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Entangled Linear Orders in the Easton's Models

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Abstract. The notion of an entangled linear order was first introduced by Avraham and Shelah [1]. Subsequently, Todorcevic [5] generalized it to higher cardinals and mentioned it is useful to solve problems such as the productivity of chain conditions and the square bracket partition relations. He also showed that if $\mathbf{wCH}(\mu)$ holds there is a 2^{μ}-entangled linear order of size 2^{μ}. From this, we can immediately observe that **GCH** implies the full existence of entangled linear orders. In this paper we will show that such full existence occurs also in the Easton's models in which we can arbitrarily determine the powers of infinite regular cardinals.

1. Entangled linear orders.

Let $(L, \leq)$ be a linear order. For $x \in [L]^n$ $(n \in N)$ we represent by $x\langle i\rangle$ the $(i+1)$-th member of x with respect to $\leq$. For disjoint $x, y \in [L]^n$ and $s \in {}^n2$, we say that the *type of entanglement* of x and y is s if $x\langle i\rangle < y\langle i\rangle \Leftrightarrow s(i)=0$ for all $i<n$. Using this notation, we introduce a notion that we mainly concern in this paper.

DEFINITION 1.1. κ is an infinite cardinal and $n \in N$. A linear order $(L, \leq)$ is called *(κ, n)-entangled*, if for any pairwise disjoint family $X \subseteq [L]^n$ of size κ and any $s \in {}^n2$ there exist $x, y \in X$ whose type of entanglement is s. $(L, \leq)$ is called *κ-entangled* if it is (κ, n)-entangled for all $n \in N$.

Find some elementary facts about entangledness below.

LEMMA 1.2. *For any infinite κ and any $n \in N$ the following hold.*

(i) *A (κ, n)-entangled linear order is also (κ, m)-entangled for all $m \leq n$.*

(ii) *All linear orders of size less than κ are κ-entangled.*

(iii) *A (κ, n)-entangled linear order of size $\lambda \geq \kappa$ is also (λ, n)-entangled.*

(iv) *A subset of a (κ, n)-entangled linear order is (κ, n)-entangled.*

(v) *All linear orders are $(\kappa, 0)$-entangled and $(\kappa, 1)$-entangled.*

(vi) *$(L, \leq)$ is a $(\kappa, 2)$-entangled linear order iff there are no disjoint $S, T \subseteq L$ of size κ that are order isomorphic or inversely isomorphic.*

(vii) *A $(\kappa, 2)$-entangled linear order does not have a discrete subset of size κ.*

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(viii) *A* $(\kappa, 2)$*-entangled linear order does not have pairwise disjoint* κ *many open intervals.*

(ix) *A* $(\kappa, 3)$*-entangled linear order has a dense subset of size less than* κ.

