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The Law of the Iterated Logarithm for the Single Point Range of Random Walk

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Dedicated to Professor Hiroshi Tanaka on his 60th birthday

Abstract. The number of the distinct points entered by a random walk once and only once in the first n steps is called the single point range up to time n. We consider the random walk on the d dimensional integer lattice. When $d \geq 4$, the auther showed a limiting behavior of the variance of the single point range and established the central limit theorem. In this note, we proved the law of the iterated logarithm in the same case.

§1. Introduction.

Let $\{S_n\}_{n=0}^{\infty}$ be a random walk on the d dimensional integer lattice $\boldsymbol{Z}^d$. To define exactly, $\{S_n\}_{n=0}^{\infty}$ is the sequence of random variables with $S_0=0$ and $S_n=\sum_{k=1}^{n} X_k$, where $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent identically distributed random variables defined on some probability space $(\Omega, \mathfrak{B}, P)$, which take values in $\boldsymbol{Z}^d$. We assume that the random walk is genuinely d dimensional, that is, the group generated by $\{x \in \boldsymbol{Z}^d; P(X_1=x)>0\}$ is isomorphic to $\boldsymbol{Z}^d$. If $P(X_1=e)=1/2d$ for each unit vector $e \in \boldsymbol{Z}^d$, we call the random walk simple.

Let $p=P(S_n \neq 0, n=1, 2, \cdots)$. The random walk is called transient if $p>0$ or, equivalently, $\sum_{n=1}^{\infty} P(S_n=0)$ converges. If $d \geq 3$, it is wellknown that the random walk is always transient.

The range of the random walk up to time n, denoted by R_n, means the number of distinct points visited by the random walk in the first n steps. It was proved by Kesten, Spitzer, and Whitman [7] that $n^{-1}R_n \to p$ *a.s.* for any random walk. Jain and Pruitt [3] showed that if $d \geq 4$ and $p<1$, there is a positive constant σ^2 such that $Var\, R_n=\sigma^2 n+o(n)$ and R_n obeys the central limit theorem. Moreover, they established the law of the iterated logarithm in [4].

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Now we consider the single point range Q_n of the random walk up to time n, that is, the number of distinct lattice points entered once and only once in the first n steps. Erdös and Taylor [1] proved that the strong law of large number holds for the single point range of the simple random walk under the normalization $p^2 n$ if $d \geq 3$ and $\pi^2 n/(\log n)^2$ if $d=2$. Pitt [5] proved that $n^{-1}Q_n \to p^2$ *a.s.* for transient random walks. In [2], Hamana showed that if $d \geq 4$ and $p<1$, there exists a positive constant μ^2 which satisfies $Var\ Q_n = \mu^2 n + o(n)$ and Q_n obeys the central limit theorem. Note that if $p=1$, then $R_n = Q_n = n$ almost surely and this case is not very interesting.

In the present paper we obtain the law of the iterated logarithm for Q_n when $d \geq 4$.

THEOREM 1. *If $p<1$ and $d \geq 4$, it holds that*

$$\limsup_{n \to \infty} \frac{Q_n - p^2 n}{\sqrt{2\mu^2 n \log\log n}} = 1 \qquad a.s.$$

and the lim inf *of this sequence is* -1 *almost surely.*

Section 2 is devoted to giving some preliminary estimates. Section 3 contains the proof of Theorem 1.

§2. Notation and preliminary results.

In this section, we will give some notation and probability estimates of some quantities related to the random walk which are given, e.g., in [3] and [7] and will play basic role in our proof of Theorem 1.

For $x \in \boldsymbol{Z}^d$, the notation $P_x(\cdot)$ will be used to denote the probability measure of events related to the random walk starting from x. When $x=0$, we will usually use $P(\cdot)$ instead of $P_0(\cdot)$. For $x, y \in \boldsymbol{Z}^d$ and $n \geq 0$, let

$$p^n(x, y) = P_x(S_n = y) = P(S_n = y - x).$$

The following lemma is essential.

LEMMA 2.1. *If $p<1$ and d is the genuine dimension of the random walk, there is a positive constant A such that*

$$p^n(0, x) \leq A n^{-d/2}$$

for all $n \geq 1$ and $x \in \boldsymbol{Z}^d$.

