s_i + r) - u(s_j + r) \| + \| u(s_i + r) - T(r)u(s_i) \| \\
&\quad\quad + \| T(r)u(s_j) - u(s_j + r) \| + \alpha(r) \| u(s_i) - u(s_j) \| \\
&\quad \leq \delta/4 + \delta/4 + \delta/4 + \delta/4 = \delta
\end{aligned}$$

for all $1 \leq i, j \leq n$. Therefore, by (3.2) we have

$$\left\| T(r)\left(\sum_{i=1}^n a_i u(s_i)\right) - \sum_{i=1}^n a_i T(r)u(s_i) \right\| \leq M\gamma^{-1}(\delta) < \varepsilon$$

for any $a_1, \cdots, a_n \geq 0$ with $\sum_{i=1}^n a_i = 1$. □

The following lemma plays a crucial role in our discussion.

LEMMA 2 [18, Lemma 7]. *For any invariant mean μ on D, u_μ belongs to $F(\mathscr{T}) \cap \bigcap_{s\in G} clco\{u(t) : t \geq s\}$.*

LEMMA 3. *Let μ and λ be invariant means on D. Then we have $u_\mu = u_\lambda$.*

PROOF. Let μ and λ be invariant means on D. Then, by Lemma 2, $u_\mu, u_\lambda \in F(\mathscr{T}) \cap \bigcap_{s\in G} clco\{u(t) : t \geq s\}$. By Lemma 1 and (2.1), for any $\varepsilon > 0$ there exist $r_0(=r_0(\varepsilon)) \in$

G and $s_0(=s_0(\varepsilon))\in G$ such that

$$\sup_{q\in G}\|u(s+q)-T(q)u(s)\|<\varepsilon/3 \tag{3.3}$$

and

$$\left\|T(t+r_0)\left(\sum_{i=1}^{n} a_i u(s_i)\right)-\sum_{i=1}^{n} a_i T(t+r_0)u(s_i)\right\|<\varepsilon/3 \tag{3.4}$$

for any $s\geq s_0, n\geq 1, a_1, \cdots, a_n\geq 0$ with $\sum_{i=1}^{n} a_i=1, s_1, \cdots, s_n\geq s_0$, and $t\in G$. Since $u_\lambda\in clco\{u(s): s\geq s_0\}$, there exists an element $\sum_{i=1}^{m} b_i u(s_i)$ in the set $co\{u(s): s\geq s_0\}$ such that

$$\left\|u_\lambda-\sum_{i=1}^{m} b_i u(s_i)\right\|<\varepsilon/3M\,. \tag{3.5}$$

Then, noting that $u_\lambda\in F(\mathscr{T})$, it follows from (3.3), (3.4), and (3.5) that

$$\begin{aligned}\left\|\sum_{i=1}^{m} b_i u(t+s_i+r_0)-u_\lambda\right\| &\leq \left\|\sum_{i=1}^{m} b_i u(t+s_i+r_0)-\sum_{i=1}^{m} b_i T(t+r_0)u(s_i)\right\| \\ &+\left\|\sum_{i=1}^{m} b_i T(t+r_0)u(s_i)-T(t+r_0)\left(\sum_{i=1}^{m} b_i u(s_i)\right)\right\| \\ &+\left\|T(t+r_0)\left(\sum_{i=1}^{m} b_i u(s_i)\right)-T(t+r_0)u_\lambda\right\| \\ &\leq \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon\end{aligned} \tag{3.6}$$

for all $t\in G$.

Fix $x^*\in X^*$ with $\|x^*\|=1$. Because μ is an invariant mean on D, we have that

$$\begin{aligned}|\langle u_\mu-u_\lambda, x^*\rangle| &= |\mu_t\langle u(t)-u_\lambda, x^*\rangle| \\ &=\left|\sum_{i=1}^{m} b_i\mu_t\langle u(t+s_i+r_0)-u_\lambda, x^*\rangle\right| \\ &=\left|\mu_t\left\langle\sum_{i=1}^{m} b_i u(t+s_i+r_0)-u_\lambda, x^*\right\rangle\right| \\ &\leq\sup_{t\in G}\left\|\sum_{i=1}^{m} b_i u(t+s_i+r_0)-u_\lambda\right\|\leq\varepsilon\end{aligned}$$

by (3.6). Since ε is arbitrary, we conclude that $u_\mu=u_\lambda$. □

REMARK 1. If the norm of X is Fréchet differentiable, Lemma 3 holds good without the assumption (3.1). See [18, Lemma 8].

