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On the Rational Approximations to $\tanh\frac{1}{k}$

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Introduction.

Let k and N_1 be positive integers with $N_1 \geq 10$, and let p_n/q_n be the n-th convergent of $\tanh(1/k)$. Let γ_{N_1}, δ_n, and $\gamma^*_{N_1}$ be defined by

$$\gamma_{N_1}=2\left(k+\frac{k+1}{N_1-1/2}\right)\left(1+\frac{\log\log(2k(N_1+1)/e)}{\log(N_1+1)}\right),$$

$$\delta_n=\frac{(k(2n+1)+2)\log\log q_n}{\log q_n},$$

and

$$\gamma^*_{N_1}=\max\{\delta_n \mid 1\leq n<N_1\},$$

respectively.

In the previous paper [1], we proved the following.

THEOREM A. *Let $k\geq 2$ be positive integers. Then*

$$\left|\tanh\frac{1}{k}-\frac{p}{q}\right|>\frac{\log\log q}{\gamma q^2\log q}$$

for all integers p and q with $q\geq 2$, where

$$\gamma\geq\max\{\gamma_{N_1},\gamma^*_{N_1}\}$$

for any positive integer $N_1\geq 10$.

COROLLARY A. *For all integers p and q with $q\geq 2$,*

$$\left|\tanh\frac{1}{2}-\frac{p}{q}\right|>\frac{\log\log q}{6q^2\log q}.$$

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COROLLARY B. *For all integers p and q with* $q\geq 2$,

$$\left|\tanh\frac{1}{3}-\frac{p}{q}\right|>\frac{\log\log q}{9q^2\log q}.$$

The purpose of this paper is to prove the following theorems.

THEOREM 1. *For any* $\varepsilon>0$, *there is an infinity of solutions of the inequality*

$$\left|\tanh\frac{1}{k}-\frac{p}{q}\right|<\left(\frac{1}{2k}+\varepsilon\right)\frac{\log\log q}{q^2\log q}$$

in integers p and q. Further, there exists a number $q'=q'(k,\varepsilon)$ *such that*

$$\left|\tanh\frac{1}{k}-\frac{p}{q}\right|>\left(\frac{1}{2k}-\varepsilon\right)\frac{\log\log q}{q^2\log q}$$

for all integers p and q with $q\geq q'$.

THEOREM 2. *For all integers p and q with* $q\geq 2$,

$$\left|\tanh 1-\frac{p}{q}\right|>\frac{\log\log q}{4q^2\log q}.$$

§1. Lemmas.

Let N_2 be a positive integer with $N_2\geq 30$. Let η_{N_2}, θ_n, and $\eta^*_{N_2}$ be defined by

$$\eta_{N_2}=2\left(1+\frac{2}{N_2-1/2}\right)\frac{\log(N_2+1)+\log\log(10(N_2+1)/13)}{\log(7(N_2+1)/10)},$$

$$\theta_n=\frac{(2n+3)\log\log q_n}{\log q_n},$$

and

$$\eta^*_{N_2}=\max\{\theta_n \mid 2\leq n<N_2\},$$

respectively.

LEMMA 1. *For all integers p and q with* $q\geq q_{N_2}$,

$$\left|\tanh 1-\frac{p}{q}\right|>\frac{\log\log q}{\eta_{N_2}q^2\log q}.$$

PROOF. We may assume that p/q is a convergent of $\tanh 1$, since otherwise

$$\left|\tanh 1-\frac{p}{q}\right|>\frac{1}{2q^2}.$$

The continued fraction of $\tanh 1$ is

$$\tanh 1=[a_0, a_1, a_2, a_3, \cdots]=[0, 1, 3, 5, \cdots].$$

In other words, $a_0=0$ and $a_n=2n-1$ for $n\geq 1$. Since $q_{n+1}=a_{n+1}q_n+q_{n-1}=(2n+1)q_n+q_{n-1}\leq 2(n+1)q_n$, we have

$$\left|\tanh 1-\frac{p_n}{q_n}\right|>\frac{1}{q_n(q_{n+1}+q_n)}\geq\frac{1}{(2n+3)q_n^2}.$$

Now we must estimate q_n. Suppose that $n\geq N_2$. Since $q_n\geq(2n-1)q_{n-1}\geq\cdots\geq\prod_{\nu=1}^{n}(2\nu-1)$, we have

$$\log q_n\geq\sum_{\nu=1}^{n}\log(2\nu-1)\geq(n-1/2)\log(2n-1)-n+1$$
$$\geq(n-1/2)\log((2n-1)/e).$$

Conversely, since $q_n\leq 2nq_{n-1}\leq 2^n n!$, we have

$$\log q_n\leq n\log 2+\sum_{\nu=1}^{n}\log\nu\leq n\log 2+(n+1)\log(n+1)-n$$
$$\leq(n+1)\log(10(n+1)/13),$$
$$\log\log q_n\leq\log(n+1)+\log\log(10(n+1)/13).$$

