TOKYO J. MATH.
VOL. 19, NO. 1, 1996

On the A. Beurling Convolution Algebra

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(Communicated by M. Maejima)

1. A. Beurling [2] considered a class of functions of $\mathbf{R}^1$, each member of which is the Fourier transform of an integrable function. The purpose of this paper is to extend his results to the class of functions on $\mathbf{R}^n$. Let us start to set notations, definitions and theorems, which we shall ask for, according to A. Beurling [2]. We consider a normed family Ω of strictly positive functions $\omega(x)$ on $\mathbf{R}^n$ which are measurable with respect to the ordinary Lebesgue measure dx, and furthermore, together with the norm $N(\omega)$, satisfy the following conditions:

(I) For each $\omega\in\Omega$, $N(\omega)$ takes a finite value,

$$0<\int\omega dx\le N(\omega)\,. \tag{1.1}$$

(II) If λ is a positive number and $\omega\in\Omega$, then $\lambda\omega\in\Omega$ and

$$N(\lambda\omega)=\lambda N(\omega)\,. \tag{1.2}$$

(III) If $\omega_1, \omega_2\in\Omega$, the sum $\omega_1+\omega_2$ as well as the convolution $\omega_1*\omega_2$ are also in Ω and

$$N(\omega_1+\omega_2)\le N(\omega_1)+N(\omega_2)\,, \tag{1.3}$$

$$N(\omega_1*\omega_2)\le N(\omega_1)N(\omega_2)\,. \tag{1.4}$$

(IV) Ω is complete under the norm N in the sense that for any sequence $\{\omega_n\}_1^\infty\subset\Omega$ such that $\sum_1^\infty N(\omega_n)<\infty$, it is satisfied that $\omega=\sum_1^\infty\omega_n$ is in Ω and

$$N(\omega)\le\sum_1^\infty N(\omega_n)\,. \tag{1.5}$$

The set of measures $\{\omega dx;\,\omega\in\Omega\}$ constitutes our starting point for the following constructions of Banach algebra and shall be referred to as a normed semi-ring of

Received September 6, 1994
Revised October 13, 1995

positive finite measures.

We associate with each ω, the Banach space $L^2_{\omega^{-1}}$ of measurable functions on $\mathbf{R}^n$ and having the norms

$$\|F\|_{L^2_{\omega^{-1}}}=\left(\int_{\mathbf{R}^n}\omega dx\int_{\mathbf{R}^n}\frac{|F|^2}{\omega}dx\right)^{1/2}. \tag{1.6}$$

From these spaces, we obtain set of functions $A^2=A^2(\mathbf{R}^n, \Omega)$ by setting

$$A^2=\bigcup_{\omega\in\Omega}L^2_{\omega^{-1}} \tag{1.7}$$

and define

$$\|F\|=\|F\|_{A^2}=\inf_{\omega\in\Omega}\|F\|_{L^2_{\omega^{-1}}}. \tag{1.8}$$

At present, these are also referred to as the so-called Beurling convolution algebra.

Now A. Beurling [2] proved the following theorem.

THEOREM I. *In the norm* (1.8), A^2 *is the Banach algebra under addition and convolution, and we have*

$$\|F_1*F_2\|\leq\|F_1\|\,\|F_2\|. \tag{1.9}$$

Let us consider the algebra A^2 which are generated by some particularly simple families of Ω. Let us consider the set $\Omega=\Omega(\mathbf{R}^n)$ of positive and summable $\omega(x)$ which are non-increasing function of $|x|$ and have the norm

$$N(\omega)=\int_{\mathbf{R}^n}\omega dx. \tag{1.10}$$

Next let us consider the family Ω_1 defined as the sub-set of Ω consisting of functions with the property:

$$\omega(0)=\lim_{x\to 0}\omega(x)<\infty. \tag{1.11}$$

The norm in Ω_1 will be defined as

$$N(\omega)=\omega(0)+\int_{\mathbf{R}^n}\omega dx. \tag{1.12}$$

The sets Ω and Ω_1 satisfy conditions (I)$\sim$(IV), and we can define the Banach algebras $A^2=A^2(\mathbf{R}^n, \Omega)$ and $\mathscr{A}^2=\mathscr{A}^2(\mathbf{R}^n, \Omega_1)$ respectively.