4. Ergodic theorems.

As in [11], a net $\{\mu_\alpha : \alpha \in I\}$ of continuous linear functionals on D is called *strongly regular* if it satisfies the following conditions;

(*a*) $\sup_{\alpha \in I} \| \mu_\alpha \| < +\infty$;

(*b*) $\lim_{\alpha \in I} \mu_\alpha(1) = 1$;

(*c*) $\lim_{\alpha \in I} \| \mu_\alpha - r_s^* \mu_\alpha \| = 0$ for every $s \in G$,

where r_s^* is the conjugate operator of r_s for $s \in G$.

The first main result of this paper is the following.

THEOREM 1. *Let D be a subspace of $m(G)$ containing a constant function* 1 *and invariant under r_s for every $s \in G$. Let C be a nonempty bounded closed convex subset of a uniformly convex real Banach space X and let $\{u(t) : t \in G\}$ be an almost-orbit of a commutative semigroup $\mathscr{T} = \{T(t) : t \in G\}$ of asymptotically nonexpansive mappings on C such that the function $t \mapsto \langle u(t), x^* \rangle$ is in D for each $x^* \in X^*$. Suppose that $u(\cdot)$ is asymptotically isometric and let $\{\mu_\alpha : \alpha \in I\}$ be a strongly regular net of continuous linear functionals on D. Then, $\int u(t+s) d\mu_\alpha(s)$ converges strongly to an element u_μ in $F(\mathscr{T})$ uniformly in $t \in G$, and $u_\mu = u_\lambda$ for invariant means μ and λ on D.*

PROOF. By Lemma 3, $u_\mu = u_\lambda$ for invariant means on μ and λ on D. Let μ be any invariant mean on D. As in the proof of Lemma 3, for any $\varepsilon > 0$ there exist $r_0(=r_0(\varepsilon)) \in G$, $s_i(=s_i(\varepsilon)) \in G$, and $b_i(=b_i(\varepsilon)) \geq 0$ with $\sum_{i=1}^m b_i = 1$ $(1 \leq i \leq m)$ such that

$$(4.1) \qquad \left\| \sum_{i=1}^m b_i u(t+s_i+r_0) - u_\mu \right\| < \varepsilon / 6 \sup_\alpha \| \mu_\alpha \|$$

for all $t \in G$. Since $\{\mu_\alpha : \alpha \in I\}$ is strongly regular, there is $\alpha_0(=\alpha_0(\varepsilon)) \in I$ such that

$$(4.2) \qquad \| \mu_\alpha - r_{r_0}^* \mu_\alpha \| < \varepsilon / 3d$$

$$(4.3) \qquad \| \mu_\alpha - r_{s_i}^* \mu_\alpha \| < \varepsilon / 6d$$

for all $i = 1, 2, \cdots, m$, and

$$(4.4) \qquad | 1 - \mu_\alpha(1) | < \varepsilon / 3d$$

for every $\alpha \geq \alpha_0$. By (4.2),

$$(4.5) \qquad \left\| \int u(s+t) d\mu_\alpha(s) - \int u(s+t+r_0) d\mu_\alpha(s) \right\|$$

$$= \left\| \int u(s+t)d\mu_\alpha(s) - \int u(s+t)d(r^*_{r_0}\mu_\alpha)(s) \right\|$$

$$\leq \sup_{q\in G} \| u(q) \| \cdot \| \mu_\alpha - r^*_{r_0}\mu_\alpha \| < \varepsilon/3$$

for all $t \in G$. Also, we have that

$$\text{(4.6)} \qquad \left\| u_\mu - \int u(s+t+r_0)d\mu_\alpha(s) \right\|$$

$$= \sup_{\|x^*\|\leq 1} \left| \left\langle u_\mu - \int u(s+t+r_0)d\mu_\alpha(s), x^* \right\rangle \right|$$

$$\leq \sup_{\|x^*\|\leq 1} \left| \langle u_\mu, x^* \rangle - \int \langle u_\mu, x^* \rangle d\mu_\alpha(s) \right|$$