PROOF. (i)–(v) are clear from Definition 1.1. Let us show (vi). Suppose $S, T \subseteq L$ are disjoint sets of size κ and $\varphi: S \to T$ is an order isomorphism (the case where φ is an inverse isomorphism is similar). Either $\{r \in S; r < \varphi(r)\}$ or $\{r \in S; r > \varphi(r)\}$ is of size κ. Without loss of generality, we assume $S' = \{r \in S; r < \varphi(r)\}$ is such. Let $X = \{\{r, \varphi(r)\} \in [L]^2; r \in S'\}$ and $s(i) = 1 - i$ for $i < 2$. Then there are no $x, y \in X$ whose type of entanglement is s. The other direction is now clear. For (vii), let $K \subseteq L$ be a discrete subset of size κ each of whose element r is isolated by (r^-, r^+). We construct inductively a sequence $\langle r_\alpha; \alpha < \kappa \rangle$ in L so that $r_\alpha \in K \setminus \bigcup_{\gamma < \alpha} \{r_\gamma^-, r_\gamma, r_\gamma^+\}$ for all $\alpha < \kappa$. Since either $r_\alpha^- < r_\alpha < r_\alpha^+ \leq r_\beta^- < r_\beta < r_\beta^+$ or $r_\beta^- < r_\beta < r_\beta^+ \leq r_\alpha^- < r_\alpha < r_\alpha^+$ holds for any distinct $\alpha, \beta < \kappa$, it is clear that $\{r_\alpha; \alpha < \kappa\}$ and $\{r_\alpha^+; \alpha < \kappa\}$ are disjoint and isomorphic, hence by (vi) we are done. For (viii), suppose, for a contradiction, $(L, \leq)$ is $(\kappa, 2)$-entangled and has a pairwise disjoint family $\{I_\alpha; \alpha < \kappa\}$ of open intervals. By (vii) we may assume each I_α has an infinite size. Choose distinct $r_\alpha^0, r_\alpha^1 \in I_\alpha$ for each $\alpha < \kappa$, then $\{r_\alpha^0; \alpha < \kappa\}$ and $\{r_\alpha^1; \alpha < \kappa\}$ are disjoint and isomorphic and we are done. For (ix), suppose $(L, \leq)$ is $(\kappa, 3)$-entangled and has no dense subset of size less than κ. Notice that $L \setminus \bar{K}$ is open and of size κ if $K \subseteq L$ is of size less than κ. Using this fact together with (vii), we can construct inductively a sequence $\langle \langle r_\alpha^0, r_\alpha^1, r_\alpha^2 \rangle; \alpha < \kappa \rangle$ of triplets from L such that $r_\alpha^0 < r_\alpha^1 < r_\alpha^2$ and $[r_\alpha^0, r_\alpha^2] \cap \overline{\bigcup_{\gamma < \alpha} \{r_\gamma^0, r_\gamma^1, r_\gamma^2\}} = \emptyset$ hold for all $\alpha < \kappa$. Now let $X = \{\{r_\alpha^0, r_\alpha^1, r_\alpha^2\}; \alpha < \kappa\}$ and define $s \in {}^3 2$ by "$s(i) = 1$ iff $i = 1$". Then there are no $x, y \in X$ with their type of entanglement s. This is a contradiction. □

Todorcevic [5] showed that the existence of an entangled linear order induces the negative solution of problems such as the productivity of a chain condition and a square bracket partition relation.

THEOREM 1.3 (Todorcevic). *If there is a* κ*-entangled linear order of size* λ *then the following* (I) *and* (II) *hold.*

(I) *If* κ *is a regular cardinal, for any* $k \in N$ *there is a poset* $\mathcal{Q}$ *such that* $\mathcal{Q}^k$ *satisfies the* κ*-cc but* $\mathcal{Q}^{k+1}$ *does not satisfy the* λ*-cc.*

(II) *The square bracket partition relation* $\lambda \to [\kappa]^2_{\aleph_0}$ *fails* (*i.e.* $[\lambda]^2$ *can be partitioned into* $\aleph_0$ *fragments* $\mathcal{W}$ *so that* $[A]^2 \cap W \neq \emptyset$ *holds for any* $A \subseteq \lambda$ *of size* κ *and any* $W \in \mathcal{W}$).

2. The maximal size of an entangled linear order.

By (ix) of Lemma 1.2, we get a necessary condition for the possible size of a κ-entangled linear order as follows.

LEMMA 2.1. *For any infinite cardinal* κ, *if there is a* κ*-entangled linear order* (*or a* $(\kappa, 3)$*-entangled linear order in fact*) *of size* λ *then*:

(A1) $\exists\mu<\kappa\ (\lambda\leq 2^{\mu})$ *holds and*

(A2) *if* $\kappa=\mu^{+}$ *and* μ *is a singular cardinal then* $\lambda\leq\mu^{<\mu}$ *holds.*

PROOF. Let $(L, \leq)$ be a κ-entangled linear order of size λ. By (ix) of Lemma 1.2, $(L, \leq)$ has a dense subset K of size $\mu<\kappa$. For $r\in L$ let us define $C_r\subseteq K$ by $C_r=\{s\in K;\ s\leq r\}$, then $r<r'$ implies $C_r\neq C_{r'}$ because some $s\in K$ between r and r' is in $C_{r'}\setminus C_r$. This means $\lambda=|L|\leq|\wp(K)|=2^{\mu}$. When $\kappa=\mu^{+}$ and μ is singular, we can have a slightly stronger condition **(A2)** for λ as follows. Let $K\subseteq L$ be as above and $\nu<\mu$ be the cofinality of μ. Then there is an increasing (with respect to $\subseteq$) sequence $\langle K_\alpha;\ \alpha<\nu\rangle$ of subsets of K such that $|K_\alpha|<\mu$ for any $\alpha<\nu$ and $K=\bigcup_{\alpha<\nu}K_\alpha$. For $\alpha<\nu$ and $r\in L$ we represent by C_r^α a set of all $s\in K_\alpha$ such that $s\leq r$. Here we choose $A_r\in[K]^{<\mu}$ for each $r\in L$ as follows.

