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Spherical Minimal Surfaces with Minimal Quadric Representation in Some Hyperquadric

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Abstract. The totally geodesic 2-sphere and the Clifford torus in S^3 are the only compact, minimal surfaces in S^3 whose quadric representations yield minimally in hyperquadrics.

1. Introduction.

In 1970, H. B. Lawson, [La], showed that every compact surface but the projective plane (which is prohibited) can be minimally immersed into S^3. Even more, he discovered that one can find compact, orientable, minimal surfaces in S^3 of arbitrary genus. New examples of compact embedded minimal surfaces in S^3 were given in [K-P-S]. Therefore the class of compact minimal surfaces in the 3-sphere is enormous. On the other hand, if we see S^3 in the Euclidean space E^4 as usual (that is embeded by its first standard immersion), then the minimal surfaces in S^3 are constructed by using eigenfunctions of their Laplacian associated to the same eigenvalue $\lambda=2$. This is clear after a well-known result of T. Takahashi, [Ta].

Therefore, if we want to study minimal surfaces in S^3 from the point of view of the spectral behaviour of their position vectors in the Euclidean space, it seems natural to immerse S^3 in the Euclidean space by using a different embedding.

The second standard immersion of S^3 in the Euclidean space can be defined in $SM(4)$, the Euclidean space of 4-order real symmetric matrices, by using the products of coordinate functions of $S^3 \subset E^4$, [T]. So minimal surfaces in S^3 can be regarded into $SM(4)$ by means of their quadric representations (see the next section for details). This idea has been exploited by several authors to give nice characterizations of some minimal surfaces in S^m (including: The totally geodesic 2-sphere in S^3, the Clifford torus in S^3, the Veronese surfaces in S^4 and S^6 respectively and the equilateral flat torus in S^5) in the context of Finite Type theory of B. Y. Chen, [Ch1], (see for instance [R], [Ba-Ch1], [Ba-Ch2] and [Ba-Ur]).

It is known that the totally geodesic 2-sphere is the only surface in S^3 whose

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quadric representation yields minimally in some hypersphere of $SM(4)$ and consequently it is of mass-symmetric in such a hypersphere. Also one can see that the quadric representation of a Clifford torus in S^3 is of mass-symmetric in some hypersphere of $SM(4)$ and yields minimally in some hyperquadric of $SM(4)$ concentric with the above mentioned hypersphere (see [Ba-Ga] and section 4). Furthermore in [Ba-Ga], the authors proved that the described situation characterizes the Clifford torus among all compact, minimal and non-totally geodesic surfaces of S^3.

However a natural and more general question appears:

PROBLEM. *Besides the Clifford torus and the totally geodesic 2-sphere, are there other compact minimal surfaces of S^3 whose quadric representation is minimal in some hyperquadric of $SM(4)$?*

In this paper we shall deal with this problem and answer this question by proving the following result:

MAIN THEOREM. *Let $x: M^2 \to S^3$ be a compact, minimal surface of the 3-dimensional unit sphere and $\Phi: M^2 \to SM(4)$ its quadric representation. Then (M^2, Φ) is minimal in some canonical hyperquadric of $SM(4)$, if and only if, (M^2, x) is either totally geodesic or the Clifford torus in S^3.*

In order to prove this result, we first observe that the hyperquadric, say Q, in which (M^2, Φ) yields minimally can not be a graph and so it is a hyperquadric with center, say Ψ_o. Then we use our main argument based in a systematic study of the nodal sets associated with the coordinate functions of (M^2, x) in E^4 (which are eigenfunctions of the Laplacian of M as we mentioned earlier) in order to get control on Ψ_o. In this manner, we are able to prove that Ψ_o is a diagonal matrix in $SM(4)$ (see Theorem 1). Then we see that Ψ_o is not only a diagonal matrix but also $\Psi_o = \frac{1}{4}I$ (I being the identity matrix of order 4 in $SM(4)$) (see Proposition 3). Finally we can prove that the center of gravity, say Φ_o, of (M^2, Φ) coincides with Ψ_o and so (M^2, Φ) is of mass-symmetric in some hypersphere of $SM(4)$. Therefore the proof of our main theorem here is reduced to the main result of [Ba-Ga] to which we have already alluded in connection with the statement of the Problem.

In view of our main theorem it is interesting to inquire whether the same kind of result can be achieved for compact minimal surfaces of S^m with $m > 2$. In this direction, we can obtain a rational uniparametric family of minimal surfaces in $S^5 \subset \boldsymbol{R}^6$ (one of whose members is the equilateral torus in S^5) whose quadric representations are minimal in certain hyperquadrics of $SM(6)$, (see Proposition 2). This shows that our result can not sharpened in this respect.

Our method does not work for minimal submanifolds of S^m with dimension greater than two. Probably, arbitrary dimension and codimension decreases the possibility of classification of this sort of submanifolds and further hypothesis would be necessary.