$$+ \sup_{\|x^*\|\leq 1} \left| \int \langle u_\mu - u(s+t+r_0), x^* \rangle d\mu_\alpha(s) \right|$$

for every $t \in G$ and $\alpha \in I$. From (4.4), the first term of (4.6) is estimated as

$$\text{(4.7)} \qquad \sup_{\|x^*\|\leq 1} \left| \langle u_\mu, x^* \rangle - \int \langle u_\mu, x^* \rangle d\mu_\alpha(s) \right|$$

$$\leq \sup_{\|x^*\|\leq 1} |\langle u_\mu, x^* \rangle| \, |1 - \mu_\alpha(1)| < \varepsilon/3$$

for all $\alpha \geq \alpha_0$. By (4.1) and (4.3), the second term of (4.6) is estimated as

$$\text{(4.8)} \qquad \sup_{\|x^*\|\leq 1} \left| \int \langle u_\mu - u(s+t+r_0), x^* \rangle d\mu_\alpha(s) \right|$$

$$= \left\| \int (u_\mu - u(s+t+r_0)) d\mu_\alpha(s) \right\|$$

$$\leq \left\| \int \left(u_\mu - \sum_{i=1}^m b_i u(s+s_i+r_0+t) \right) d\mu_\alpha(s) \right\|$$

$$+ \left\| \int \sum_{i=1}^m b_i u(s+s_i+r_0+t) d\mu_\alpha(s) - \int u(s+t+r_0) d\mu_\alpha(s) \right\|$$

$$\leq \sup_{q\in G} \left\| u_\mu - \sum_{i=1}^m b_i u(q+s_i+r_0) \right\| \cdot \sup_\alpha \| \mu_\alpha \|$$

$$+ \left\| \int \sum_{i=1}^m b_i u(s+r_0+t) d(r^*_{s_i}\mu_\alpha)(s) - \int \sum_{i=1}^m b_i u(s+r_0+t) d\mu_\alpha(s) \right\|$$

$$<\varepsilon/6+\sup_{q\in G}\|u(q)\|\cdot\max_{1\le i\le m}\|r^*_{s_i}\mu_\alpha-\mu_\alpha\|$$

$$<\varepsilon/6+\varepsilon/6=\varepsilon/3$$

for all $\alpha\ge\alpha_0$ and $t\in G$. Then, it follows from (4.7) and (4.8) that

$$\left\| u_\mu-\int u(s+t+r_0)d\mu_\alpha(s)\right\|<2\varepsilon/3$$

for all $\alpha\ge\alpha_0$ and $t\in G$. Consequently, combining this with (4.5), we have that

$$\left\| \int u(s+t)d\mu_\alpha(s)-u_\mu\right\|<\varepsilon$$

for every $\alpha\ge\alpha_0$ and $t\in G$. □

Throughout the rest of this section, suppose that X is a real uniformly convex Banach space and that C is a nonempty bounded closed convex subset of X. Using Theorem 1, we can get the strong ergodic theorems for asymptotically nonexpansive mappings and semigroups.

Let $N=\{0, 1, 2, \cdots\}$ and let $Q=\{q_{n,m}\}_{n,m\in N}$ be a matrix satisfying the following conditions:

(*a*) $$\sup_{n\ge 0}\sum_{m=0}^{\infty}|q_{n,m}|<+\infty\,;$$

(*b*) $$\lim_{n\to\infty}\sum_{m=0}^{\infty}q_{n,m}=1\,;$$

(*c*) $$\lim_{n\to\infty}\sum_{m=0}^{\infty}|q_{n,m+1}-q_{n,m}|=0\,.$$

Then, Q is called a *strongly regular matrix*.

Let T be an asymptotically nonexpansive mapping from C into itself satisfying (1.1) and let $\{x_n\}$ be an almost-orbit of T, i.e.,

$$\lim_{n\to\infty}\left[\sup_{m\ge 0}\|x_{n+m}-T^m x_n\|\right]=0\,.$$

Put $G=\{0, 1, 2, \cdots\}$, $\mathscr{T}=\{T^i : i\in G\}$, and $D=m(G)$ in Theorem 1. We obtain the following corollary. For its proof, see [11] and [18].