From now on, let us restrict to functions which are defined on real lines. Let us begin with $A^2=A^2(\mathbf{R}^1, \Omega)$ and recall that

(1.13) $$\|F\|_{A^2}=\inf_{\omega\in\Omega}\left(\int_{-\infty}^{\infty}\omega dx\int_{-\infty}^{\infty}\frac{|F|^2}{\omega}dx\right)^{1/2},$$

where Ω consists of summable positive ω which are non-increasing functions of $|x|$. By capitals $F(x)$, $G(x)$, $\cdots$ we shall denote elements of A^2, while $f(t)$, $g(t)$, $\cdots$ will be their Fourier transforms in the definition

(1.14) $$f(t)=\int_{-\infty}^{\infty}e^{-itx}F(x)dx\,.$$

Let us remark that $A^2\subset L^1$ by the inequality

$$\int_{-\infty}^{\infty}|F|dx\leq\left(\int_{-\infty}^{\infty}\omega\delta\varphi\int_{-\infty}^{\infty}\frac{|F|^2}{\omega}dx\right)^{1/2},$$

and so the Fourier transform f of $F\in A^2$ is well defined. The ring of Fourier transforms f of $F\in A^2$ will be denoted by $\tilde{A}^2$ and will have the norm

(1.15) $$\|f\|=\|f\|_{\tilde{A}^2}=\|F\|_{A^2}\,.$$

It is clear that by (1.9), if f and g belong to $\tilde{A}^2$, then fg also does to $\tilde{A}^2$ and satisfy the inequality:

(1.16) $$\|fg\|\leq\|f\|\,\|g\|\,.$$

Other notations that will be used in the paper are

(1.17) $$\eta(\alpha)=\eta(\alpha,f)=\sqrt{\frac{1}{\pi}\int_{-\infty}^{\infty}|f(t+\alpha)-f(t)|^2dt}\,,$$

(1.18) $$\Delta_\alpha f(t)=f(t+\alpha)-f(t)\,,\qquad\text{and}$$

(1.19) $$A(f)=\int_{-\infty}^{\infty}\eta(\alpha,f)\frac{d\alpha}{|\alpha|^{3/2}}\,.$$

This integral $A(f)$ will be an important tool for the analysis of A^2 as shown by the following theorem.

THEOREM II. *A function f belongs to the ring $\tilde{A}^2$ if and only if*:
(a) *f is continuous,*
(b) $\lim_{t\to\infty}f(t)=0$,
(c) $A(f)<\infty$.

Under these conditions, f is the Fourier transform of an $F\in A^2$, and the following inequalities hold:

(1.20) $$\|F\|_{A^2}<A(f)<5\|F\|_{A^2}$$

provided $f \neq 0$.

Next, a function $K(z)$ defined in the whole complex-plane and having the properties

$$|K(z')-K(z)| \leq |z'-z|, \qquad K(0)=0 \tag{1.21}$$

will in the sequel be called a contractor. Furthermore, a function $g(t)$ shall be called a contraction of $f(t)$ if for all t and t'

$$|g(t)| \leq |f(t)|, \qquad |g(t')-g(t)| \leq |f(t')-f(t)|. \tag{1.22}$$

We shall also consider conditions such as

$$|g(t)| \leq \sum_{\nu=1}^{N} |f_\nu(t)|, \qquad |g(t')-g(t)| \leq \sum_{\nu=1}^{N} |f_\nu(t')-f_\nu(t)|, \tag{1.23}$$

and in that case we shall say that g is a contraction of the series $\sum_{\nu=1}^{N} f_\nu$.

One of the most striking consequences of Theorem II is that each contractor K defines a bounded operator on $\tilde{A}^2$. In fact if f is continuous and tends to 0 for $t \to \pm\infty$, the same is true of $g(x)=K(f(x))$. Clearly, $A(g) \leq A(f)$ and Theorem II yields

$$\|K(f)\| < 5\|f\|. \tag{1.24}$$

The continuity thoerem for $\tilde{A}^2$ is now a consequence of the following stronger

THEOREM III. *Let g be a contraction of the series $\sum_{\nu=1}^{N} f_\nu$ where each f_ν belongs to $\tilde{A}^2$. Then*

$$g \in \tilde{A}^2, \qquad \|g\| \leq k \sum_{\nu=1}^{N} \|f_\nu\|, \tag{1.25}$$

where k (<5) is a constant depending only on the dimension n.

If, in a sequence $\{g_n\}$, each function is a contraction of $\sum_{\nu=1}^{N} f_\nu$, then the assumption

$$\lim_{n\to\infty} M(g_n)=0 \tag{1.26}$$

implies

$$\lim_{n\to\infty} \|g_n\|=0, \tag{1.27}$$

where $M(g)$ denotes the essential supremum of g.

This property of a space will be referred to as the principle of uniform contraction, whereas the implication (1.25) alone will be called the principle of contraction.