During a visit to the University of Ioannina, we posed the problem of finding new methods for dealing with the classification of minimal hypersurfaces of the sphere with minimal quadric representation in some hyperquadric. Our natural feelings would have expected a richer family than that of the 2-dimensional case. However, the results of Hasanis-Vlachos in [H-V2], extending our characterization of the Clifford torus in [Ba-Ga], express a plausible state of affairs which stands in contradiction to our native geometrical intuition. Nevertheless the problem for minimal hypersurfaces of S^m is still open. It is expected that rather weak additional assumptions will suffice to restrict the domain of possibilities to those with which we are familiar.

2. The second standard immersion of S^3.

First of all, we describe the most important of the notions discussed in subsequent paragraphs. We start then by enumerating the main facts about the second standard immersion of S^3 and conclude by explaining the idea of quadric representation.

Let $\boldsymbol{R}^4$ be the Euclidean space of dimension 4, endowed as usual with the inner product $\langle u, v\rangle = u \cdot v^t$ for any $u, v \in \boldsymbol{R}^4$, where a vector in $\boldsymbol{R}^4$ is regarded as a 1-row matrix and v^t denotes the transpose of v. Thus, we have that the unit 3-sphere centered at the origin of $\boldsymbol{R}^4$ is given by $S^3 = \{u \in \boldsymbol{R}^3 \mid \langle u, u\rangle = 1\}$. Let $SM(4) = \{P \in gl(4, \boldsymbol{R}) \mid P^t = P\}$ be the space of symmetric 4×4 matrices over $\boldsymbol{R}$ endowed with the metric $g(P, Q) = \frac{1}{2}\operatorname{tr}(P \cdot Q)$ for any $P, Q \in SM(4)$. Consider the mapping $f: S^3 \to SM(4)$ defined by $f(u) = u^t \cdot u$, then f is an isometric immersion which is actually the second standard immersion of S^3.

For each point $u \in S^3$, the normal space of S^3 in $SM(4)$ at u (or more precisely at $f(u)$) is given by

$$T_u^\perp S^3 = \{P \in SM(4) \mid u \cdot P = \mu u \text{ for some } \mu \in \boldsymbol{R}\}\ .$$

In particular, we have that $f(u) \in T_u^\perp S^3$.

If $\bar\sigma$ denotes the second fundamental form of f, then

$$\bar\sigma(X, Y) = X^t \cdot Y + Y^t \cdot X - 2\langle X, Y\rangle f(u)$$

for any $X, Y \in T_u S^3$. It is also well known, that $\bar\sigma$ is parallel and satisfies the following properties (see for instance [R]):

$$g(\bar\sigma(X, Y), \bar\sigma(V, W)) = 2\langle X, Y\rangle\langle V, W\rangle + \langle X, V\rangle\langle Y, W\rangle + \langle X, W\rangle\langle Y, V\rangle\ ,$$

$$\bar A_{\bar\sigma(X, Y)} V = 2\langle X, Y\rangle V + \langle X, V\rangle Y + \langle Y, V\rangle X\ ,$$

$$g(\bar\sigma(X, Y), f(u)) = -\langle X, Y\rangle\ ,$$

$$g(\bar\sigma(X, Y), I) = 0\ ,$$

where $\bar A$ is the Weingarten map of f, X, Y, V, W are tangent vectors of S^3 and I is the identity matrix.

It is also known that S^3 is immersed via f as a minimal submanifold of a hypersphere of $SM(4)$ centered at $\frac{1}{4}I$ and of radius $r=\sqrt{3/8}$.

Now, let us consider an isometric immersion of a Riemannian surface M into the 3-sphere, $x: M^2 \to S^3$. We can combine both immersions, x and f to obtain an isometric immersion $\Phi = f \circ x: M \to SM(4)$, and since the coordinates of Φ depend quadratically on the coordinates of x, we say that Φ is the *quadric representation* of x (or of M). Also, $\Phi(M)$ will be called the quadric representation image of M (see [Di] for some properties related with this topic).

3. Basic properties.

With a view to future applications, we wish to establish some properties of our immersion. Let $x: M \to S^3 \subset E^4$ be a compact, minimal surface in the unit 3-sphere and consider its quadric representation $\Phi: M \to SM(4)$. We already know that $\Phi(M)$ is contained in some hypersphere of $SM(4)$ with center $\frac{1}{4}I$ (I being the identity matrix of order 4). The center of gravity of Φ is $\Phi_o = (a_{ij})$ where

$$a_{ij} = \frac{1}{\mathrm{vol}(M)} \int_M x_i x_j dv\,. \tag{1}$$

Let Q be a canonical hyperquadric, that is, a hyperquadric which in the coordinate system $u_{ij} = x_i x_j$ takes one of the following forms:

$$\sum_{i \le j} r_{ij}(u_{ij} - b_{ij})^2 = k\,,$$

$$\sum_{i \le j, i \ne 3} r_{ij}(u_{ij} - b_{ij})^2 + 2u_{33} = 0\,,$$

r_{ij}, b_{ij}, $k \in \mathbf{R}$.

Now let assume that (M, Φ) is minimal in some canonical hyperquadric, say Q, with center $\Psi_o = (b_{ij})$. As we shall see (Remark 1) one can assume without loss of generality that Q is given in the coordinate system $u_{ij} = x_i x_j$ by

$$\sum_{i \le j} r_{ij}(u_{ij} - b_{ij})^2 = k \tag{2}$$

for some real constants r_{ij} and k.