COROLLARY 2. *Suppose that* $\lim_{n\to\infty}\|x_n-x_{n+k}\|$ *exists uniformly in* $k=0, 1, 2, \cdots$. *Then, the following hold*:

(i) $n^{-1}\sum_{i=0}^{n-1}x_{i+k}$ *converges strongly to some point of* $F(T)$, *as* $n\to\infty$, *uniformly in* $k=0, 1, 2, \cdots$.

(ii) *If* $Q=\{q_{n,m}\}_{n,m\in N}$ *is a strongly regular matrix, then* $\sum_{m=0}^{\infty}q_{n,m}x_{m+k}$ *converges* *strongly to some point of* $F(T)$, *as* $n \to \infty$, *uniformly in* $k=0, 1, 2, \cdots$.

(iii) $(1-r)\sum_{i=0}^{\infty} r^i x_{i+k}$ *converges strongly to some point of* $F(T)$, *as* $r \uparrow 1$, *uniformly in* $k=0, 1, 2, \cdots$.

REMARK 2. If X is a real Hilbert space, $C=-C$, and T is odd (i.e., $-Tx=T(-x)$ for all $x \in C$) asymptotically nonexpansive, the assumption of Corollary 2 is satisfied.

Let $Q : \boldsymbol{R}^+ \times \boldsymbol{R}^+ \to \boldsymbol{R}$ be a function satisfying the following conditions:

$$(a) \qquad \sup_{s \geq 0} \int_0^\infty |Q(s, t)|\, dt < +\infty ;$$

$$(b) \qquad \lim_{s \to \infty} \int_0^\infty |Q(s, t)|\, dt = 1 ;$$

$$(c) \qquad \lim_{s \to \infty} \int_0^\infty |Q(s, t+h) - Q(s, t)|\, dt = 0 \qquad \text{for all } h \geq 0 .$$

Then, $Q(\cdot, \cdot)$ is called a *strongly regular kernel*. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be an asymptotically nonexpansive semigroup on C satisfying (1.2) and let a function $u(\cdot) : \boldsymbol{R}^+ \to C$ be an almost-orbit of $\mathscr{T}$, i.e.,

$$\lim_{s \to \infty} \left[\sup_{t \geq 0} \| u(t+s) - T(t)u(s) \| \right] = 0 .$$

Put $G = \boldsymbol{R}^+$, $\mathscr{T} = \{T(t) : t \in G\}$, and $D = \{v \in m(G) : v(\cdot)$ is a strongly measurable function on $G\}$ in Theorem 1. We get the following results.

COROLLARY 3. *Suppose that* $\lim_{t \to \infty} \| u(t) - u(t+h) \|$ *exists uniformly in* $h \geq 0$ *and that* $u(\cdot)$ *is strongly measurable. Then, the following hold*:

(i) $s^{-1} \int_0^s u(t+h)dt$ *converges strongly to some point of* $F(\mathscr{T})$, *as* $s \to \infty$, *uniformly in* $h \geq 0$.

(ii) *If* $Q(\cdot, \cdot)$ *is a strongly regular kernel, then* $\int_0^\infty Q(s, t)u(t+h)dt$ *converges strongly to some point of* $F(\mathscr{T})$, *as* $s \to \infty$, *uniformly in* $h \geq 0$.

(iii) $\lambda \int_0^\infty e^{-\lambda t} u(t+h)dt$ *converges strongly to some point of* $F(\mathscr{T})$, *as* $\lambda \downarrow 0$, *uniformly in* $h \geq 0$.

5. Concluding remarks.

In this section, we briefly investigate the ergodic theorem for affine semigroups in general Banach spaces.

Throughout this section, we suppose that C is a nonempty closed convex subset of a real Banach space X, G a commutative topological semigroup with the identity. Now we consider a commutative semigroup $\mathscr{T} = \{T(t) : t \in G\}$ on C which satisfies the following conditions:

(d) For each $t \in G$, $T(t)$ is affine, i.e.,

$$T(t)(\alpha x+(1-\alpha)y)=\alpha T(t)x+(1-\alpha)T(t)y \qquad \text{for all } 0 \leq \alpha \leq 1 \text{ and } x, y \in C;$$

(e) There exists a constant $M \geq 0$ such that

$$\| T(t)x - T(t)y \| \leq M \| x-y \| \qquad \text{for all } t \in G \text{ and } x, y \in C.$$

Then, we call $\mathscr{T}=\{T(t): t \in G\}$ an *affine semigroup on C* for simplicity.