- If there is $\alpha<\nu$ such that $\sup_L C_r^\alpha=r$ then let A_r be C_r^α for such α.
- If there is no such $\alpha<\nu$ then choose a sequence $\langle s_\alpha;\ \alpha<\nu\rangle$ from K so that $C_r^\alpha\leq s_\alpha<r$ holds for any $\alpha<\nu$ and let $A_r=\{s_\alpha;\ \alpha<\nu\}$.

It is clear that $\sup_L A_r=r$ for all $r\in L$, hence if $r\neq r'$ then $A_r\neq A_{r'}$ for all $r, r'\in L$. So $\lambda=|L|\leq|[K]^{<\mu}|=\mu^{<\mu}$. □

It is well-known that the countable chain condition is a productive property if $\mathbf{MA}_{\aleph_1}$ (*the Martin's Axiom*) holds, hence there is no uncountable $\aleph_1$-entangled linear order by (I) of Theorem 1.3. And moreover it can be observed by (i), (vi) and (ix) of Lemma 1.2 that **OCA** (*the Open Coloring Axiom*) implies the non-existence of an uncountable $(\aleph_1, 3)$-entangled linear order. Each of these facts says it is consistent that **(A1)** and **(A2)** does not give the sufficient condition for the size of an entangled linear order. But if **GCH** holds they give it by the following theorem.

THEOREM 2.2 (Todorcevic). *Let* μ *be an infinite cardinal such that* $\mathrm{w}\mathbf{CH}(\mu)$ *holds (i.e.* $2^{<\mu}=\mu$*). If* $\kappa\leq 2^{\mu}$ *and* $\mathrm{cf}\kappa=\mathrm{cf}2^{\mu}$, *there is a* κ*-entangled linear order of size* κ.

COROLLARY. *If* **GCH** *holds, there is a* κ*-entangled linear order of size* λ *iff* **(A1)** *and* **(A2)** *hold.*

PROOF OF COROLLARY. We first observe that **(A1)** and **(A2)** are equivalent to the following (a) and (b) under **GCH**.

(a) If κ is a successor cardinal then $\lambda\leq\kappa$.

(b) If κ is a limit cardinal then $\lambda<\kappa$.

So, by Lemma 2.1 and (ii) of Lemma 1.2 it is sufficient to show that there are μ^{+}-entangled linear order of size μ^{+} for any infinite cardinal μ. But it is clear from Theorem 2.2 since $2^{<\mu}=\mu$ and $2^{\mu}=\mu^{+}$ by **GCH**. □

This means that entangled linear orders fully exist under **GCH**. Our main aim in this paper is to show such full existence of entangled linear orders occurs also in the Easton's models, in which we can arbitrarily determine the powers of infinite regular cardinals.

3. The Easton's models.

This section is devoted to a short survey of the Easton's models in order to introduce some elementary facts about them that are needed in the next section. For more detailed information a reader should consult W. B. Easton's original paper [2] or the chapter 3 of Jech [3].

Let M be a c.t.m. of **ZFC**+**GCH**+**TWA**, where **TWA** is *the Total Well-ordering Axiom* which says that the whole universe can be well-ordered by some class relation. We represent by Ord^M, Card^M and Reg^M the class of all ordinals, infinite cardinals and infinite regular cardinals in M respectively. A class function $\boldsymbol{F}$ from Reg^M to Card^M in M is called an *index function* when the following hold.

- o $\mu < \boldsymbol{F}(\mu)$ for all $\mu \in \mathrm{Reg}^M$.
- o If $\nu \leq \mu$ then $\boldsymbol{F}(\nu) \leq \boldsymbol{F}(\mu)$ for all $\mu, \nu \in \mathrm{Reg}^M$.
- o $\mu < (\mathrm{cf}\,\boldsymbol{F}(\mu))^M$ for all $\mu \in \mathrm{Reg}^M$.

For a given index function $\boldsymbol{F}$ we define a class poset as follows.