For $x, y \in \boldsymbol{Z}^d$, let $G(x, y)$ denote the Green function defined by

$$G(x, y) = \sum_{n=0}^{\infty} p^n(x, y).$$

For a transient random walk, $G(x, y) \leq G(0, 0) = 1/p < \infty$. The following useful bounds about the Green function are given in Jain-Pruitt [3].

LEMMA 2.2. *If $p<1$ and $d\geq 3$, then*

$$\sum_{x\in \boldsymbol{Z}^d} p^n(0, x)\{G(u, x)+G(x, u)\}=O(n^{1-d/2})$$

uniformly for $u\in \boldsymbol{Z}^d$.

For $x\in \boldsymbol{Z}^d$, τ_x will denote the first hitting time of x, defined by

$$\tau_x=\inf\{n\geq 1;\ S_n=x\}\ .$$

If there are no positive integers with $S_n=x$, then $\tau_x=\infty$. The taboo probabilities are defined by

$$p_z^n(x, y)=P_x(S_n=y,\ \tau_z\geq n)\ .$$

By Lemma 2.1,

$$P_x(n<\tau_y<\infty)=\sum_{l=n+1}^{\infty} p_y^l(x, y)\leq C_1 n^{1-d/2} \tag{2.1}$$

uniformly for all $x, y\in \boldsymbol{Z}^d$.

LEMMA 2.3. *If $p<1$ and $d\geq 3$, then*

$$P_x(n<\tau_x<\infty,\ \tau_0<\infty)\leq C_2 n^{1-d/2}\{G(0, x)+G(x, 0)\}$$

for $x\in \boldsymbol{Z}^d$ and $x\neq 0$.

§3. The proof of Theorem 1.

For $m<k\leq n$, define

$$Z(k; m, n)=\begin{cases}1 & \text{if}\quad S_k\neq S_\alpha \text{ for any } \alpha\in(m, n] \text{ and } \alpha\neq k\ ,\\ 0 & \text{otherwise}\ .\end{cases}$$

Using this indicator, Q_n can be expressed by $\sum_{k=1}^{n} Z(k; 0, n)$.

Form a sequence $\{n_i\}$ of positive integers by taking for $k=1, 2, \cdots$ all integers in the interval $[2^{2k}, 2^{2k+2})$ which are of the form $2^{2k}+j\{[k^{-1}2^k]+1\}$ where j is a nonnegative integer. It is clear that at most $3k2^k$ members of the sequence are in $[2^{2k}, 2^{2k+2})$ for each k. For the sake of convenience, we put $n_0=0$.

Since

$$|Q_n-EQ_n-Q_{n_i}+EQ_{n_i}|\leq n-n_i=o(n_i^{1/2})$$

if $n_i\leq n<n_{i+1}$, it is enough to prove along the sequence $\{n_i\}$. We also need to see EQ_n can be replaced by $p^2 n$. However, it is valid since

$$EQ_n=\sum_{j=1}^{n} P_0(\tau_0>n-j)P_0(\tau_0>j)=p^2n+O(\log n)\,,$$

which is obtained by (2.1).

Before proving Theorem 1, we introduce several random variables and give two lemmas.

For $m<k\leq n\leq l$, let

$$W(k;m,n,l)=Z(k;m,n)-Z(k;m,l)\,,$$

and for $p\leq m<k\leq l$,

$$V(k;p,m,l)=Z(k;m,l)-Z(k;p,l)\,.$$

These are also indicator random variables:

$$W(k;m,n,l)=\begin{cases}1 & \text{if } S_k\neq S_\alpha \text{ for any } \alpha\in(m,n] \text{ and } \alpha\neq k,\\ & \text{and there is a } t\in(n,l] \text{ such that } S_t=S_k\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

$$V(k;p,m,l)=\begin{cases}1 & \text{if } S_k\neq S_\alpha \text{ for any } \alpha\in(m,l] \text{ and } \alpha\neq k\,,\\ & \text{and there is a } t\in(p,m] \text{ sucht that } S_t=S_k\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

For $0\leq j<i$, put

$$A_j=\sum_{k=n_j+1}^{n_{j+1}} Z(k;n_j,n_{j+1})\,,\qquad W_j=\sum_{k=n_j+1}^{n_{j+1}} W(k;n_j,n_{j+1},n_i)\,,$$

and

$$V_j=\sum_{k=n_j+1}^{n_{j+1}} V(k;0,n_j,n_i).$$

Then we can easily show

$$Q_{n_i}=\sum_{j=0}^{i-1} A_j-\sum_{j=0}^{i-1} W_j-\sum_{j=0}^{i-1} V_j\,.$$

We will show that the term involving A's is the dominating one among the three terms on the right hand side.

