TOKYO J. MATH.
VOL. 18, No. 1, 1995

Arithmetic Holomorphic Functions of Exponential Type on the Product of Half Planes

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1. Introduction and notations.

In [2] Bazylewicz considered arithmetic entire functions of exponential type in C^m (i.e. $f(N^m) \subset \mathcal{O}_A$, $\mathcal{O}_A$ is the set of algebraic integers.). In this paper we generalize Bazylewicz's result to holomorphic functions of exponential type in the product of half planes using the theory of analytic functionals with non-compact carrier. We adhere the following notations. N is the set of integers greater than 1. Let K be a number field over Q with degree $[K:Q]=d=r+2s$. Let $K^{(i)}$ $(1 \leq i \leq r)$ be real conjugate fields and $\overline{K}^{(r+j)}=K^{(r+s+j)}$ $(1 \leq j \leq s)$ be complex conjugate fields of K. We put $\delta=d$ if $K \subset R$, $\delta=d/2$ if $K \not\subset R$. For an algebraic integer α, we put $\overline{|\alpha|}=\max|\alpha_i|$ (where α_i's are conjugates of α over Q). $\mathcal{O}_A$ denotes the ring of algebraic integers and $\mathcal{O}_K$ is the ring of algebraic integers in K. For $p(z)=\sum a_n z^n \in K[z]$, we put $p^{(j)}(z)=\sum a_n^{(j)} z^n \in K^{(j)}[z]$, where $a_n^{(j)}$ are conjugates of a_n. $\tau(F)$ denotes the transfinite diameter of compact set F in C. For the details of transfinite diameter, we refer the reader to [1]. $\overline{F}$ is complex conjugate of $F \subset C$. $C-F$ is the complement of F in C. S^a denotes closure of S. $H_L(z)=\sup_{\zeta \in L} \mathrm{Re}\langle z, \zeta\rangle$ is the supporting function of L in C^m, where $\langle z, \zeta\rangle=z_1\zeta_1+\cdots+z_m\zeta_m$ for $z=(z_1, \cdots, z_m)$, $\zeta=(\zeta_1, \cdots, \zeta_m) \in C^m$. $L_i=\mathrm{pr}_i(L)$ is i-th projection of set L in C^m.

Following Theorem 1 is our main result.

THEOREM 1. *There exists a finite set* ϑ of $\mathcal{O}_A^m$ *(direct product of* $\mathcal{O}_A$*) having following property: Suppose that* $0 \leq k' < 1$ *and* $f(z)$ *satisfies*

(i) $f(z)$ *is holomorphic in* $\prod_{i=1}^m \{z_i:\ \mathrm{Re}\, z_i < -k'\}$,

(ii) *For any* $\varepsilon>0$ *and* $\varepsilon'>0$, *there exists* $C_{\varepsilon,\varepsilon'} \geq 0$ *such that*

$$|f(z)| \leq C_{\varepsilon,\varepsilon'} \exp(H_L(z)+\varepsilon|z|) \qquad (\mathrm{Re}\, z_i \leq -k'-\varepsilon'),$$

where L is a closed convex set contained in

$$\prod_{i=1}^m \{\zeta_i \in C: |\mathrm{Im}\,\zeta_i| \leq b_i < \pi,\ \mathrm{Re}\,\zeta_i \geq a_i\},$$

Received October 26, 1993
Revised July 12, 1994

(iii) $f((-N)^m) \subset O_K$,

(iv) $\limsup_{\|n\| \to \infty} \frac{1}{\|n\|} \log \overline{|f(-n)|} \le c$ (*c is a positive constant*),

where $\|n\| = n_1 + \cdots + n_m$, $n = (n_1, \cdots, n_m) \in N^m$.

If $\tau(\exp(-L_i)) < -c(\delta - 1)$ $(1 \le i \le m)$, *then* $f(z) = \sum P_b(z_1, \cdots, z_m) b^z$, *where* $P_b \in K(\vartheta)[z_1, \cdots, z_m]$, $b = (b_1, \cdots, b_m) \in \vartheta$ *such that* $\log b_i \in -L_i$ $(i = 1, \cdots, m)$.