DEFINITION 3.1. Let $\boldsymbol{F}$ be an index function. We define a class poset $\boldsymbol{P}$ with the inverse inclusion ordering as a collection of all $p \in M$ satisfying the following (a)–(c).

(a) p is a function such that $\mathrm{dom}\,p \subseteq \mathrm{Reg}^M \times \mathrm{Ord}^M \times \mathrm{Ord}^M$ and $\mathrm{ran}\,p \subseteq \{0, 1\}$.

(b) If $\langle \mu, \alpha, \xi \rangle \in \mathrm{dom}\,p$ then $\alpha < \boldsymbol{F}(\mu)$ and $\xi < \mu$.

(c) $|\{\langle \nu, \alpha, \xi \rangle \in \mathrm{dom}\,p\,;\, \nu \leq \mu\}|^M < \mu$ for all $\mu \in \mathrm{Reg}^M$.

For any $H \subseteq \boldsymbol{P}$ (possibly H is not a class in M) and a class $A \subseteq \mathrm{Reg}^M$ in M, a *restriction* $H \upharpoonright A$ *of* H *to* A is a set $\{p \in \boldsymbol{P}\,;\, \mathrm{dom}\,p \subseteq A \times \mathrm{Ord}^M \times \mathrm{Ord}^M\}$. In particular we also write H^μ or $H^{\leq\mu}$ for $H \upharpoonright A$ when A is respectively $\{\mu\}$ or $\{\nu \in \mathrm{Reg}^M\,;\, \nu \leq \mu\}$ for $\mu \in \mathrm{Reg}^M$. So the class $M^{\boldsymbol{P}}$ of all $\boldsymbol{P}$-names is defined as $\bigcup \{M^{\boldsymbol{P}^{\leq\mu}}\,;\, \mu \in \mathrm{Reg}^M\}$. Next, let us generalize the notion of generic filters to our class poset $\boldsymbol{P}$. We call $G \subseteq \boldsymbol{P}$ an *M-generic filter on* $\boldsymbol{P}$ if G is a filter on $\boldsymbol{P}$ such that $G \cap D \neq \varnothing$ for every class D in M which is dense in $\boldsymbol{P}$. It is clear that if G is an M-generic filter on $\boldsymbol{P}$ and $\mu \in \mathrm{Reg}^M$ then $G^{\leq\mu}$ is an M-generic filter on $\boldsymbol{P}^{\leq\mu}$ in the usual sense, hence each $M[G^{\leq\mu}] = \{\mathrm{val}(\sigma, G^{\leq\mu})\,;\, \sigma \in M^{\boldsymbol{P}^{\leq\mu}}\}$ is a c.t.m. of **ZFC** with its ordinals the same as M's.

Here, we fix an M-generic filter G on $\boldsymbol{P}$ and let an *Easton's model* $M[G]$ *for* $\boldsymbol{F}$ be $\bigcup \{M[G^{\leq\mu}]\,;\, \mu \in \mathrm{Reg}^M\}$, then $M[G]$ is a c.t.m. of **ZFC** and the following lemma holds.

LEMMA 3.2. *M, $\boldsymbol{F}$, $\boldsymbol{P}$ and G are as above. For any $\kappa \in \mathrm{Reg}^M$, a function $f \in M[G]$ from κ to Ord^M belongs to $M[G^{\leq\kappa}]$ in fact.*

By Lemma 3.2, a generic extension by $\boldsymbol{P}$ does not change cofinalities and cardinalities since $\boldsymbol{P}^{\leq\kappa}$ satisfies the κ^+-*cc* in M and hence preserves all cofinalities $\geq \kappa^+$ for any $\kappa \in \mathrm{Reg}^M$.

The most important property of an Easton's model is that the powers of infinite regular cardinals in it are given by its index function.

THEOREM 3.3 (Easton). *For any regular cardinal μ in $M[G]$ (hence in M) it holds that*:

$$M[G]\models 2^{\mu}=F(\mu)\,.$$

Moreover it is known that the singular cardinals hypothesis (i.e. $2^{\nu}<\mu$ implies $\mu^{\nu}=\mu^{+}$ for any singular cardinal μ of cofinality ν) holds in $M[G]$. So we can completely calculate the cardinal exponentation in $M[G]$. Here we mention just the form of $\mu^{<\mu}$, which will be needed later.