We define $h: SM(4) \to \mathbf{R}$ by $h(u_{ij}) = \sum_{i \le j} r_{ij}(u_{ij} - b_{ij})^2$ and denote by H the mean curvature vector field of (M, Φ). Then one has

$$\Delta \phi = -2H\,. \tag{3}$$

Since (M, Φ) is minimal in Q, H must be normal to Q in $SM(4)$, so there exists a smooth function μ on M with

$$H = \mu \nabla h \tag{4}$$

where ∇h denotes the gradient of h in $SM(4)$ restricted to (M, Φ)

Notice that since (M, Φ) has non-zero constant mean curvatue, namely $|H|^2 = 3$, then $\mu(p) \neq 0$ at any point p of M. Therefore we combine (3) with (4) to get

$$\Delta(x_i x_j) = -4\mu r_{ij}(x_i x_j - b_{ij}) . \tag{5}$$

From now on, we will consider that (M, x) is a non-totally geodesic surface in S^3. Consequently $r_{ii} \neq 0$, otherwise we use (5) to see that $x_i^2 = c_i$ is constant on M and so $c_i = 0$ which is impossible.

LEMMA 1. *The following properties hold*:

$$\sum_{i=1}^{4} b_{ii} = 1 , \quad r_{ij} \neq 0 , \quad r_{11} = r_{22} = r_{33} = r_{44} = r .$$

PROOF. First we use (5) to get $\int_M \mu(x_i^2 - b_{ii}) dv = 0$ and so $\int_M \mu dv = \int_M \mu dv (\sum_{i=1}^4 b_{ii})$ which proves the first formula.

Next let us assume $r_{ij} = 0$ then $x_i x_j$ must be a constant and so it must be zero because x_i are eigenfunctions of M. Set $U = \{p \in M \mid x_j(p) \neq 0\}$, if U is a non-empty open subset of M, then we apply (5) to obtain $4\mu r_{ii} b_{ii} = 0$ on U and so $b_{ii} = 0$. Now we come back to (5) to get

$$\Delta x_i^2 = -4\mu r_{ii} x_i^2 \tag{6}$$

which is impossible because (M, x) is not totally geodesic in S^3.

Finally we use again (5) to have $4\mu \sum_{i=1}^4 r_{ii}(x_i^2 - b_{ii}) = 0$ and because μ does not vanish, M is contained in the hypersurface of E^4 given by

$$\sum_{i=1}^{4} r_{ii}(x_i^2 - b_{ii}) = 0 . \tag{7}$$

Then we apply Δ to this relation to prove that M also satisfies

$$\sum_{i=1}^{4} r_{ii}^2(x_i^2 - b_{ii}) = 0 . \tag{8}$$

Since M is contained in S^3 we use an easy argument involving (7) and (8) to see that r_{ii} $1 \leq i \leq 4$ are roots of a second order polynomial. Now it is easy to see that the only possibility that one has is that described in the statement. (Q.E.D.)

Now we can combine (5) with the fact that $\Delta x_i = 2x_i$, to get the following useful equation

$$\langle \nabla x_i, \nabla x_j \rangle = 2(1 + r_{ij}\mu) x_i x_j - 2\mu r_{ij} b_{ij} \tag{9}$$

where ∇x_i denotes the gradient of x_i in M.

The mean curvature vector field H of (M, Φ) regarded in $SM(4)$ can be written as

$$H=\frac{1}{2}\sum_{i=1}^{2}\bar{\sigma}(E_i, E_i)=E_1^tE_1+E_2^tE_2-2x^tx \tag{10}$$

where $\{E_1, E_2\}$ denotes an orthonormal basis on the tangent plane of M. We set $\Omega=\Delta\Phi=-2H$ which can be certainly regarded as a field of endomorphisms on $\boldsymbol{R}^4$ along M and in particular

$$\Omega(x)=x\cdot\Omega=4x\,. \tag{11}$$

This relation gives us

$$\mu\left(x_j\sum_{i=1}^{4}r_{ji}x_i^2-\sum_{i=1}^{4}r_{ji}b_{ji}x_i\right)+x_j=0 \tag{12}$$

where $1\leq j\leq 4$.

REMARK 1. If $F: SM(4)\to\boldsymbol{R}$ is a differentiable map and we suppose that (M, Φ) is minimal in some hypersurface $F^{-1}(c)$, then its mean curvature vector field H satisfies

$$H=\mu\nabla F$$

where ∇F denotes the gradient of F in $SM(4)$ restricted to (M, Φ) and μ is nothing but a smooth function on M which is non-zero anywhere. The above equation can be written as follows:

$$\Delta(x_ix_j)=-2\mu\frac{\partial F}{\partial u_{ij}}$$

where $u_{ij}=x_ix_j$, $1\leq i, j\leq 4$. This shows that $F^{-1}(c)$ can not be a graph. For this reason we have assumed that Q is a hyperquadric with center.