Firstly, we estimate the term involving W's by the similar technique used in [4].

LEMMA 3.1. *If* $p<1$, $d\geq 4$, *and* $2^{2k}\leq n_i<2^{2k+2}$, *then*

$$Var\left(\sum_{j=0}^{i-1} W_j\right)\leq C_3k^3 2^k\,.$$

PROOF.

$$\begin{aligned}
&Var\left(\sum_{j=0}^{i-1} W_j\right) \\
&=\sum_{j=0}^{i-1}\sum_{k=n_j+1}^{n_{j+1}} Var\, W(k; n_j, n_{j+1}, n_i) \\
&\quad +2\sum_{j=0}^{i-1}\sum_{k=n_j+2}^{n_{j+1}}\sum_{l=n_j+1}^{k-1} Cov(W(k; n_j, n_{j+1}, n_i), W(l; n_j, n_{j+1}, n_i)) \\
&\quad +2\sum_{j=1}^{i-1}\sum_{l=0}^{j-1}\sum_{k=n_j+1}^{n_{j+1}}\sum_{h=n_l+1}^{n_{l+1}} Cov(W(k; n_j, n_{j+1}, n_i), W(h; n_l, n_{l+1}, n_i)) \\
&=\mathrm{I}+\mathrm{II}+\mathrm{III}\,.
\end{aligned}$$

We employ the lemmas in Section 2 and Markov property to estimate each term. For $n_j<k\le n_{j+1}$,

$$\begin{aligned}
Var\, W(k; n_j, n_{j+1}, n_i) &\le EW(k; n_j, n_{j+1}, n_i)\le EW(k; k, n_{j+1}, n_i) \\
&\le \sum_{x\ne 0} p^{n_{j+1}-k}(0, x)P_x(\tau_0<\infty)\,.
\end{aligned}$$

Hence, noting $P_x(\tau_0<\infty)\le G(x, 0)$, we have from Lemma 2.2

$$\mathrm{I}\le C_4\sum_{j=0}^{i-1}\sum_{k=n_j+1}^{n_{j+1}} (n_{j+1}-k+1)^{-1}\le C_5 i\log n_i\,.$$

For $n_j<l<k\le n_{j+1}$,

$$\begin{aligned}
Cov(W(k; n_j, n_{j+1}, n_i), W(l; n_j, n_{j+1}, n_i)) &\le EW(k; n_j, n_{j+1}, n_i)W(l; n_j, n_{j+1}, n_i) \\
&\le \sum_{x\ne 0} p^{k-l}(0, x)P_x(n_{j+1}-k<\tau_x<\infty, \tau_0<\infty) \\
&\le C_6(n_{j+1}-k+1)^{-1}(k-l)^{-1}\,.
\end{aligned}$$

In the last estimate we used Lemma 2.2 and 2.3. Then we obtain

$$\begin{aligned}
\mathrm{II} &\le C_7\sum_{j=0}^{i-1}\{\log(n_{j+1}-n_j)\}^2 \\
&\le C_7 i(\log n_i)^2\,.
\end{aligned}$$

The estimate of III is slightly complicated. Since $W(h; n_l, n_{l+1}, n_i)=W(h; n_l, n_{l+1}, n_j)+W(h; n_l, n_j, k)+W(h; n_l, k, n_i)$ for $n_l<h\le n_{l+1}\le n_j<k\le n_{j+1}$, and $W(h; n_l, n_{l+1}, n_j)$ and $W(k; n_j, n_{j+1}, n_i)$ are independent, it holds

$$\begin{aligned}
Cov(W(k; n_j, n_{j+1}, n_i), W(h; n_l, n_{l+1}, n_i)) &\le EW(k; n_j, n_{j+1}, n_i)W(h; n_l, n_j, k) \\
&\quad +EW(k; n_j, n_{j+1}, n_i)W(h; n_l, k, n_i)\,.
\end{aligned}$$