LEMMA 3.4. *In $M[G]$, the cardinal exponentation $\mu^{<\mu}$ for a singular cardinal μ is calculated as follows.*

$$\mu^{<\mu}=\begin{cases}\mu^{+} & \textit{if } \mu \textit{ is strongly limit}\\ 2^{<\mu} & \textit{otherwise}\,.\end{cases}$$

4. Entangled linear orders in the Easton's models.

Now we will show that **(A1)** and **(A2)** of Lemma 2.1 also give a sufficient condition for the existence of a κ-entangled linear order of size λ in $M[G]$. Throughout this section F is any index function and $M[G]$ is an Easton's model for F.

Let $\mathrm{Ord}^{M[G]}$, $\mathrm{Card}^{M[G]}$ and $\mathrm{Reg}^{M[G]}$ denote the class of all ordinals, infinite cardinals and infinite regular cardinals in $M[G]$ respectively. By the argument in the previous section, they are identical with Ord^{M}, Card^{M} and Reg^{M}. For $\mu\in\mathrm{Reg}^{M[G]}$ and $\alpha<F(\mu)$, we represent by f^{α}_{μ} *the* $\langle\mu,\alpha\rangle$*-th generic function* that is a function from μ to $\{0,1\}$ defined as:

$$f^{\alpha}_{\mu}(\xi)=\varepsilon \Leftrightarrow \exists p\in G^{\leq\mu}\qquad (\langle\mu,\alpha,\xi\rangle\in\mathrm{dom}\,p\wedge p(\langle\mu,\alpha,\xi\rangle)=\varepsilon)$$

for any $\xi<\mu$ and $\varepsilon\in\{0,1\}$. We can easily check the well-definedness of this definition. And a standard forcing argument shows that if $\langle\mu,\alpha\rangle\neq\langle\nu,\beta\rangle$ then there is an ordinal $\eta<\min(\mu,\nu)$ such that $f^{\alpha}_{\mu}(\eta)\neq f^{\beta}_{\nu}(\eta)$. So we represent the least such ordinal by $\eta(f^{\alpha}_{\mu},f^{\beta}_{\nu})$.

DEFINITION 4.1. For $\mu\in\mathrm{Card}^{M[G]}$ we define a linear order $(L_{\mu},\leq_{\mu})$ as follows.
- $L_{\mu}=\{\langle\nu,\beta\rangle\in\mathrm{Reg}^{M[G]}\times\mathrm{Ord}^{M[G]}\,;\,\nu\leq\mu\wedge\beta<F(\nu)\}$.
- For any $\langle\nu,\beta\rangle,\langle\nu',\beta'\rangle\in L_{\mu}$, $\langle\nu,\beta\rangle\leq_{\mu}\langle\nu',\beta'\rangle$ iff $\langle\nu,\beta\rangle=\langle\nu',\beta'\rangle$ or otherwise $f^{\beta}_{\nu}(\eta(f^{\beta}_{\nu},f^{\beta'}_{\nu'}))<f^{\beta'}_{\nu'}(\eta(f^{\beta}_{\nu},f^{\beta'}_{\nu'}))$.

$(L_{\mu},\leq_{\mu})$ is in $M[G^{\leq\mu}]$ in fact.

It is clear from definition that if $\nu\leq\mu$ then $(L_{\nu},\leq_{\nu})$ is a suborder of $(L_{\mu},\leq_{\mu})$, so we will omit the subscript of the ordering from now on.

LEMMA 4.2. *Let $\mu\in\mathrm{Reg}^{M[G]}$ and $\kappa=\mu^{+}$. Then $(L_{\mu},\leq)$ is a κ-entangled linear order in $M[G]$.*

To show this lemma, notice that any $X\subseteq[L_{\mu}]^{n}$ (where $n\in N$) of size κ in $M[G]$

belongs to $M[G^{\leq\kappa}]$ by Lemma 3.2. Then $(L_\mu, \leq)$ is κ-entangled in $M[G]$ iff so is it in $M[G^{\leq\kappa}]$. And since $\boldsymbol{P}^{\leq\kappa}$ is canonically isomorphic to $\boldsymbol{P}^{\kappa}\times\boldsymbol{P}^{\leq\mu}$, we get $M[G^{\leq\kappa}]=M[G^{\kappa}][G^{\leq\mu}]$ where G^{κ} is an M-generic filter on $\boldsymbol{P}^{\kappa}$ and $G^{\leq\mu}$ is an $M[G^{\kappa}]$-generic filter on $\boldsymbol{P}^{\leq\mu}$. We will first introduce the lemma which plays an important role in the proof of Lemma 4.2.