4. Some examples.

We postpone for the time being the proofs of the main results and first, as a kind of illustrative exercise, we give examples of minimal surfaces of S^5 whose quadric representation is minimal in hyperquadrics. This digression is in fact more than a preliminary exercise. We shall see, Proposition 2, that our main theorem has no equivalent in higher codimension.

Let a be any non-zero, positive real number and define a surface in E^4 by

$$M_a: x(u, v)=(\cos u\cos av, \cos u\sin av, \sin u\cos v, \sin u\sin v)\,. \tag{13}$$

These surfaces were studied by H. B. Lawson, [La], and they are minimal surfaces in $S^3\subset E^4$. In particular when $a=1$, one obtains the Clifford torus in $S^3\subset E^4$.

The Laplacian of M_a with the induced metric is given by

$$\Delta=-\frac{\partial^2}{\partial u^2}-\frac{1}{g}\frac{\partial^2}{\partial v^2}-\frac{1}{g}(1-a^2)\cos u\sin u\frac{\partial}{\partial u} \tag{14}$$

where $g=a^2\cos^2 u+\sin^2 u$, (the determinant of the first fundamental form of M_a).

Now we consider the quadric representation of M_a, Φ_a: $M_a \to SM(4)$ defined by

$$\Phi_a(x(u, v))=x^t(u, v)\cdot x(u, v)=(u_{ij}(u, v)=x_i(u, v)\cdot x_j(u, v)) \tag{15}$$

where $1\leq i, j\leq 4$.

A long and direct computation shows that along M_a one has

$$\Delta u_{11}=6u_{11}-\frac{2a^2}{g}u_{22}-\frac{2u_{11}}{u_{11}+u_{22}}, \tag{16}$$

$$\Delta u_{22}=6u_{22}-\frac{2a^2}{g}u_{11}-\frac{2u_{22}}{u_{11}+u_{22}}. \tag{17}$$

Also notice that $g=a^2(u_{11}+u_{22})+u_{33}+u_{44}$.

It is clear that (M_a, Φ_a) is minimal in a hyperquadric, say $F^{-1}(0)$, where F is some polynomial of degree two on $SM(4)$ if and only if there exists some smooth function, say μ, on M_a such that

$$\Delta\Phi_a=-2\mu\nabla F. \tag{18}$$

In particular if (M_a, Φ_a) is minimal in $F^{-1}(0)$, we use (16), (17) and (18) to get

$$\begin{aligned}&\frac{\partial F}{\partial u_{22}}(6u_{11}g(u_{11}+u_{22})-2a^2u_{11}(u_{11}+u_{22})-2u_{22}g)\\ &\qquad=\frac{\partial F}{\partial u_{11}}(6u_{22}g(u_{11}+u_{22})-2a^2u_{11}(u_{11}+u_{22})-2u_{22}g).\end{aligned} \tag{19}$$

A new direct computation, involving the fact that $\partial F/\partial u_{11}$ and $\partial F/\partial u_{22}$ are polynomials of degree one and (18), shows that the only possibility is $a=1$. Therefore we have:

PROPOSITION 1. *(M_a, Φ_a) is minimal in some hyperquadric of $SM(4)$ if and only if $a=1$ and so (M_1, Φ_1) is the Clifford torus in $S^3\subset E^4$.*

REMARK 2. Notice that our main theorem is an extension of the statement proved in the above proposition.

The next example will show that our main theorem is the best possible in the sense that it does not admit extension to minimal surfaces in $S^m\subset E^{m+1}$. In fact we are going to obtain a rational uniparametric family of minimal surfaces in $S^5\subset E^6$ whose quadric representations are minimal in hyperquadrics of $SM(6)$. Let t be a real number with $0<t\leq\frac{1}{2}$ and $\sqrt{2t}\in Q$. Define a surface in E^6 by

$$M_t:\ x(\theta,\tau)=\frac{1}{\sqrt{2(2-t)}}\Bigg(\cos\theta,\ \sin\theta,\ \cos\tau,\ \sin\tau,$$
$$\sqrt{2(1-t)}\,\cos\frac{\theta+\tau}{\sqrt{2t}},\ \sqrt{2(1-t)}\,\sin\frac{\theta+\tau}{\sqrt{2t}}\Bigg). \tag{20}$$

These surfaces were studied by K. Kenmotsu, [Ke], and they are minimal flat tori in $S^5 \subset E^6$. In particular when $t=\frac{1}{2}$ one obtains the so-called equilateral torus. The Laplacian of M_t with the induced flat metric is given by

$$\Delta=-2\left(\frac{\partial^2}{\partial\theta^2}-2(1-t)\frac{\partial^2}{\partial\theta\partial\tau}+\frac{\partial^2}{\partial\tau^2}\right). \tag{21}$$