Since

$$l_1(x)=\frac{\log\log(10(x+1)/13)}{\log(x+1)}\qquad(x\geq 25)$$

and

$$l_2(x)=\frac{\log(x+1)}{\log(7(x+1)/10)}\qquad(x\geq 1)$$

are strictly decreasing functions and $7(x+1)/10\leq(2x-1)/e$ $(x\geq 30)$, we have

$$\log\log q_n\leq(1+l_1(N_2))\log(n+1)\leq l_2(N_2)(1+l_1(N_2))\log((2n-1)/e).$$

From these inequalities, we find

$$\frac{\log\log q_n}{\log q_n} \leq l_2(N_2)\cdot\frac{1+l_1(N_2)}{n-1/2} \leq 2\left(1+\frac{2}{N_2-1/2}\right)\frac{\log(N_2+1)}{\log(7(N_2+1)/10)}$$
$$\cdot\left(1+\frac{\log\log(10(N_2+1)/13)}{\log(N_2+1)}\right)\cdot\frac{1}{2n+3} = \frac{\eta_{N_2}}{2n+3}.$$

Therefore,

$$\left|\tanh 1 - \frac{p_n}{q_n}\right| > \frac{\log\log q_n}{\eta_{N_2} q_n^2 \log q_n}.$$

This completes the proof.

LEMMA 2. *For all integers p and q with* $q \geq 2$,

$$\left|\tanh 1 - \frac{p}{q}\right| > \frac{\log\log q}{\eta q^2 \log q},$$

where

$$\eta \geq \max\{\eta_{N_2}, \eta^*_{N_2}\}$$

for any positive integer $N_2 \geq 30$.

PROOF. It suffices only to consider that p/q is an n-th convergent of $\tanh 1$. From the definition of $\eta^*_{N_2}$, we have the following inequalities

$$\left|\tanh 1 - \frac{p_n}{q_n}\right| > \frac{1}{(2n+3)q_n^2} = \frac{\log\log q_n}{\theta_n q_n^2 \log q_n} \geq \frac{\log\log q_n}{\eta^*_{N_2} q_n^2 \log q_n} \qquad (2 \leq n < N_2).$$

And from Lemma 1, we have

$$\left|\tanh 1 - \frac{p_n}{q_n}\right| > \frac{\log\log q_n}{\eta_{N_2} q_n^2 \log q_n} \qquad (n \geq N_2).$$

This completes the proof.

§2. Proof of Theorem 1.

We prove the first statement. The continued fraction of $\tanh(1/k)$ is

$$\tanh\frac{1}{k} = [a_0, a_1, a_2, a_3, \cdots] = [0, k, 3k, 5k, \cdots].$$

In other words, $a_0 = 0$ and $a_n = k(2n-1)$ for $n \geq 1$. Let n be a sufficiently large integer to ensure the validity of the later argument.

Case 1: $k \geq 2$. Since $q_{n+1} = a_{n+1}q_n + q_{n-1} > k(2n+1)q_n$, we have

$$\left| \tanh\frac{1}{k} - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{k(2n+1)q_n^2}.$$

Now we must estimate q_n. Since $\log q_n > n - 1/2$, we have $\log\log q_n > \log(n - 1/2)$. And conversely, $\log q_n \leq (n+1)\log(2k(n+1)/e)$. From these inequalities, we find

$$\frac{\log\log q_n}{\log q_n} > \frac{\log(n-1/2)}{(n+1)\log(2k(n+1)/e)} > \frac{\kappa_1}{k(2n+1)}$$

for any positive $\kappa_1 < 2k$. Therefore,

$$\left| \tanh\frac{1}{k} - \frac{p_n}{q_n} \right| < \frac{\log\log q_n}{\kappa_1 q_n^2 \log q_n}.$$

Case 2: $k = 1$. Since $q_{n+1} = a_{n+1}q_n + q_{n-1} > (2n+1)q_n$, we have

$$\left| \tanh 1 - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{(2n+1)q_n^2}.$$

Now we must estimate q_n. Since $\log q_n > n - 1/2$, we have $\log\log q_n > \log(n - 1/2)$. And conversely, $\log q_n \leq (n+1)\log(10(n+1)/13)$. From these inequalities, we find

$$\frac{\log\log q_n}{\log q_n} > \frac{\log(n-1/2)}{(n+1)\log(10(n+1)/13)} > \frac{\kappa_2}{2n+1}$$

for any positive $\kappa_2 < 2$. Therefore,

$$\left| \tanh 1 - \frac{p_n}{q_n} \right| < \frac{\log\log q_n}{\kappa_2 q_n^2 \log q_n}.$$

The second statement of Theorem 1 follows immediately from Theorem A and Lemma 1. This completes the proof.

§3. Proof of Theorem 2.

For $N_2 = 30$, we have $\eta_{30} = 3.18414\cdots$ and $\eta_{30}^* = \theta_5 = 3.55703\cdots$. Hence we can choose η so that $\eta = 4$. Then Theorem 2 follows immediately from Lemma 2. This completes the proof.

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