A. Beurling [2] also considered the same problem as Theorem II for the algebra $\mathscr{A}^2=\mathscr{A}^2(\mathbf{R}_1, \Omega_1)$. For the norm of $F \in \mathscr{A}^2$ we have

$$\|F\|_{\mathscr{A}^2} = \inf_{\omega\in\Omega_1} \left(\left(\omega(0)+\int_{-\infty}^{\infty} \omega dx\right)\int_{-\infty}^{\infty} \frac{|F|^2}{\omega}\, d\omega\right)^{1/2}. \tag{1.28}$$

Then he first proved:

THEOREM IV. *The space $\mathscr{A}^2$ is the intersection of A^2 and L^2, and the norms in these spaces satisfy inequalities*

$$\|F\|_{\mathscr{A}^2} > \|F\|_{A^2}, \tag{1.29}$$

$$\|F\|_{\mathscr{A}^2} > \|F\|_{L^2}, \tag{1.30}$$

$$\|F\|_{\mathscr{A}^2} < \|F\|_{A^2} + \|F\|_{L^2}. \tag{1.31}$$

By $\tilde{\mathscr{A}}^2$ we shall denote the ring of Fourier transform f of $F \in \mathscr{A}^2$ with the norm $\|f\|_{\tilde{\mathscr{A}}^2} = \|F\|_{\mathscr{A}^2}$. Combining Theorems II and IV we have

THEOREM V. *A function f belongs to $\tilde{\mathscr{A}}^2$ if and only if*:

(a) *f is continuous,*

(b) *$f \in L^2$,*

(c) *$A(f) < \infty$.*

Under these conditions the following inequalities hold:

$$\|F\|_{\mathscr{A}^2} < A(f) + \frac{1}{\sqrt{2\pi}} \|f\|_{L^2} < 6\|F\|_{\mathscr{A}^2} \tag{1.32}$$

provided $f \neq 0$.

From this we see that the principle of uniform contraction is valid in $\tilde{A}^2$ with constant $k = 6$.

These are outlines of principal results due to A. Beurling [2], of which we shall quote.

2. The present authors considered that it is worth-while to extend theorems on the real line $\mathbf{R}^1$ to those on the n-dimensional euclidian space $\mathbf{R}^n$. This is the purpose of this paper.

Now let us remark that x, y, t, α and β are n-dimensional vectors in $\mathbf{R}^n$ and let us denote tx or xy the inner product. Let us also remark that $\Omega = \Omega(\mathbf{R}^n)$ denotes the set of positive measurable $\omega(x)$ which are non-increasing function of $|x|$. Let us consider the algebra $A^2 = A^2(\mathbf{R}^n, \Omega)$ with the norm

$$\|F\| = \|F\|_{A^2} = \inf_{\omega \in \Omega} \left(\int_{\mathbf{R}^n} \omega(x) dx \int_{\mathbf{R}^n} \frac{|F(x)|^2}{\omega(x)} dx \right)^{1/2}. \tag{2.1}$$

Let us consider by $\tilde{A}^2$ the set of function f which is the Fourier transform of $F \in A^2$.

Let us also define the notation as follows:

$$\eta(\alpha) = \eta(\alpha, f) = \sqrt{\left(\frac{1}{\pi}\right)^n \int_{\mathbf{R}^n} |\Delta_\alpha^n f(t)|^2 dt}, \tag{2.2}$$

where $\Delta_\alpha^n f$ is the vector difference along α, that is

$$\Delta_\alpha^n f(t)=\sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}f(t+(n-k)\alpha)\,, \tag{2.3}$$

and

$$A(f)=\int_{\mathbf{R}^n}\eta(\alpha, f)\frac{d\alpha}{|\alpha|^{3n/2}}\,. \tag{2.4}$$

Then we shall prove the following theorem:

THEOREM 1. *A function f belongs to the ring $\tilde{A}^2$ if and only if*:
(a) *f is continuous,*
(b) $\lim_{|t|\to\infty}f(t)=0$,
(c) $A(f)<\infty$.

Under these conditions, f is the Fourier transform of $F\in A^2$, and the following inequalities hold:

$$c_n\|F\|_{A^2}\le A(f)\le d_n\|F\|_{A^2}\,, \tag{2.5}$$

provided $f\neq 0$, where c_n and d_n are positive constants which are independent of f.

PROOF. In the first we shall prove that our conditions are sufficient and imply $c_n\|F\|\le A(f)$. From (c) and the definition of $\eta(\alpha, f)$ and $A(f)$, it follows that $\Delta_\alpha^n f(t)$ is in $L^2(\mathbf{R}^n)$ for α belonging to a certain set E whose complement is of measure zero. We point out also that $\eta(\alpha)>0$ for $\alpha\neq 0$ except in the trivial case $f=0$ which we exclude. In fact, if $\eta(\alpha)=0$ for some $\alpha\neq 0$, then under the assumption (a) we have $\Delta_\alpha^n f(t)=0$ $(\forall t)$ which under the condition (b) implies $f(t)=0$ $(\forall t)$.