Let us consider the quadric representation (M_t, Φ_t) of M_t into $SM(6)$. We put $\Phi_t=(u_{ij}=x_i\cdot x_j)$, $1\le i,j\le 6$. From a direct, long computation we obtain $\Delta\Phi_t$ which can be described as follows:

$$\Delta\left(u_{ii}-\frac{1}{4(2-t)}\right)=8\left(u_{ii}-\frac{1}{4(2-t)}\right),$$
$$\Delta\left(u_{jj}-\frac{1-t}{2(2-t)}\right)=8\left(u_{jj}-\frac{1-t}{2(2-t)}\right),$$
$$\Delta u_{12}=8u_{12}, \qquad \Delta u_{13}=4u_{13}+4(1-t)u_{24}, \quad \Delta u_{14}=4u_{14}-4(1-t)u_{23},$$
$$\Delta u_{15}=4u_{15}-\frac{4t}{\sqrt{2t}}u_{26}, \qquad \Delta u_{16}=4u_{16}+\frac{4t}{\sqrt{2t}}u_{25},$$
$$\Delta u_{23}=4u_{23}-4(1-t)u_{14}, \qquad \Delta u_{24}=4u_{24}+4(1-t)u_{13},$$
$$\Delta u_{25}=4u_{25}+\frac{4t}{\sqrt{2t}}u_{16}, \quad \Delta u_{26}=4u_{26}-\frac{4t}{\sqrt{2t}}u_{15}, \quad \Delta u_{34}=8u_{34},$$
$$\Delta u_{35}=4u_{35}-\frac{4t}{\sqrt{2t}}u_{46}, \qquad \Delta u_{36}=4u_{36}+\frac{4t}{\sqrt{2t}}u_{45},$$
$$\Delta u_{45}=4u_{45}+\frac{4t}{\sqrt{2t}}u_{36}, \quad \Delta u_{46}=4u_{46}-\frac{4t}{\sqrt{2t}}u_{35}, \quad \Delta u_{56}=8u_{56},$$

where $1\le i\le 4$ and $5\le j\le 6$.

Let us define $F_t\colon SM(6)\to \boldsymbol{R}$ by

$$F_t((u_{ij}))=\frac{1}{2}\sum_{i=1}^{4}\left(u_{ii}-\frac{1}{4(2-t)}\right)^2+\frac{1}{2}\sum_{j=5}^{6}\left(u_{jj}-\frac{1-t}{2(2-t)}\right)^2$$
$$+\tfrac{1}{2}(u_{12}^2+u_{34}^2+u_{56}^2)+\tfrac{1}{4}(u_{13}^2+u_{24}^2)+\tfrac{1}{4}(u_{14}^2+u_{23}^2+u_{15}^2+u_{26}^2+u_{16}^2)$$
$$+\tfrac{1}{4}(u_{25}^2+u_{35}^2+u_{46}^2+u_{36}^2+u_{45}^2)+\tfrac{1}{2}(1-t)(u_{13}u_{24}-u_{14}u_{23})$$

$$+\frac{t}{2\sqrt{2t}}(u_{16}u_{25}+u_{36}u_{45}-u_{15}u_{26}-u_{35}u_{46})\,.$$

It is not difficult to see that $\Delta\Phi_t=8\nabla F_t$ along M_t which proves that (M_t, Φ_t) is minimal in some hyperquadric $F_t^{-1}(r)$ for all t.

PROPOSITION 2. *For a given $t\in \boldsymbol{R}$ satisfying $0<t\leq\frac{1}{2}$ and $\sqrt{2t}\in\boldsymbol{Q}$, M_t is a minimal flat torus in $S^5\subset E^6$ whose quadric representation (M_t, Φ_t) is minimal in some hyperquadric of $SM(6)$.*

This proposition is particularly noteworthy because, as a consequence, the codimension of the surface has a very clear geometrical influence and leaves open the problem for surfaces of S^m, $m>3$.

5. The nodal lines.

We must undertake in this section a somewhat complicated way of studying the nodal sets of the eigenfunctions x_i on M, in order to get control on the center $\Psi_o=(b_{ij})$ of the hyperquadric Q in which (M, Φ) yields minimally.

First of all, under the assumption that Ψ_o is not a diagonal matrix, we derive a group of lemmas in which a role is played by the geometry of the nodal sets of the coordinate functions. Then, by using our work [Ba-Ga] we show how the conclusions of these lemmas would lead to contradiction, so that Ψ_o must be a diagonal matrix.

Denote by $C_j=x_j^{-1}(0)=\{p\in M\,|\,x_j(p)=0\}$ the nodal set associated to the eigenfunctions x_j, $1\leq j\leq 4$, on M. Let us assume that Ψ_o is not a diagonal matrix so that without loss of generality we consider $b_{12}\neq 0$.

LEMMA 2. *$C_1=x_1^{-1}(0)$ is a great circle of the unit 2-sphere $S^2=S^3\cap\{x_1=0\}$. Moreover μ is a constant on C_1 and it satisfies*

$$r\mu=-3/2\,. \tag{22}$$

PROOF. We start by using the first equation of (12) and the above mentioned fact that μ does not vanish to see that C_1 must satisfy

$$x_2^2+x_3^2+x_4^2=1\,,$$
$$r_{12}b_{12}x_2+r_{13}b_{13}x_3+r_{14}b_{14}x_4=0\,. \tag{23}$$