The bound of each term of the right hand side can be derived by a simple calculation. Let $r_n = P_0(n < \tau_0 < \infty)$ for $n \geq 1$. We have

$$\begin{aligned} EW(k; n_j, n_{j+1}, n_i)W(h; n_l, n_j, k) &\leq r_{n_{j+1}-k} r_{n_j-h} \\ &\leq C_8(n_{j+1}-k+1)^{-1}(n_j-h+1)^{-1} \end{aligned}$$

and

$$\begin{aligned} EW(k; n_j, n_{j+1}, n_i)W(h; n_l, k, n_i) &\leq \sum_{x \neq 0} p^{k-h}(0, x)P_x(n_{j+1}-k < \tau_x < \infty, \tau_0 < \infty) \\ &\leq C_9(n_{j+1}-k+1)^{-1}(k-h+1)^{-1} . \end{aligned}$$

Since $k > n_j$,

$$\begin{aligned} \mathrm{III} &\leq C_{10} \sum_{j=1}^{i-1} \sum_{l=0}^{j-1} \sum_{k=n_j+1}^{n_{j+1}} \sum_{h=n_l+1}^{n_{l+1}} (n_{j+1}-k+1)^{-1}(n_j-h+1)^{-1} \\ &= C_{10} \sum_{j=1}^{i-1} \sum_{k=n_j+1}^{n_{j+1}} \sum_{h=1}^{n_j} (n_{j+1}-k+1)^{-1}(n_j-h+1)^{-1} \\ &\leq C_{11} i (\log n_i)^2 . \end{aligned}$$

If $2^{2k} \leq n_i < 2^{2k+2}$, then

$$i(\log n_i)^2 \leq C_{12}(2k+2)^2 \sum_{j=0}^{k+1} 3j2^j \leq C_{13}k^3 2^k . \tag{3.1}$$

This completes the proof of the lemma. □

Next we give an estimate of the term involving V's.

LEMMA 3.2. *If $p < 1$, $d \geq 4$, and $2^{2k} \leq n_i < 2^{2k+2}$, then*

$$Var\left(\sum_{j=0}^{i-1} V_j\right) \leq C_{14} k^6 2^k .$$

PROOF.

$$\begin{aligned} Var\left(\sum_{j=0}^{i-1} V_j\right) &= \sum_{j=0}^{i-1} \sum_{k=n_j+1}^{n_{j+1}} Var\, V(k; 0, n_j, n_i) \\ &\quad + 2\sum_{j=0}^{i-1} \sum_{k=n_j+2}^{n_{j+1}} \sum_{l=n_j+1}^{k-1} Cov(V(k; 0, n_j, n_i), V(l; 0, n_j, n_i)) \\ &\quad + 2\sum_{j=1}^{i-1} \sum_{l=0}^{j-1} \sum_{k=n_j+1}^{n_{j+1}} \sum_{h=n_l+1}^{n_{l+1}} Cov(V(k; 0, n_j, n_i), V(h; 0, n_l, n_i)) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} . \end{aligned}$$

The first two terms can be computed in the same way as Lemma 3.1 by reversing the random walk and we have $\mathrm{I} \leq C_{15} i \log n_i$ and $\mathrm{II} \leq C_{16} i (\log n_i)^2$. Hence I, $\mathrm{II} \leq C_{17} k^3 2^k$ if $2^{2k} \leq n_i < 2^{2k+2}$ by (3.1).