Using the Plancherel theorem, we shall define for all $\alpha\in E$ a function $F_\alpha(x)$ by setting

$$(e^{-i\alpha x}-1)^n F_\alpha(x)=\text{l.i.m.}_{N\to\infty}\left(\frac{1}{2\pi}\right)^n\int_{|t|<N}e^{itx}\Delta_\alpha^n f(t)dt\,.$$

If β is another number in E, both sides of the identity

$$\Delta_\beta^n\Delta_\alpha^n f(t)=\Delta_\alpha^n\Delta_\beta^n f(t)$$

belong to L^2, and we obtain by taking the inverse Fourier transforms,

$$(e^{-i\beta x}-1)^n(e^{-i\alpha x}-1)^n F_\alpha(x)=(e^{-i\alpha x}-1)^n(e^{-i\beta x}-1)^n F_\beta(x)\,.$$

Hence, $F_\alpha(x)=F(x)$ is independent of α, and the Parseval relation yields

$$\eta^2(\alpha)=2^n\int_{\mathbf{R}^n}|F(x)|^2\sin^{2n}\frac{\alpha x}{2}dx\,. \tag{2.6}$$

Therefore, we have

$$A(f)=\int_{\mathbf{R}^n}\frac{d\alpha}{\eta(\alpha)|\alpha|^{3n/2}}\int_{\mathbf{R}^n}2^n|F(x)|^2\sin^{2n}\frac{\alpha x}{2}\,dx$$
$$=\int_{\mathbf{R}^n}|F(x)|^2\int_{\mathbf{R}^n}\frac{2^n\sin^{2n}(\alpha x/2)}{\eta(\alpha)|\alpha|^{3n/2}}\,d\alpha dx\,.$$

Let us define

(2.7)
$$\frac{1}{\mu(x)}=\int_{\mathbf{R}^n}\frac{2^n\sin^{2n}(\alpha x/2)}{\eta(\alpha)|\alpha|^{3n/2}}\,d\alpha\,,\qquad\text{and}$$

(2.8)
$$\omega(x)=\int_{|\alpha|<2/|x|}\frac{\eta(\alpha)}{|\alpha|^{n/2}}\,d\alpha\,.$$

Using the Schwartz inequality, we have

$$\left(\int_{|\alpha|<2/|x|}\frac{\sqrt{2}^{\,n}|\sin^n(\alpha x/2)|}{|\alpha|^n}\,d\alpha\right)^2=\left(\int_{|\alpha|<2/|x|}\frac{\sqrt{2}^{\,n}\eta(\alpha)^{1/2}|\sin^n(\alpha x/2)|}{|\alpha|^{n/4}\eta(\alpha)^{1/2}|\alpha|^{3n/4}}\,d\alpha\right)^2$$
$$\leq\int_{|\alpha|<2/|x|}\frac{\eta(\alpha)}{|\alpha|^{n/2}}\,d\alpha\int_{|\alpha|<2/|x|}\frac{2^n\sin^{2n}(\alpha x/2)}{\eta(\alpha)|\alpha|^{3n/2}}\,d\alpha\leq\frac{\omega(x)}{\mu(x)}\,.$$

On the other hand, replacing α by $2\beta/|x|$, using $\sin t>ct$ ($c=\sin 1$, $0<t<1$) and denoting β_1 the first co-ordinates of β, then we have

$$\int_{|\alpha|<2/|x|}\frac{\sqrt{2}^{\,n}|\sin^n(\alpha x/2)|}{|\alpha|^n}\,d\alpha=\int_{|\beta|<1}\frac{\sqrt{2}^{\,n}|\sin^n(\beta x/|x|)|}{|\beta|^n}\,d\beta$$
$$>(c\sqrt{2}\,)^n\int_{|\beta|<1}\left(\frac{|\beta_1|}{|\beta|}\right)^n d\beta=\sqrt{b_n}\,,\qquad\text{say}\,.$$

Combining these estimations, we have

$$1/\mu(x)\geq b_n/\omega(x)\,.$$

Therefore we have

$$A(f)=\int_{\mathbf{R}^n}\frac{|F(x)|^2}{\mu(x)}\,dx\geq b_n\int_{\mathbf{R}^n}\frac{|F(x)|^2}{\omega(x)}\,dx$$

and so

(2.9)
$$\int_{\mathbf{R}^n}\frac{|F(x)|^2}{\omega(x)}\,dx\leq b_n^{-1}A(f)\,.$$

We have also

$$\int_{\mathbf{R}^n}\omega(x)dx=\int_{\mathbf{R}^n}\int_{|\alpha|<2/|x|}\frac{\eta(\alpha)}{|\alpha|^{n/2}}d\alpha dx=\int_{\mathbf{R}^n}\frac{\eta(\alpha)}{|\alpha|^{n/2}}d\alpha\int_{|x|<2/|\alpha|}dx$$
$$=\frac{\sqrt{\pi}^{\,n}}{\Gamma(n/2+1)}\int_{\mathbf{R}^n}\left(\frac{2}{|\alpha|}\right)^n\frac{\eta(\alpha)}{|\alpha|^{n/2}}d\alpha=\frac{2^n\sqrt{\pi}^{\,n}}{\Gamma(n/2+1)}\int_{\mathbf{R}^n}\frac{\eta(\alpha)}{|\alpha|^{3n/2}}d\alpha$$
$$=a_nA(f)\,,\qquad \text{say}\,.$$