LEMMA 4.3. *Let* $\mu\in\mathrm{Reg}^{M[G]}$ *and* $\kappa=\mu^{+}$. *Then the following* (i) *and* (ii) *hold in* $M[G^{\kappa}]$.

(i) $2^{\nu}=\nu^{+}$ *for any infinite cardinal* $\nu\leq\mu$.

(ii) $\boldsymbol{P}^{\leq\mu}$ *satisfies the strong* κ-*cc* (*i.e. every subset of* $\boldsymbol{P}^{\leq\mu}$ *of size* κ *includes a pairwise compatible set of size* κ).

PROOF. Since $\boldsymbol{P}^{\kappa}$ is κ-*closed* and G^{κ} is an M-generic filter on $\boldsymbol{P}^{\kappa}$, all subsets of μ in $M[G^{\kappa}]$ belong to M. Then (i) is clear because **GCH** holds in M. For (ii) we work in $M[G^{\kappa}]$. Let $\{p_\gamma;\gamma<\kappa\}$ be any subset of $\boldsymbol{P}^{\leq\mu}$. Since $|\mathrm{dom}p_\gamma|<\mu$ for all $\gamma<\kappa$ and $\mu^{<\mu}=\mu<\kappa$ holds in $M[G^{\kappa}]$ by (i) and the regularity of μ, there are $A\subseteq\kappa$ of size κ and d of size $<\mu$ such that $\mathrm{dom}p_\gamma\cap\mathrm{dom}p_\delta=d$ for any distinct $\gamma,\delta\in A$ by Δ-system lemma (see Lemma 4.4 below). Moreover we can find $B\subseteq A$ of size κ so that $p_\gamma\restriction d=p_\delta\restriction d$ for any $\gamma,\delta\in B$ because $|{}^{d}2|\leq 2^{<\mu}=\mu<\kappa=\mathrm{cf}\kappa$ by (i). $\{p_\gamma;\gamma\in B\}$ is clearly pairwise compatible. □

We used the Δ-system lemma for higher cardinals in the proof above. For the readers' convenience we will state it here. The following form of this lemma is seen in Kunen [4].

LEMMA 4.4 (Δ-system Lemma). *Let* μ *be an infinite cardinal and* κ *be a regular cardinal larger than* μ *such that* $\nu^{<\mu}<\kappa$ *for all* $\nu<\kappa$. *If* $\{a_\gamma;\gamma<\kappa\}$ *is a family of sets of size* $<\mu$ *then there are* $A\subseteq\kappa$ *of size* κ *and a set* d *of size* $<\mu$ *such that* $\{a_\gamma;\gamma\in A\}$ *forms a* Δ-*system with a root* d (*i.e.* $a_\gamma\cap a_\delta=d$ *for any distinct* $\gamma,\delta\in A$).

Now we will prove Lemma 4.2. By the argument above, it is sufficient to show that $(L_\mu,\leq)$ is a κ-entangled linear order in $M[G^{\kappa}][G^{\leq\mu}]$. Fix $n\in\boldsymbol{N}$ and let $X=\{x_\gamma;\gamma<\kappa\}$ be a pairwise disjoint subset of $[L_\mu]^n$ in $M[G^{\kappa}][G^{\leq\mu}]$. Let $\dot{X}$ be a $\boldsymbol{P}^{\leq\mu}$-name for X in $M[G^{\kappa}]$ (i.e. $\dot{X}\in M[G^{\kappa}]^{\boldsymbol{P}^{\leq\mu}}$ such that $\mathrm{val}(\dot{X},G^{\leq\mu})=X$) and $p_0\in G^{\leq\mu}$ be a condition such that:

$$[p_0\Vdash_{\boldsymbol{P}^{\leq\mu}}(\dot{X}\subseteq[L_\mu]^n\wedge\forall x,y\in\dot{X}\,(x\neq y\Rightarrow x\cap y=\varnothing))]^{M[G^{\kappa}]}.$$