We will estimate the third term III. For $n_l < h \leq n_{l+1} \leq n_j < k \leq n_{j+1}$, it holds that $V(k; 0, n_j, n_i) = V(k; h, n_j, n_i) + V(k; 0, h, n_i)$. Let

$$U(h; 0, n_l, n_{l+1}, n_i) = V(h; 0, n_l, n_{l+1}) - V(h; 0, n_l, n_i)\,.$$

Then we obtain

$$\begin{aligned} Cov(V(k; 0, n_j, n_i), V(h; 0, n_l, n_i)) = {} & Cov(V(k; 0, h, n_i), V(h; 0, n_l, n_i)) \\ & + Cov(V(k; h, n_j, n_i), V(h; 0, n_l, n_{l+1})) \\ & - Cov(V(k; h, n_j, n_i), U(h; 0, n_l, n_{l+1}, n_i))\,. \end{aligned}$$

The first two terms can be bounded in the same fashion as Lemma 3.1. Then

$$\begin{aligned} Cov(V(k; 0, h, n_i), V(h; 0, n_l, n_i)) & \leq \sum_{x \neq 0} p^{k-h}(0, x) P_x(h - n_l \leq \tau_x < \infty, \tau_0 < \infty) \\ & \leq C_{18}(k-h)^{-1}(h-n_l)^{-1}\,. \end{aligned}$$

Since $V(k; h, n_j, n_i) = V(k; h, n_{l+1}, n_i) + V(k; n_{l+1}, n_j, n_i)$,

$$\begin{aligned} Cov(V(k; h, n_j, n_i), V(h; 0, n_l, n_{l+1})) & \leq P_0(h - n_l \leq \tau_0 < h) P_0(k - n_{l+1} \leq \tau_0 < k-h) \\ & \leq C_{19}(k - n_{l+1})^{-1}(h - n_l)^{-1} \end{aligned}$$

by using the independence of $V(k; n_{l+1}, n_j, n_i)$ and $V(h; 0, n_l, n_{l+1})$. When $h \leq n_{l+1}$, it holds that $(k-h)^{-1} \leq (k - n_{l+1})^{-1}$. Hence for $2^{2k} \leq n_i < 2^{2k+2}$ the sum of the first two terms is not greater than constant multiple of

$$\begin{aligned} & \sum_{j=1}^{i-1} \sum_{l=0}^{j-1} \sum_{k=n_j+1}^{n_{j+1}} \sum_{h=n_l+1}^{n_{l+1}} (k - n_{l+1})^{-1}(h - n_l)^{-1} \\ & = \sum_{l=0}^{i-2} \sum_{k=n_{l+1}+1}^{n_i} \sum_{h=n_l+1}^{n_{l+1}} (k - n_{l+1})^{-1}(h - n_l)^{-1} \\ & = O\{i(\log n_i)^2\} = O(k^3 2^k)\,. \end{aligned}$$

Lastly, we estimate the remaining term of III. Recall $r_n = P_0(n < \tau_0 < \infty)$. Then, noting the definition of the random variables $V(k; h, n_j, n_i)$ and $U(h; 0, n_l, n_{l+1}, n_i)$, we have

$$\begin{aligned} & Cov(V(k; h, n_j, n_i), U(h; 0, n_l, n_{l+1}, n_i)) \\ & \qquad \geq -EV(k; h, n_j, n_i) EU(h; 0, n_l, n_{l+1}, n_i)\,, \\ & \qquad\quad EV(k; h, n_j, n_i) \leq r_{k - n_j - 1}\,, \end{aligned}$$

and

$$EU(h;0,n_l,n_{l+1},n_i)\le r_{n_{l+1}-h-1}r_{h-n_l-1}\,.$$

Hence the remaining term of III is not greater than

$$2\sum_{l=0}^{i-2}\sum_{j=l+1}^{i-1}\sum_{k=n_j+1}^{n_{j+1}}\sum_{h=n_l+1}^{n_{l+1}} r_{k-n_j-1}r_{n_{l+1}-h-1}r_{h-n_l-1}$$

$$=O\left(\sum_{l=0}^{i-2}\sum_{j=l+1}^{i-1}(n_{l+1}-n_l)^{-1}\log(n_{l+1}-n_l)\log(n_{j+1}-n_j)\right).$$