Therefore we have

$$\int_{\mathbf{R}^n}\omega(x)dx=a_nA(f)\,, \tag{2.10}$$

where $a_n=2^n\sqrt{\pi}^{\,n}/\Gamma(n/2+1)$. From (2.9) and (2.10), we obtain

$$\|F\|^2\leq\int_{\mathbf{R}^n}\omega(x)dx\int_{\mathbf{R}^n}\frac{|F(x)|^2}{\omega(x)}dx\leq\frac{a_n}{b_n}A(f)^2$$

and so F belongs to the class A^2 and we have

$$c_n\|F\|\leq A(f)\,, \tag{2.11}$$

where $c_n=\sqrt{b_n/a_n}>0$, which depends only on n and not on f.

In the last we shall prove that f is actually the Fourier transform of F. In accordance with the definition of F and the fact that f is continuous, we have for all α and t

$$\Delta_n^\alpha f(t)=\int_{\mathbf{R}^n}e^{-itx}(e^{-i\alpha x}-1)^nF(x)dx\,.$$

The Riemann-Lebesgue theorem and hypothesis (b): $\lim_{|\alpha|\to\infty}f(t+(n-k)\alpha)=0$ $(0\leq k<n)$, t being fixed, imply that

$$f(t)=\int_{\mathbf{R}^n}e^{-itx}F(x)dx\,, \tag{2.12}$$

which is the desired relation.

Now we shall prove the necessity of our condition. Let F be a function of the class A^2 and f be its Fourier transform. Then it remains only to prove that $A(f)\leq d_n\|F\|$. The proof of this inequality depends essentially on the following lemma.

LEMMA. *Let $0<a<n<b$, and let $\omega_0(|x|)$ be non-increasing in $|x|$ and summable on $\mathbf{R}^n$. Then $\omega_0(|x|)$ possesses a majorant $\omega^*(|x|)$ with the properties that $|x|^a\omega^*(|x|)$ is non-increasing, $|x|^b\omega^*(|x|)$ is non-decreasing in $|x|$ respectively and we have*

(2.13) $$\int_{\mathbf{R}^n} \omega^*(|x|)dx \le \frac{b(2n-a)}{(n-a)(b-n)} \int_{\mathbf{R}^n} \omega_0(|x|)dx\,.$$

PROOF. The one-dimensional case is due to A. Beurling [1], and in our case, functions which we consider are radial and so the lemma can be proved by running along the same lines as his. Therefore we shall describe only the different aspect from his proof. As for the same aspect as his, the reader will be referred to his original paper [1].

Let us denote $B=\{x\in\mathbf{R}^n; |x|\le 1\}$ and $S=\{x\in\mathbf{R}^n; |x|=1\}$. Let us denote dS the area element on S. Let us denote also $|R|$ the measure of the set R.

Without loss of generality, we shall assume that $\omega_0(r)$ $(r=|x|)$ is continuous on the left, and let us consider the family of functions $\omega(r)$ such that $\omega_0(r)\le\omega(r)$ and $r^b\omega(r)$ is non-decreasing. Let us denote $\omega_1(r)$ the least function of this family. This defines the least majorant of $\omega_0(r)$. Write

$$\bigcup_i (r_{0i}, r_{1i}) = \{r;\, \omega_1(r) > \omega_0(r)\}\,,$$

where (r_{0i}, r_{1i}) are non overlapping intervals. Let (r_0, r_1) be one of them. Let $T=\{x\in\mathbf{R}^n; r_0\le|x|\le r_1\}$, then we have

$$r_0^b\omega_0(r_0)=r_1^b\omega_0(r_1)\,,$$

$$\omega_1(r)=(r_0/r)^b\omega_0(r_0) \quad (r_0\le r\le r_1)\,.$$

Let $R_T=\{(x, s);\, 0\le|x|\le r_0,\ \omega_0(r_0)\ge s\ge\omega_0(r_1)\}$, then

$$|R_T|=|B|(\omega_0(r_0)-\omega_0(r_1))r_0^n\,, \qquad \text{and}$$

$$A_T=\int_{r_0\le|x|\le r_1}\omega_1(x)dx=\int_{r_0}^{r_1}\int_S\left(\frac{r_0}{r}\right)^b\omega_0(r_0)r^{n-1}dSdr\,,$$

$$=|S|r_0^b\omega_0(r_0)\int_{r_0}^{r_1} r^{n-b-1}dr=|S|\omega_0(r_0)\frac{r_0^b}{b-n}(r_0^{n-b}-r_1^{n-b})\,.$$