By standard argument we can find a set $\{\dot{x}_\gamma;\gamma<\kappa\}$ of $\boldsymbol{P}^{\leq\mu}$-names in $M[G^{\kappa}]$ such that $\mathrm{val}(\dot{x}_\gamma,G^{\leq\mu})=x_\gamma$ for any $\gamma<\kappa$. In the rest of this proof we will work in $M[G^{\kappa}]$ and show that for any given $s\in{}^{n}2$ the set of all $q\in\boldsymbol{P}^{\leq\mu}$ which force for some $\gamma,\delta<\kappa$ that:

"the type of entanglement of $\dot{x}_\gamma$ and $\dot{x}_\delta$ is s"

is dense below p_0. If we achieve it, the proof is completed by the $M[G^{\kappa}]$-genericity of

$G^{\leq\mu}$ and $p_0 \in G^{\leq\mu}$.

We first introduce a new notion. For $q \in \boldsymbol{P}^{\leq\mu}$ we say q *has a sectionwise even domain* if for any $\nu \leq \mu$ there are $\zeta < \nu$ and $A \subseteq \boldsymbol{F}(\nu)$ of size $< \nu$ such that:

$$\operatorname{dom} q \cap (\{\nu\} \times \boldsymbol{F}(\nu) \times \nu) = \{\nu\} \times A \times \zeta .$$

We represent ζ and A in the above definition by $\zeta_q(\nu)$ and $A_q(\nu)$. It is clear that the set of all $q \in \boldsymbol{P}^{\leq\mu}$ with sectionwise even domains is dense in $\boldsymbol{P}^{\leq\mu}$. Now fix $s \in {}^n2$ and we will begin. Let $p \in \boldsymbol{P}^{\leq\mu}$ be any condition below p_0. For each $\gamma < \kappa$ we can choose $q_\gamma \leq p$ with a sectionwise even domain and $z_\gamma \in {}^nL_\mu$ such that $q_\gamma \Vdash_{\boldsymbol{P}^{\leq\mu}} (\dot{x}_\gamma\langle i\rangle = z_\gamma(i))$ for any $i < n$. Without loss of generality, we will assume $\beta_\gamma(i) \in A_{q_\gamma}(\nu_\gamma(i))$, where $z_\gamma(i) = \langle \nu_\gamma(i), \beta_\gamma(i)\rangle$, for any $\gamma < \kappa$ and $i < n$. Then by (ii) of Lemma 4.3 there is $B \subseteq \kappa$ of size κ such that $\{q_\gamma;\, \gamma \in B\}$ is pairwise compatible. Furthermore by some iterative use of the Pigeon Hole Principle we reduce B to B' of size κ such that the following hold.

(a) There are fixed $\nu_0, \cdots, \nu_{n-1} \leq \mu$ such that $\nu_\gamma(i) = \nu_i$ for all $\gamma \in B'$ and $i < n$.

(b) There are fixed $\zeta_0 < \nu_0, \cdots, \zeta_{n-1} < \nu_{n-1}$ such that $\zeta_{q_\gamma}(\nu_i) = \zeta_i$ for all $\gamma \in B'$ and $i < n$.

(c) There are fixed $g_0 \in {}^{\zeta_0}2, \cdots, g_{n-1} \in {}^{\zeta_{n-1}}2$ such that $q_\gamma(\langle \nu_i, \beta_\gamma(i), \xi\rangle) = g_i(\xi)$ for all $\gamma \in B'$, $i < n$ and $\xi < \zeta_i$.

So pick distinct $\gamma, \delta \in B'$ and let $\bar{q} = q_\gamma \cup q_\delta$. It is clear that $\bar{q} \in \boldsymbol{P}^{\leq\mu}$, and since $\bar{q} \leq p_0$ we get $z_\gamma(i) \neq z_\delta(j)$ for any $i, j < n$. Moreover, by (a) and (b) we can easily certify that $\langle \nu_\gamma(i), \beta_\gamma(i), \zeta_i\rangle, \langle \nu_\delta(i), \beta_\delta(i), \zeta_i\rangle \notin \operatorname{dom}\bar{q}$ for all $i < n$. Then we define an extension $q \in \boldsymbol{P}^{\leq\mu}$ of $\bar{q}$ as

$$\begin{aligned} q = \bar{q} &\cup \{\langle\langle \nu_\gamma(i), \beta_\gamma(i), \zeta_i\rangle, s(i)\rangle \,;\, i < n\} \\ &\cup \{\langle\langle \nu_\delta(i), \beta_\delta(i), \zeta_i\rangle, 1 - s(i)\rangle \,;\, i < n\} . \end{aligned}$$