In what follows, let $\mathscr{T}=\{T(t): t \in G\}$ be an affine semigroup on C and let $\{u(t): t \in G\}$ be an almost-orbit of $\mathscr{T}$.

For each $\varepsilon>0$ and $r \in G$, we set

$$F_\varepsilon(T(r))=\{x \in C: \| T(r)x-x \| \leq \varepsilon\}.$$

$F_\varepsilon(T(r))$ may be possibly empty. $F_\varepsilon(T(r))$ is closed convex because $T(r)$ is affine. Since G is commutative, there exists a net $\{\lambda_\alpha: \alpha \in I\}$ of finite means on G such that

$$\lim_{\alpha \in I} \| \lambda_\alpha - r_s^* \lambda_\alpha \| = 0, \tag{5.1}$$

for every $s \in G$, where I is a directed set. See [6]. We can prove the following lemma for this net $\{\lambda_\alpha: \alpha \in I\}$, which is analogous to [18, Lemma 6]. Its proof is a simple modification, and hence we omit it.

LEMMA 4. *Suppose that $F(\mathscr{T})$ $(\equiv \bigcap_{s \in G} F(T(s))) \neq \varnothing$. For any $\varepsilon>0$ and $q \in G$ there exists $\alpha_0(\varepsilon, q) \in I$ satisfying*

$$\int u(w+s)d\lambda_\alpha(s) \in F_\varepsilon(T(q))$$

for all $\alpha \geq \alpha_0(\varepsilon, q)$ and $w \in G$.

PROPOSITION. *Let $\{u(t): t \in G\}$ be an almost-orbit of $\mathscr{T}=\{T(t): t \in G\}$ and suppose that the weak closure of $\{u(t): t \in G\}$ is weakly compact. Then, $F(\mathscr{T}) \neq \varnothing$.*

PROOF. Let $\{\lambda_\alpha: \alpha \in I\}$ be a net of finite means on G satisfying (5.1). Since $\int u(t)d\lambda_\alpha(t)$ is contained in $clco\{u(t): t \in G\}$ which is weakly compact by the assumption, there exists a subnet $\{\lambda_{\alpha_\beta}: \beta \in J\}$ of the net $\{\lambda_\alpha: \alpha \in I\}$ such that w-$\lim_{\beta \in J} \int u(t)d\lambda_{\alpha_\beta}(t)=y$. Then, $y \in C$. We will show that $y \in F(\mathscr{T})$.

Let $\varepsilon>0$ be arbitrarily given. Fix $x^* \in X^*$ with $\| x^* \|=1$. Let $s \in G$. Since $T(s)$ is affine, note that w-$\lim_{\beta \in J} \int T(s)u(t)d\lambda_{\alpha_\beta}(t)=$ w-$\lim_{\beta \in J} T(s)(\int u(t)d\lambda_{\alpha_\beta}(t))=T(s)y$. Also, $\{T(r)u(p): p, r \in G\}$ is bounded by the assumption. Put $d=\sup_{p, r \in G} \| T(r)u(p) \|$. By (2.1), there exists $s_0(=s_0(\varepsilon)) \in G$ such that

$$\sup_{t \in G} \| u(t+q)-T(t)u(q) \| < \varepsilon/5 \tag{5.2}$$

for all $q \geq s_0$. Considering what is mentioned above and (5.1), we can choose

$\beta(=\beta(\varepsilon, s))\in J$ so that

$$\| \lambda_{\alpha_\beta} - r^*_{s+s_0}\lambda_{\alpha_\beta} \| < \varepsilon/5d\,, \tag{5.3}$$

$$\| \lambda_{\alpha_\beta} - r^*_{s_0}\lambda_{\alpha_\beta} \| < \varepsilon/5d\,, \tag{5.4}$$

$$\left| \left\langle y - \int u(t)\lambda_{\alpha_\beta}(t), x^* \right\rangle \right| < \varepsilon/5\,, \tag{5.5}$$

and

$$\left| \left\langle \int T(s)u(t)d\lambda_{\alpha_\beta}(t) - T(s)y, x^* \right\rangle \right| < \varepsilon/5\,. \tag{5.6}$$