2. Transformations of analytic functionals with non-compact carriers.

Let L be a closed convex set in C^m bounded in imaginary direction. We define the test function space $Q(L; k')$ as follows:

$$Q(L; k') = \liminf_{\varepsilon > 0, \varepsilon' > 0} Q_b(L_\varepsilon; k' + \varepsilon'),$$

$$Q_b(L_\varepsilon; k' + \varepsilon') = \{ f \in H(\mathring{L}_\varepsilon) \cap C(L_\varepsilon^a) : \sup_{\zeta \in L_\varepsilon} | f(\zeta) \exp((k' + \varepsilon')\zeta) | < \infty \},$$

where L_ε is the ε-neighbourhood of L, $H(\mathring{L}_\varepsilon)$ and $C(L_\varepsilon^a)$ denote the space of holomorphic functions in $\mathring{L}_\varepsilon$ (interior of L_ε) and the space of continuous functions in L_ε^a (closure of L_ε) respectively. $Q'(L : k')$ denotes the dual space of $Q(L : k')$. An element of $Q'(L : k')$ is called an analytic functional with carrier L and of type k'. Now we define Fourier-Borel transform $\tilde{T}(z)$ of $T \in Q'(L : k')$ as follows:

$$\tilde{T}(z) = \langle T_\zeta, \exp(\langle z, \zeta \rangle) \rangle .$$

The following Paley-Wiener type theorem characterizes Fourier-Borel transform of $Q'(L : k')$.

THEOREM 2. ([4]) *Let* $T \in Q'(L : k')$. *Then* $\tilde{T}(z)$ *is a holomorphic function in* $D = \prod_{i=1}^m \{ z_i \in C : \operatorname{Re} z_i < -k' \}$ *and satisfies the following estimate: for any* $\varepsilon > 0$, $\varepsilon' > 0$, *there exists* $C_{\varepsilon, \varepsilon'} \ge 0$ *such that*

$$(*) \qquad | \tilde{T}(z) | \le C_{\varepsilon,\varepsilon'} \exp(H_L(z) + \varepsilon |z|) \qquad (\operatorname{Re} z_i \le -k' - \varepsilon') .$$

Conversely, if a holomorphic function $f(z)$ *in* D *satisfies* (*), *then* $f(z)$ *is a Fourier-Borel transform of some* $T \in Q'$ $(L : k')$.

To define Avanissian-Gay transform, we put following assumptions (1) and (2):

(1) $0 \le k' < 1$,

(2) $pr_i(L) \subset \{ \zeta_i \in C : |\operatorname{Im} \zeta_i| \le b_i < \pi, \operatorname{Re} \zeta_i \ge a_i \}$ $(i = 1, \cdots, m)$ for some b_i and a_i.

Avannisian-Gay transform $G_T(w)$ of $T \in Q'$ $(L : k')$ is defined as follows:

$$G_T(w) = \langle T_\zeta, \prod_{i=1}^m (1 - w_i \exp(\zeta_i))^{-1} \rangle ,$$

where $w=(w_1, \cdots, w_m) \in \prod_{i=1}^m (C-\exp(-L_i)^a)$.

Avannisian-Gay transform has the following properties.

PROPOSITION 1 ([3], [4]).

(i) *$G_T(w)$ is holomorphic in $\prod_{i=1}^m (C-\exp(-L_i)^a)$.*

(ii) $G_T(w)=(-1)^m \sum_{n \in N^m} \tilde{T}(-n_1, -n_2, \cdots, -n_m) w_1^{-n_1} w_2^{-n_2} \cdots w_m^{-n_m}$

where $n=(n_1, \cdots, n_m) \in N^m$.

(iii) *For any $\varepsilon>0$ and $\varepsilon'>0$, there exists $C_{\varepsilon,\varepsilon'} \geq 0$ such that $|G_T(w)| \leq C_{\varepsilon,\varepsilon'} |w_1|^{-k'-\varepsilon'} \cdots |w_m|^{-k'-\varepsilon'}$ $(b_i+\varepsilon \leq |\arg w_i| \leq \pi,\ 1 \leq i \leq m)$.*

(iv) (*Inversion formula*)

$$\tilde{T}(z)=(2\pi i)^{-m} \int_\Gamma G_T(w_1, w_2, \cdots, w_m) w_1^{-z_1-1} \cdots w_m^{-z_m-1} dw_1 dw_2 \cdots dw_m,$$

where $\Gamma=\Gamma_1 \times \cdots \times \Gamma_m$, $\Gamma_i (1 \leq i \leq m)$ is the positively oriented boundary of sector with vertex 0 surrounding $\exp(-L_i)^a$.