This implies that

$$\frac{A_T}{|R_T|}=\frac{|S|}{|S|}\frac{r_0^b\omega_0(r_0)(r_0^{n-b}-r_1^{n-b})}{(\omega_0(r_0)-\omega_0(r_1))r_0^n}\frac{1}{b-n}$$

$$=\frac{n}{b-n}\frac{1-(r_0/r_1)^{b-n}}{1-(r_0/r_1)^b}\le\frac{n}{b-n}\,,$$

where we have used the relation $r_0^b\omega_0(r_0)=r_1^b\omega_0(r_1)$. We get

$$\int_{\omega_0(|x|)<\omega_1(|x|)} \omega_1(|x|)dx = \sum A_T \leq \frac{n}{b-n} \sum |R_T| ,$$
$$\leq \frac{n}{b-n} \int_{\mathbf{R}^n} \omega_0(|x|)dx ,$$

where the summation is taken over the whole interval $\bigcup_i (r_{0i}, r_{1i})$. Hence,

$$\int_{\mathbf{R}^n} \omega_1(|x|)dx = \left(\int_{\omega_0<\omega_1} + \int_{\omega_0=\omega_1} \right) \omega_1(|x|)dx ,$$
$$= \frac{n}{b-n} \int_{\mathbf{R}^n} \omega_0(|x|)dx + \int_{\mathbf{R}^n} \omega_0(|x|)dx .$$

Therefore we have

$$\int_{\mathbf{R}^n} \omega_1(|x|)dx \leq \frac{b}{b-n} \int_{\mathbf{R}^n} \omega_0(|x|)dx . \tag{2.14}$$

Similarly, $\omega_1(r)$ posseses a least majorant $\omega^*(r)$ such that $r^a\omega^*(r)$ is non-increasing and $r^b\omega^*(r)$ is non-decreasing. The last result is proved as follows. If $\omega^*(r) \neq \omega_1(r)$, the set where $\omega^*(r) > \omega_1(r)$ is formed by a sequence of non overlapping intervals, and if (r_0, r_1) is one of them, then we have

$$r_0^a \omega_1(r_0) = r_1^a \omega_1(r_1) ,$$
$$\omega^*(r) = (r_0/r)^a \omega_1(r_0) \qquad (r_0 \leq r \leq r_1) .$$

Since $r^a\omega^*(r) = r_0^a\omega_1(r_0) = \textit{constant}$ in this interval and r^{b-a} is increasing, so $r^b\omega^*(r)$ is increasing. On the complementary set, $\omega^*(r) = \omega_1(r)$ and so $r^b\omega^*(r) = r^b\omega_1(r)$ is non-decreasing. Therefore $r^b\omega^*(r)$ is non-decreasing on the whole $r > 0$. This also implies that

$$\frac{\int_{r_0<|x|<r_1} \omega^*(|x|)dx}{\int_{r_0<|x|<r_1} \omega_1(|x|)dx} = \frac{\int_{r_0}^{r_1} r^{n-1}(r_0/r)^a \omega_1(r_0)dr}{\int_{r_0}^{r_1} r^{n-1}\omega_1(r)dr} ,$$
$$\leq \frac{1-(r_0/r_1)^{n-a}}{1-(r_0/r_1)^n} \frac{n}{n-a} \leq \frac{n}{n-a} ,$$

where we have used that the $\omega_1(r)$ is non-increasing. Therefore we obtain as before

$$\int_{\mathbf{R}^n} \omega^*(|x|)dx \leq \frac{2n-a}{n-a} \int_{\mathbf{R}^n} \omega_1(|x|)dx . \tag{2.15}$$

By combining inequalities (2.14) and (2.15), the proof of the lemma is complete.

Now let us apply the lemma to ω for $a = n-1/2$, $b = n+1/2$. Then we get the majorant ω^* and we have

$$\int_{\mathbf{R}^n} \omega^*(|x|)dx \leq (2n+1)^2 \int_{\mathbf{R}^n} \omega(|x|)dx .$$

By the Schwartz inequality, we have

$$A(f)^2 = \left\{ \int_{\mathbf{R}^n} \frac{\eta(\alpha)}{|\alpha|^{3n/2}} d\alpha \right\}^2$$

$$= \left\{ \int_{\mathbf{R}^n} \frac{\eta(\alpha)}{\omega^*(2/|\alpha|)^{1/2} |\alpha|^{n/2}} \frac{\omega^*(2/|\alpha|)^{1/2}}{|\alpha|^n} d\alpha \right\}^2$$

$$\leq \int_{\mathbf{R}^n} \frac{\eta(\alpha)^2}{\omega^*(2/|\alpha|)|\alpha|^n} d\alpha \int_{\mathbf{R}^n} \frac{\omega^*(2/|\alpha|)}{|\alpha|^{2n}} d\alpha .$$