Since $b_{12}\neq 0$, the last formulae prove the first statement of the lemma. In particular one can parametrize C_1 as $\alpha_1(s)=(0, x_2(s), x_3(s), x_4(s))$ in such a way that $|\alpha_1'(s)|=1$ and

$$x_i''(s)=-x_i(s) \tag{24}$$

where $2\leq i\leq 4$. Then we systematically use (9) to prove

$$|\nabla x_1(s)|^2 = -2\mu(s) r b_{11} \neq 0\,, \tag{25}$$

$$\nabla x_j(s) = x_j'(s)\alpha_1'(s) + \frac{r_{1j}b_{1j}}{rb_{11}}\nabla x_1(s)\,. \tag{26}$$

Once more we use (9) and combine it with (26) to obtain

$$x_j'(s)x_k'(s) = 2(1 + r_{jk}\mu(s))x_j(s)x_k(s) + 2\mu(s)c_{jk} \tag{27}$$

with $c_{jk} = r_{1j}r_{1k}b_{1j}b_{1k}/(rb_{11}) - r_{jk}b_{jk}$.

In particular

$$(x_j'(s))^2 = 2(1 + r\mu(s))x_j^2(s) + 2\mu(s)c_{jj} \tag{28}$$

and summing up these identities for j running from 2 to 4 we get the constancy of μ along C_1.

Next we combine (24) with (27) and (28) respectively to obtain

$$(3 + 2r_{jk}\mu)x_j(s)x_k(s) = 0\,, \tag{29}$$

$$(3 + 2r\mu)x_j(s)^2 = 0\,. \tag{30}$$

Now we already know that C_1 is a great circle of S^2, then it must intersect to any other great circle of S^2, in particular to those defined by $x_j = 0$, $2 \leq j \leq 4$. Now (30) proves (22). (Q.E.D.)

LEMMA 3. *The following relations hold*:

$$b_{13} = b_{14} = b_{23} = b_{24} = b_{34} = 0\,, \qquad r_{12}^2 b_{12}^2 - r^2 b_{11}b_{22} = 0\,,$$
$$r_{34} = r, \qquad b_{33} = b_{44} = 1/3\,.$$

PROOF. Using (29) and (30) in (27) and (28) we have

$$x_j'(s)x_k'(s) = -x_j(s)x_k(s) + 2\mu c_{jk}\,, \tag{31}$$

$$(x_j'(s))^2 = -x_j(s)^2 + 2\mu c_{jj}\,. \tag{32}$$

On the other hand one may consider the quadric representation of $C_1 = S^1$ whose center of gravity, say $\bar{\Phi}_o$, is a diagonal matrix. So if $\bar{\Delta}$ denotes its Laplacian, one uses (23) and (31) to obtain

$$\bar{\Delta}(x_j(s)x_k(s)) = -\frac{d^2}{ds^2}(x_j(s)x_k(s)) = 4x_j(s)x_k(s) - 4\mu c_{jk}\,.$$

Thus if $j \neq k$

$$0 = \int_{C_1} \bar{\Delta}(x_j(s)x_k(s))ds = -8\pi\mu c_{jk}$$

which proves that

$$c_{jk}=0 \tag{33}$$

where $j \neq k$.

Equation (24) allows us to write $x_j(s)=m_j\cos s+n_j\sin s$, $2 \le j \le 4$, for certain real constants m_j and n_j. Hence we use (31) and (33) to prove

$$m_j m_k + n_j n_k = 0 \tag{34}$$

where $j \neq k$. Now it is easy to see that (34) implies that some $x_j(s)$, $2 \le j \le 4$, vanishes identically along $C_1 = x_1^{-1}(0)$. Since we are assuming that $b_{12} \neq 0$, it is clear from (23) that $x_2(s)$ vanishes identically on C_1 and so $C_1 = C_2$. Consequently $b_{13}=b_{14}=0$. Also one uses the same argument on $C_2 = C_1$ to obtain $b_{23}=b_{24}=0$. Furthermore we can choose $x_3(s)=\cos s$ and $x_4(s)=\sin s$, along C_1 and then use them in (31) and (32) to get $b_{34}=0$, $r_{12}^2 b_{12}^2 - r^2 b_{11} b_{22} = 0$ and $b_{33}=b_{44}=1/3$. Finally combining the information we have just obtained with (12) we have

$$\mu(r\cos^2 s + r_{34}\sin^2 s) + \frac{3}{2} = 0\,, \qquad \mu(r_{34}\cos^2 s + r\sin^2 s) + \frac{3}{2} = 0\,,$$

in particular, we obtain $r_{34}=r$ and the proof is complete. (Q.E.D.)

Our next step is the study of the nodal set $C_4 = x_4^{-1}(0)$ associated with the eigenfunction x_4 on M. C_4 consists of a finite number of circles, [Che], but actually it can be checked that it is regular. Thus we parametrize C_4 by $\alpha_4(s)=(x_1(s), x_2(s), x_3(s), 0)$ with $|\alpha_4'(s)|=1$. From (9), one has $\langle \nabla x_j(s), \nabla x_4(s)\rangle = 0$, $1 \le j \le 3$, along C_4 and so $\nabla x_j(s) = x_j'(s)\alpha_4'(s)$, $1 \le j \le 4$. Once more we use (9) to obtain

$$x_1'(s)x_2'(s) = 2(1+r_{12}\mu(s))x_1(s)x_2(s) - 2\mu(s)r_{12}b_{12}\,, \tag{35}$$