3. Proof of Theorem 1.

In this section, we will prove Theorem 1 using analytic functionals with non-compact carrier. Following Proposition 2 is essential in our discussion.

PROPOSITION 2 ([2]). *Let $S_{i,j} (1 \leq i \leq m)$ be compact sets in C and $\tau_{i,j}=\tau(S_{i,j})$ are their transfinite diameters. Suppose that $S_{i,j}$ satisfies the following conditions:*

(i) $S_{i,j}=\bar{S}_{i,j} (1 \leq j \leq r)$ (i.e. $S_{i,j} \subset R$), $S_{i,j+r}=\bar{S}_{i,r+j+s}$ $(1 \leq j \leq s)$,

(ii) $\prod_{j=1}^d \tau_{i,j}<1$ $(1 \leq i \leq m)$.

We assume that $g^{(j)}(z)=\sum_{n \in N^m} a_n^{(j)} z^{-n}$ is holomorphic in $\prod_{i=1}^m (C-S_{i,j})$, where $a_n^{(j)}$'s are algebraic integers in $K^{(j)}$. Then there exist polynomials $P(z_1, z_2, \cdots, z_m)$, $Q_i(z_i)$ satisfying the following conditions:

(1) *$Q_i(z_i)$ are monic (coefficient of highest degree term is unit),*

(2) $\deg_{z_i} P(z_1, z_2, \cdots, z_m)<\deg Q_i(z_i)$,

(3) $g^{(j)}(z)=P^{(j)}(z_1, z_2, \cdots, z_m)/\prod_{i=1}^m Q_i^{(j)}(z_i)$.

PROOF OF THEOREM 1. By Theorem 2, there exists $T \in Q'(L:k')$ such that $f(z)=\tilde{T}(z)$. We put $g(w)=G_T(w)$. By (ii) in Proposition 1 we have $g(w)=(-1)^m \sum_{n \in N^m} f(-n) w^{-n}$. Now we put

$$g^{(j)}(w)=(-1)^m \sum_{n \in N^m} f(-n)^{(j)} w^{-n} \qquad (1 \leq j \leq d).$$

Each $g^{(j)}(w)$ is holomorphic in $\prod_{i=1}^m \{|w_i|>e^c\}$. Since $g(w)$ is Avanissian-Gay transform of T, $g(w)$ and $\bar{g}(w)=\sum_{n \in N^m} \overline{f(-n)} w^{-n}$ are holomorphic in $\prod_{i=1}^m (C-S_i)$, $\prod_{i=1}^m (C-\bar{S}_i)$ respectively. Here $S_i=\exp(-L_i)^a$. We define

$$S_{i,j}=\begin{cases} S_i & \text{if } K=K^{(j)} \\ \overline{S_i} & \text{if } \overline{K}=K^{(j)} \\ \{w\,;\,|w_i|\le e^c\}\,, & \text{other case}\,. \end{cases}$$

Then $g^{(j)}$ and $S_{i,j}$ satisfy all assumptions in Proposition 2. So we have

$$G_T(w)=g(w)=P(w_1, w_2, \cdots, w_m)\Big/\prod_{i=1}^{m} Q_i(w_i)\,,$$

where P and Q_i's are polynomials satisfying the conditions in Proposition 2. By inversion formula (iv) in Proposition 1 and the residue theorem, we obtain

$$\begin{aligned} f(z)&=(2\pi i)^{-m}\int_{\Gamma} G_T(w)w_1^{-z_1-1}\cdots w_m^{-z_m-1}dw_1dw_2\cdots dw_m \\ &=(2\pi i)^{-m}\int_{\Gamma} P(w_1, \cdots, w_m)/\prod_{i=1}^{m} Q_i(w_i)w_1^{-z_1-1}\cdots w_m^{-z_m-1}dw_1\cdots dw_m \\ &=\sum_{b_i:\text{algebraic integer}} P_b(z)b^z\,. \end{aligned}$$

q.e.d.

References

[1] L. V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill (1973).

[2] A. Bazylewicz, Critère de reconnaisabilité de fonctions analytiques et fonctions entières arithmétiques, Acta Arith. **51** (1989), 311–319.

[3] M. Morimoto and K. Yoshino, A uniqueness theorem for holomorphic functions of exponential type, Hokkaido Math. J. **7** (1978), 259–270.

[4] P. Sargos and M. Morimoto, Transformations des fonctionnelles analytiques à porteurs non-compacts, Tokyo J. Math. **4** (1981), 457–492.

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