For the second integral of the last formula, let us put $\alpha = re$, $r = |\alpha|$ and $e \in S$, then we have

$$\int_{\mathbf{R}^n} \frac{\omega^*(2/|\alpha|)}{|\alpha|^{2n}} d\alpha = |S| \int_0^\infty \frac{\omega^*(2/r)}{r^{2n}} r^{n-1} dr$$

$$= \frac{|S|}{2^n} \int_0^\infty \omega^*(u) u^{n-1} du = \frac{1}{2^n} \int_{\mathbf{R}^n} \omega^*(|x|)dx .$$

For the first integral of the last formula, we have

$$\int_{\mathbf{R}^n} \frac{\eta(\alpha)^2}{\omega^*(2/|\alpha|)|\alpha|^n} d\alpha = 2^n \int_{\mathbf{R}^n} |F(x)|^2 \int_{\mathbf{R}^n} \frac{\sin^{2n}(\alpha x/2)}{\omega^*(2/|\alpha|)|\alpha|^n} d\alpha dx .$$

Let us put $\alpha = re$, $e \in S$ and $r = |\alpha|$ similarly, and next let us put $\rho = |x| r/2$. Then we can estimate the second integral of the last formula. We have

$$\int_{\mathbf{R}^n} \frac{\sin^{2n}(\alpha x/2)}{\omega^*(2/|\alpha|)|\alpha|^n} d\alpha = \int_0^\infty \int_S \frac{\sin^{2n}(rex/2)}{\omega^*(2/r) r^n} r^{n-1} dS dr$$

$$= \left(\int_0^1 \int_S + \int_1^\infty \int_S \right) \frac{\sin^{2n}(\rho e x/|x|)}{\omega^*(|x|/\rho)} \frac{1}{\rho} dS d\rho$$

$$= \frac{1}{\omega^*(|x|)} \left(\int_0^1 \int_S \frac{\omega^*(|x|)}{\omega^*(|x|/\rho)} \frac{\sin^{2n}(\rho e x/|x|)}{\rho} dS d\rho \right.$$

$$\left. + \int_1^\infty \int_S \frac{\omega^*(|x|)}{\omega^*(|x|/\rho)} \frac{\sin^{2n}(\rho e x/|x|)}{\rho} dS d\rho \right) .$$

Here, since $t^b \omega^*(t)$ is non-decreasing, from $|x|^b \omega^*(|x|) \leq (|x|/\rho)^b \omega^*(|x|/\rho)$ $(0 < \rho \leq 1)$, we have

$$\frac{\omega^*(|x|)}{\omega^*(|x|/\rho)} \leq \frac{1}{\rho^b} = \frac{1}{\rho^{n+1/2}} \qquad (0<\rho\leq 1),$$

and since $t^a\omega^*(t)$ is non-increasing, from $|x|^a\omega^*(|x|)\leq(|x|/\rho)^a\omega^*(|x|/\rho)$ $(\rho\geq 1)$, we have

$$\frac{\omega^*(|x|)}{\omega^*(|x|/\rho)} \leq \frac{1}{\rho^a} = \frac{1}{\rho^{n-1/2}} \qquad (\rho\geq 1).$$

Therefore the last formula does not exceed the following one

$$\frac{1}{\omega^*(|x|)}\left(\int_0^1\int_S \frac{1}{\rho^{n+1/2}}\frac{\sin^{2n}(\rho ex/|x|)}{\rho}dSd\rho+\int_1^\infty\int_S \frac{1}{\rho^{n-1/2}}\frac{\sin^{2n}(\rho ex/|x|)}{\rho}dSd\rho\right)$$

$$\leq \frac{1}{\omega^*(|x|)}|S|\left\{\int_0^1 \frac{\rho^{n-1}}{\rho^{1/2}}d\rho+\int_1^\infty \frac{1}{\rho^{n+1/2}}d\rho\right\}=\frac{4}{2n-1}\frac{|S|}{\omega^*(|x|)}.$$

Thus we have

$$A(f)^2\leq \frac{4}{2n-1}|S|\int_{\mathbf{R}^n}\omega^*(|x|)dx\int_{\mathbf{R}^n}\frac{|F(x)|^2}{\omega^*(|x|)}dx.$$

Since ω^* is the majorant of ω, we have

(2.16) $$A(f)\leq d_n\|F\|,$$

where

$$d_n=\sqrt{\frac{4(2n+1)^2}{2n-1}|S|}=\sqrt{\frac{8(2n+1)^2\sqrt{\pi}^{\,n}}{(2n-1)\Gamma(n/2)}}.$$

This completes the proof of Theorem 1.

Next we shall generalize Theorem III to functions on the n-dimensional euclidean space $\mathbf{R}^n$.