Clearly $q \leq \bar{q} \leq p$ and the following holds for each $i < n$ where $\dot{f}^\beta_\nu$ denotes the $\boldsymbol{P}^{\leq\mu}$-name for f^β_ν which is constructed in the canonical way.

$$\begin{aligned} q \Vdash_{\boldsymbol{P}^{\leq\mu}} [&\forall\xi < \zeta_i\, (\dot{f}^{\beta_\gamma(i)}_{\nu_\gamma(i)}(\xi) = g_i(\xi)) \wedge \dot{f}^{\beta_\gamma(i)}_{\nu_\gamma(i)}(\zeta_i) = s(i) \\ &\wedge \forall\xi < \zeta_i\, (\dot{f}^{\beta_\delta(i)}_{\nu_\delta(i)}(\xi) = g_i(\xi)) \wedge \dot{f}^{\beta_\delta(i)}_{\nu_\delta(i)}(\zeta_i) = 1 - s(i)] . \end{aligned}$$

Therefore γ, δ and q are what we needed. □

The next lemma says that we can omit the assumption "μ is regular" in Lemma 4.2.

LEMMA 4.5. *Let $\mu \in \mathrm{Card}^{M[G]}$ and $\kappa = \mu^+$. Then $(L_\mu, \leq)$ is a κ-entangled linear order in $M[G]$.*

PROOF. We have already shown in the case where μ is regular. Let $\mu \in \mathrm{Card}^{M[G]}$ be singular in $M[G]$. Notice that $L_\mu = \bigcup\{L_\nu\,;\, \nu \in \mathrm{Reg}^{M[G]} \wedge \nu < \mu\}$. For any $n \in N$ and any $X \subseteq [L_\mu]^n$ of size κ we can find $Y \subseteq X$ of size κ and $\nu \in \mathrm{Reg}^{M[G]}$ less than μ such that $Y \subseteq [L_\nu]^n$ by the regularity of κ. Then there are $x, y \in Y$ with any type of entanglement because $(L_\nu, \leq)$ is κ-entangled by Lemma 4.2 and (iii) of Lemma 1.2. □

So we have got the following theorem.

THEOREM 4.6. *Let $M[G]$ be an Easton's model for an index function F. Then the necessary and sufficient condition for the existence of a κ-entangled linear order of size λ in $M[G]$ is* **(A1)** *and* **(A2)**.

PROOF. The necessity is by Lemma 2.1. To show the sufficiency we may assume κ is an infinite successor cardinal μ^+, since the limit cases are trivial by the successor cases. If μ is regular then the size of a κ-entangled linear order $(L_\mu, \leq)$ is $F(\mu)=2^\mu$. On the other hand, if μ is singular it is computed as $|L_\mu|=\sup\{F(\nu)\,;\,\nu\in\mathrm{Reg}^{M[G]}\wedge\nu<\mu\}=\sup\{2^\nu\,;\,\nu\in\mathrm{Reg}^{M[G]}\wedge\nu<\mu\}=2^{<\mu}$. So, by Lemma 3.4 and the singular cardinals hypothesis in $M[G]$, the rest case is where μ is strongly limit and $\lambda=\mu^+=2^\mu$. But it is clear from Theorem 2.2 because $\mathbf{wCH}(\mu)$ holds. $\square$

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References

[1] U. AVRAHAM and S. SHELAH, Martin's axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic, Israel J. Math., **38** (1981), 161–176.
[2] W. B. EASTON, Powers of regular cardinals, Ann. Math. Logic, **1** (1970), 139–178.
[3] T. JECH, *Set Theory*, Academic Press (1978).
[4] K. KUNEN, *Set Theory, An Introduction to Independence Proofs*, North-Holland (1980).
[5] S. TODORCEVIC, Remarks on chain conditions in products, Compositio Math., **55** (1985), 295–302.

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