From (5.3),

$$\begin{aligned} &\left\| \int u(t)d\lambda_{\alpha_\beta}(t) - \int u(t+s_0+s)\lambda_{\alpha_\beta}(t) \right\| \\ &\quad = \left\| \int u(t)\lambda_{\alpha_\beta}(t) - \int u(t)d(r^*_{s+s_0}\lambda_{\alpha_\beta})(t) \right\| \\ &\quad \le \sup_{q\in G} \| u(q) \| \cdot \| \lambda_{\alpha_\beta} - r^*_{s+s_0}\lambda_{\alpha_\beta} \| < \varepsilon/5\,. \end{aligned} \tag{5.7}$$

By (5.4),

$$\begin{aligned} &\left\| \int T(s)u(t+s_0)d\lambda_{\alpha_\beta}(t) - \int T(s)u(t)d\lambda_{\alpha_\beta}(t) \right\| \\ &\quad \le \sup_{p,r\in G} \| T(r)u(p) \| \cdot \| \lambda_{\alpha_\beta} - r^*_{s_0}\lambda_{\alpha_\beta} \| < \varepsilon/5\,. \end{aligned} \tag{5.8}$$

Also, by (5.2) we have

$$\begin{aligned} &\left\| \int u(t+s_0+s)d\lambda_{\alpha_\beta}(t) - \int T(s)u(t+s_0)d\lambda_{\alpha_\beta}(t) \right\| \\ &\quad \le \sup_{t\in G,\, q\ge s_0} \| u(t+q) - T(t)u(q) \| < \varepsilon/5\,. \end{aligned}$$

Consequently, combining this with (5.5)–(5.8) we get

$$\begin{aligned} |\langle y - T(s)y, x^* \rangle| \le &\left| \left\langle y - \int u(t)d\lambda_{\alpha_\beta}(t), x^* \right\rangle \right| \\ &+ \left\| \int u(t)d\lambda_{\alpha_\beta}(t) - \int u(t+s_0+s)d\lambda_{\alpha_\beta}(t) \right\| \\ &+ \left\| \int u(t+s_0+s)d\lambda_{\alpha_\beta}(t) - \int T(s)u(t+s_0)d\lambda_{\alpha_\beta}(t) \right\| \end{aligned}$$

$$+\left\| \int T(s)u(t+s_0)d\lambda_{\alpha_\beta}(t) - \int T(s)u(t)d\lambda_{\alpha_\beta}(t) \right\|$$
$$+\left| \left\langle \int T(s)u(t)d\lambda_{\alpha_\beta}(t) - T(s)y, x^* \right\rangle \right|$$
$$< \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = \varepsilon .$$

Since ε is arbitrary, $y = T(s)y$ for all $s \in G$. Therefore, $F(\mathscr{T}) \neq \emptyset$. □

We can show the next lemma by Lemma 4 and Proposition in the same way as the proof of [18, Lemma 7].

LEMMA 5. *Suppose that a subspace D of $m(G)$ satisfies the property (*) in Section 2 and that the weak closure of $\{u(t) : t \in G\}$ is weakly compact. For any invariant mean μ on D, u_μ belongs to $F(\mathscr{T}) \cap \bigcap_{s \in G} clco\{u(t) : t \geq s\}$.*

Using Lemma 5, we have the following lemma similarly as the proof of Lemma 3.

LEMMA 6. *Suppose that the assumption in Lemma 5 is satisfied. Let μ and λ be invariant means on D. Then we have $u_\mu = u_\lambda$.*

The second main result of this paper is the following.

THEOREM 2. *Let D be a subspace of $m(G)$ containing a constant function 1 and invariant under r_s for every $s \in G$. Let C be a nonempty closed convex subset of a real Banach space X and let $\{u(t) : t \in G\}$ be an almost-orbit of an affine semigroup $\mathscr{T} = \{T(t) : t \in G\}$ on C such that the function $t \mapsto \langle u(t), x^* \rangle$ is in D for all $x^* \in X^*$ and let $\{\mu_\alpha : \alpha \in I\}$ be a strongly regular net of continuous linear functionals on D. If the weak closure of $\{u(t) : t \in G\}$ is weakly compact, then $\int u(t+s)d\mu_\alpha(s)$ converges strongly to an element u_μ in $F(\mathscr{T})$ uniformly in $t \in G$, and $u_\mu = u_\lambda$ for invariant means μ and λ on D.*

ACKNOWLEDGEMENT. The author thanks to Professor I. Miyadera for his constant encouragement and advice.