A function $g(t)$ shall be called a contraction of $f(t)$ if for all t and α

(2.17) $$|g(t)|\leq|f(t)|, \qquad |\Delta_\alpha^n g(t)|\leq|\Delta_\alpha^n f(t)|.$$

We shall also consider conditions such as

(2.18) $$|g(t)|\leq\sum_{\nu=1}^N|f_\nu(t)|, \qquad |\Delta_\alpha^n g(t)|\leq\sum_{\nu=1}^N|\Delta_\alpha^n f_\nu(t)|,$$

and in this case we shall say that g is a contraction of the series $\sum_{\nu=1}^N f_\nu$.

Then we have

THEOREM 2. *Let g be a contraction of the series $\sum_{\nu=1}^N f_\nu$ where each f_ν belongs to $\tilde{A}^2$. Then*

$$g \in \tilde{A}^2, \qquad \|g\| \le k \sum_{\nu=1}^{N} \|f_\nu\|, \tag{2.19}$$

where k is a constant depending only on the dimension n.

If, in a sequence $\{g_n\}$, each funcion is a contraction of $\sum_{\nu=1}^{N} f_\nu$, then the assumption

$$\lim_{n\to\infty} M(g_n) = 0 \tag{2.20}$$

implies

$$\lim_{n\to\infty} \|g_n\| = 0 . \tag{2.21}$$

PROOF. The proof of this theorem can be done by running the same lines as that of Theorem III.

We shall generalize Theorems IV and V to functions on the n-dimensional euclidean space $\mathbf{R}^n$.

Let us consider the algebra $\mathscr{A}^2 = \mathscr{A}^2(\mathbf{R}^n, \Omega_1)$. For the norm of $F \in \mathscr{A}^2$ we have

$$\|F\|_{\mathscr{A}^2} = \inf_{\omega\in\Omega_1} \left(\left(\omega(0) + \int_{\mathbf{R}^n} \omega dx\right) \int_{\mathbf{R}^n} \frac{|F|^2}{\omega} dx\right)^{1/2} . \tag{2.22}$$

Then we shall first prove

THEOREM 3. *The space $\mathscr{A}^2$ is the intersection of A^2 and L^2, and the norms in these spaces satisfy the inequalities*

$$\|F\|_{\mathscr{A}^2} > \|F\|_{A^2}, \tag{2.23}$$

$$\|F\|_{\mathscr{A}^2} > \|F\|_{L^2}, \tag{2.24}$$

$$\|F\|_{\mathscr{A}^2} < \|F\|_{A^2} + \|F\|_{L^2} . \tag{2.25}$$

PROOF. The proof of this theorem can be done by running the same lines as that of Theorem IV.

By $\tilde{\mathscr{A}}^2$, we shall denote the ring of Fourier transform f of $F \in \mathscr{A}^2$ with norm $\|f\| = \|F\|$. Then we have

THEOREM 4. *A function f belongs to $\tilde{\mathscr{A}}^2$ if and only if:*

(a) *f is continuous,*

(b) *$f \in L^2$,*

(c) *$A(f) < \infty$.*

Under these conditions the following inequalities hold:

$$\|F\|_{\mathscr{A}^2} < A(f) + (1/\sqrt{2\pi})^n \|f\|_{L^2} < (d_n + 1)\|F\|_{\mathscr{A}^2} \tag{2.26}$$

provided $f \neq 0$.

PROOF. The proof can be done by Theorems 1 and 3. However there is something different aspect in the proof of sufficiency of the conditions (a), (b) and (c). For the sake of completeness, we shall point it out.

Since we assume that f belongs to L^2 instead of $\lim_{|t|\to\infty} f(t)=0$, we can define its Fourier transform F by the Plancherel theorem directly:

$$F(x)=\underset{N\to\infty}{\mathrm{l.i.m.}}\left(\frac{1}{2\pi}\right)^n \int_{|t|<N} e^{itx} f(t)dt\,,$$

and the Parseval relation yields

$$\eta^2(\alpha)=2^n\int_{\mathbf{R}^n} |F(x)|^2 \sin^{2n}\frac{\alpha x}{2}\,dx\,.$$

From this formula, if $\eta(\alpha)=0$ $(\alpha\neq 0)$, we obtain $F(x)=0$ (a.e.x) and so $f(t)=0$ $(\forall t)$. Therefore we have $\eta(\alpha)>0$ for $\alpha\neq 0$ except in the trivial case $f=0$ which we shall exclude. After that, we can run the same lines as the proof of Theorem 1, we get $F\in A^2$ and so $F\in\mathscr{A}^2$ and $f\in\tilde{\mathscr{A}}^2$ respectively.

From this we see that the principle of uniform contraction is valid in $\tilde{\mathscr{A}}^2$ with constant $k=d_n+1$.

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