If $2^{2k}\le n_i<2^{2k+2}$, then

$$\sum_{j=0}^{i-1}\log(n_{j+1}-n_j)\le C_{20}i\log n_i\le C_{21}k^2 2^k$$

and

$$\sum_{l=0}^{i-2}(n_{l+1}-n_l)^{-1}\log(n_{l+1}-n_l)\le\sum_{\alpha=0}^{k}\sum_{\beta:2^{2\alpha}\le n_\beta<2^{2\alpha+2}}(n_{\beta+1}-n_\beta)^{-1}\log(n_{\beta+1}-n_\beta)$$

$$\le C_{22}\sum_{\alpha=1}^{k}\{3\alpha 2^\alpha(\alpha^{-1}2^\alpha)^{-1}\log(\alpha^{-1}2^\alpha)\}$$

$$\le C_{23}k^4\,.$$

Therefore we obtain $\mathrm{III}\le C_{24}k^6 2^k$. This completes the proof of the lemma. □

Since we derive the estimates of negligible terms, the proof will be completed along the same line as Theorem in [4].

PROOF OF THEOREM 1. Let $B_j=W_j+V_j$ for $j=1,2,\cdots$. When $2^{2k}\le n_i<2^{2k+2}$, we have

$$P\left(\left|\sum_{j=0}^{i-1}(B_j-EB_j)\right|>\varepsilon n_i^{1/2}\right)\le\varepsilon^{-2}n_i^{-1}Var\left(\sum_{j=0}^{i-1}B_j\right)\le C_{25}k^6 2^{-k}$$

for each $\varepsilon>0$ by using Lemma 3.1, Lemma 3.2, and Chebyshev's inequality. Now we introduce a subsequence $\{n_{\mu_i}\}$ by taking every k^9th member of $\{n_i\}$ in the interval $[2^{2k},2^{2k+2})$. There are at most $3k^{-8}2^k$ members in $[2^{2k},2^{2k+2})$. Since

$$\sum_{i=0}^{\infty}P\left(\left|\sum_{j=0}^{\mu_i-1}(B_j-EB_j)\right|>\varepsilon n_{\mu_i}^{1/2}\right)\le\sum_{k=1}^{\infty}(C_{25}k^6 2^{-k}\times 3k^{-8}2^k)<\infty\,,$$

we can conclude B part converges to zero almost surely along the subsequence by Borel-Cantelli Lemma. We need to prove that this implies convergence to zero along the original sequence $\{n_i\}$. For $2^{2k}\le n_{\mu_m}<n_i<n_{\mu_{m+1}}\le 2^{2k+2}$,

$$\sum_{j=0}^{\mu_m-1}(B_j-EB_j)-\sum_{j=\mu_m}^{i-1}EB_j\le\sum_{j=0}^{i-1}(B_j-EB_j)$$
$$\le\sum_{j=0}^{\mu_{m+1}-1}(B_j-EB_j)+\sum_{j=i}^{\mu_{m+1}-1}EB_j$$

since $B_j\ge 0$ for each $j\ge 0$. So we only need to obtain the estimate of EB_j. If $2^{2k}\le n_j<2^{2k+2}$,

$$EB_j=\sum_{k=n_j+1}^{n_{j+1}}(EW(k;n_j,n_{j+1},n_i)+EV(k;0,n_j,n_i))$$
$$\le C_{26}\log(n_{j+1}-n_j)\le C_{27}k\,.$$

Moreover, both $\mu_{m+1}-i$ and $i-\mu_m$ are less than or equal to k^9. Thus

$$\sum_{j=\mu_m}^{i-1}EB_j,\quad\sum_{j=i}^{\mu_{m+1}-1}EB_j\le C_{27}k^{10}=o(2^k)$$

and we have finished to show the convergence to zero along the original sequence.

It remains to check that A part satisfies the Kolmogorov condition ([5] page 272). $\{A_j\}$ is the sequence of independent random variables and the distribution of A_j coincides with that of $Q_{n_{j+1}-n_j}$ for each j. Hence

$$Var\left(\sum_{j=0}^{i-1}A_j\right)=\mu^2n_i+o(n_i)\,.$$

For $2^{2k}\le n_i<2^{2k+2}$,

$$|A_i-EA_i|\le n_{i+1}-n_i\le k^{-1}2^k+1=o\left\{\left(\frac{n_i}{\log\log n_i}\right)^{1/2}\right\}.$$

Accordingly we complete the proof of Theorem 1. □

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