References

[1] J. B. BAILLON, Quelques propriétés de convergence asymptotique pour les semi-groupes de contractions impaires, C.R. Acad. Sci. Paris Sér. A-B, **283** (1976), 75–78.

[2] J. B. BAILLON, Quelques propriétés de convergence asymptotique pour les contractions impaires, C.R. Acad. Sci. Paris Sér. A-B, **283** (1976), 587–590.

[3] R. E. BRUCK, On the almost-convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak ω-limit set, Israel J. Math., **29** (1978), 1–16.

[4] R. E. BRUCK, On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces, Israel J. Math., **38** (1981), 304–314.

[5] R. E. BRUCK, Asymptotic behavior of nonexpansive mappings, *Fixed Points and Nonexpansive Mappings*, Amer. Math. Soc. Contemporary Math., **18** (1983), 1–47.

[6] M. M. DAY, Amenable semigroups, Illinois J. Math., **1** (1957), 509–544.

[7] W. F. EBERLEIN, Abstract ergodic theorems and weak almost periodic functions, Trans. Amer. Math. Soc., **67** (1949), 217–240.

[8] K. GOEBEL and W. A. KIRK, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc., **35** (1972), 171–174.

[9] J. A. GOLDSTEIN, S. OHARU and A. VOGT, Affine semigroups on Banach spaces, Hiroshima Math. J., **18** (1988), 433–450.

[10] N. HIRANO, K. KIDO and W. TAKAHASHI, Asymptotic behavior of commutative semigroups of nonexpansive mappings in Banach spaces, Nonlinear Analysis, **10** (1986), 229–249.

[11] N. HIRANO, K. KIDO and W. TAKAHASHI, Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces, Nonlinear Analysis, **12** (1988), 1269–1281.

[12] K. KIDO and W. TAKAHASHI, Mean ergodic theorems for semigroups of linear operators, J. Math. Anal. Appl., **103** (1984), 387–394.

[13] K. KIDO and W. TAKAHASHI, Means on commutative semigroups and nonlinear ergodic theorems, J. Math. Anal. Appl., **111** (1985), 585–605.

[14] K. KOBAYASI and I. MIYADERA, On the strong convergence of the Cèsaro means of contractions in Banach spaces, Proc. Japan Acad. Ser. A, **56** (1980), 245–249.

[15] G. G. LORENTZ, A contribution to the theory of divergent sequences, Acta Math., **80** (1948), 167–190.

[16] I. MIYADERA and K. KOBAYASI, On the asymptotic behavior of almost-orbits of nonlinear contraction semigroups in Banach spaces, Nonlinear Analysis, **6** (1982), 349–365.

[17] H. OKA, On the nonlinear mean ergodic theorems for asymptotically nonexpansive mappings in Banach spaces, Publ. Res. Inst. Math. Sci. (Kôkyûroku), **730** (1990), 1–20.

[18] H. OKA, Nonlinear ergodic theorems for commutative semigroups of asymptotically nonexpansive mappings, to appear in Nonlinear Analysis.

[19] S. REICH, A note on the mean ergodic theorem for nonlinear semigroups, J. Math. Anal. Appl., **91** (1983), 547–551.

[20] W. M. RUESS and W. H. SUMMERS, Weak almost periodicity and the strong ergodic limit theorem for contraction semigroups, Israel J. Math., **64** (1988), 139–157.

[21] W. TAKAHASHI, *Nonlinear Functional Analysis* (in Japanese), Kindaikagaku (Tokyo, 1988).

[22] W. TAKAHASHI and P. J. ZHANG, Asymptotic behavior of almost-orbits of semigroups of Lipschitzian mappings in Banach spaces, Kodai Math. J., **11** (1988), 129–140.

Present Address:
DEPARTMENT OF MATHEMATICS, SCHOOL OF EDUCATION, WASEDA UNIVERSITY
NISHI-WASEDA, SHINJUKU-KU, TOKYO 169–50, JAPAN