$$x_1'(s)x_3'(s) = 2(1+r_{13}\mu(s))x_1(s)x_3(s)\,, \tag{36}$$

$$x_2'(s)x_3'(s) = 2(1+r_{23}\mu(s))x_2(s)x_3(s)\,, \tag{37}$$

$$(x_1'(s))^2 = 2(1+r\mu(s))x_1^2(s) - 2r\mu(s)b_{11}\,, \tag{38}$$

$$(x_2'(s))^2 = 2(1+r\mu(s))x_2^2(s) - 2r\mu(s)b_{22}\,, \tag{39}$$

$$(x_3'(s))^2 = 2(1+r\mu(s))x_3^2(s) - 2r\mu(s)b_{33}\,, \tag{40}$$

along C_4.

One can combine (38), (39) and (40) with Lemmas 1 and 3 to prove that $\mu(s)$ must be a constant along C_4, namely $r\mu = -3/2$. Therefore (38), (39) and (40) turn to

$$(x_1'(s))^2 = -x_1^2(s) + 3b_{11}\,, \tag{41}$$

$$(x_2'(s))^2 = -x_2^2(s) + 3b_{22}\,, \tag{42}$$

$$(x_3'(s))^2 = -x_3^2(s) + 1\,. \tag{43}$$

LEMMA 4. *The nodal line $C_4 = x_4^{-1}(0)$ consists of a finite number of great circles in the 2-sphere $S^3 \cap \{x_4 = 0\}$.*

PROOF. According to a well-known result of S. Y. Cheng, [Che], we already know that C_4 consists of a finite number of circles immersed in $M \subset S^3$. Therefore it is enough to prove that $(d^2/ds^2)x_i(s) = -x_i(s)$, $1 \le i \le 3$, along C_4.

First we use (43), (41) and (42) to see that $x_i''(s)x_i'(s) = -x_i(s)x_i'(s)$, $1 \le i \le 3$. Thus we will prove that $x_i'(s) \ne 0$ on an open subset which is dense in C_4 and then use continuity.

Since $\alpha_4(s)$ is regular we see from (43) that, s_o is a critical point of $x_3(s)$ if and only if $x_3(s_o) = \pm 1$ and so $\alpha_4(s_o) = (0, 0, \pm 1, 0)$. Consequently, $x_3(s)$ has isolated critical points on C_4 which proves that $x_3''(s) = -x_3(s)$ along C_4.

Next let us prove that $1 + r_{13}\mu \ne 0$ and the same argument works for $1 + r_{23}\mu \ne 0$. In fact, otherwise (36) proves that $x_1'(s)x_3'(s) = 0$ along C_4 and so $x_1'(s)$ vanishes identically on an open subset, dense in C_4. This fact combined with (35) proves that $x_1(s)$ and $x_2(s)$ are constant on C_4, which is impossible.

Finally let us consider $\boldsymbol{R} = \{p \in C_4 \mid x_3(p) \ne 0\}$. It is an open subset which is dense in C_4. In fact since $b_{34} = 0$, one can use (9) along C_4 to see that $C_4 - \boldsymbol{R}$ can not contain any open subset. It is also clear that $x_1'(s) \ne 0$ and $x_2'(s) \ne 0$ on $\boldsymbol{R}$ otherwise either (37) or (36) would imply that either $x_1(s)$ and $x_1'(s)$ at a point vanish or $x_2(s)$ and $x_2'(s)$ vanish, which is impossible. (Q.E.D.)

Now we use the last lemma and come back to (37), (35) and (36) to get

$$(3 + 2r_{ij}\mu)(x_i(s)x_j(s))' = 0 \tag{44}$$

where $1 \le i \ne j \le 3$.

If $3 + 2r_{ij}\mu \ne 0$, then $x_i(s)x_j(s)$ must be constant on C_4. Since Lemma 4 shows that $x_i(s)$ are eigenfunctions of the Laplacian of C_4, such a constant is zero and so $x_i(s)x_j(s)$ vanishes identically on C_4. Therefore on the open subset $U = \{p \in C_4 \mid x_j(p) \ne 0\}$, $x_i(s)$ vanishes identically and this is impossible by (9) and $b_{j4} = 0$, $1 \le j \le 3$. Consequently $r_{12}\mu = r_{13}\mu = r_{23}\mu = -3/2$. Now we use this information in (12) to get $\mu r = -3/2$ in $M - C_3$. But that is also true along C_3 because we can use the same argument as that we did along C_4. Therefore we prove that μ is a constant on M.

THEOREM 1. *Let (M, x) be a compact minimal surface in the unit 3-sphere $S^3 \subset E^4$. If its quadric representation is minimal in some canonical hyperquadric Q of $SM(4)$ with center Ψ_o, then Ψ_o is a diagonal matrix.*

PROOF. If Ψ_o were not a diagonal matrix, then we already know that the smooth function μ must be some non-zero constant on M.

On the other hand, from (5) we have

$$b_{ij} = \frac{1}{\mathrm{vol}(M)} \int_M x_i x_j \, dv = a_{ij}$$

which would prove that the center of gravity Φ_o of Φ coincides with the center Ψ_o of Q. Now we use the lemma and the main theorem of [Ba-Ga] to see that (M, x) would be a Clifford torus in S^3 and this is impossible because in that case $\Phi_o = \Psi_o$ would be a diagonal matrix. (Q.E.D.)

6. Proof of the main theorem.

We turn now to a circle of questions centered around how one is to decide whether the centre of (M, Φ), Φ_o, coincides with that of Q, Ψ_o, or not. According to the last section, we can suppose from now on that Ψ_o is a diagonal matrix. The aim is to prove that Ψ_o is not only a diagonal matrix but also a multiple of the identity matrix. In order to prove that we first notice that all nodal curves associated to the eigenfunctions x_i have the same behaviour and so without loss of generality we will consider the nodal set C_1. From (9) we have $|\nabla x_1|^2 = -2\mu r b_{11} > 0$ along C_1, so C_1 is a regular curve which can be parametrized by the length of the arc as $\alpha_1(s) = (0, x_2(s), x_3(s), x_4(s))$. Then we use (9) once more to get

$$\nabla x_j(s) = x'_j(s)\alpha'_1(s)\,, \tag{45}$$

$$x'_j(s)x'_k(s) = 2(1 + r_{jk}\mu(s))x_j(s)x_k(s)\,, \tag{46}$$

$$(x'_j(s))^2 = 2(1 + r\mu(s))x_j^2(s) - 2\mu(s)rb_{jj} \tag{47}$$

where $2 \leq j \neq k \leq 4$.

LEMMA 5. *The smooth function μ is constant along the nodal lines $C_j = x_j^{-1}(0)$. Namely $\mu = -1/(2rb_{jj})$ along C_j.*

PROOF. Without loss of generality one can prove the lemma for $j = 1$. But this follows automatically from (47). (Q.E.D.)

Now (47) can be written as

$$(x'_j(s))^2 = \frac{2b_{11} - 1}{b_{11}} x_j^2(s) + \frac{b_{jj}}{b_{11}}\,. \tag{48}$$

LEMMA 6. *There exists an open subset U in C_1 in which one of the coordinate functions $x_j(s)$, $2 \leq j \leq 4$, must be a non-zero constant.*

PROOF. If $1 + r_{34}\mu = 0$, then we use (46) to get

$$x'_3(s)x'_4(s) = 0 \tag{49}$$

and so consider $U = \{p \in C_1 \mid x'_4(p) \neq 0\}$, an open subset of C_1 on which $x_3(s)$ is a non-zero constant because of (48) and $b_{33} \neq 0$.

In the case $1 + r_{34}\mu \neq 0$, one can use systematically (46) and (48) to obtain

$$\left(\frac{2(1+r_{23}\mu)(1+r_{24}\mu)}{1+r_{34}\mu}+\frac{1-2b_{11}}{b_{11}}\right)x_2^2(s)x_3(s)x_4(s)=\frac{b_{22}}{b_{11}}x_3(s)x_4(s)$$

and so take $U=\{p\in C_1 \mid x_3(p)x_4(p)\neq 0\}$ to have that $x_2(s)$ must be a non-zero constant on U. (Q.E.D.)

REMARK 3. One should compare the last lemma with the behaviour of the nodal lines for the coordinate functions of the Clifford torus in S^3.

From now on and without loss of generality we will assume $x_2(s)$ is a non-zero constant, say k, on an open subset U of C_1. Then we use again (48) and (47) to get

$$r_{23}=r_{24}=-1/\mu\,, \tag{50}$$

$$k^2=\frac{b_{22}}{1-2b_{11}}\,. \tag{51}$$

Since $b_{13}=b_{14}=0$, we see from (9) that $x_3(s)$ and $x_4(s)$ can not vanish identically on U. Consequently one can parametrize U as follows

$$\beta(s)=\left(0,\, k,\, R\cos\frac{s}{R},\, R\sin\frac{s}{R}\right) \tag{52}$$

where $R^2=1-k^2$.

Then we use (52) in (48) to obtain

$$\sin^2\frac{s}{R}=\frac{R^2(2b_{11}-1)}{b_{11}}\cos^2\frac{s}{R}+\frac{b_{33}}{b_{11}}\,,$$

$$\cos^2\frac{s}{R}=\frac{R^2(2b_{11}-1)}{b_{11}}\sin^2\frac{s}{R}+\frac{b_{44}}{b_{11}}\,,$$

and so $b_{33}=b_{44}=b_{11}$.

The same argument works on any nodal line so that $b_{ii}=1/4$ for $1\le i\le 4$. Therefore we use Lemma 1 to conclude the following:

PROPOSITION 3. *The center of the hyperquadric Q is $\Psi_o=\frac{1}{4}I$, where I denotes the identity matrix of order* 4.

Now we make a straightforward long computation involving (12) in order to show that μ must be a non-zero real constant along M. Then we use (5) and (1) to prove that $\Phi_o=\Psi_o$. But this means that (M,Φ) is of mass-symmetric in a hypersphere of $SM(4)$, hence by using the main result of [Ba-Ga], we obtain the main theorem.

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