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# On Edge Transitive Circulant Graphs

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**Abstract.** A classification is given for edge transitive circulant graphs whose complements are also edge transitive. In particular, it is shown that the only self-complementary symmetric circulant graphs are certain Paley graphs with a prime number of vertices. Similar results on digraphs are also obtained.

## 1. Introduction.

In this paper we classify the family of circulant graphs which are edge transitive and whose complements are also edge transitive. We show that if a graph $G$ is circulant and both $G$ and $\bar{G}$ are edge transitive, then $G$ is isomorphic to one of the following

1. $mK_n$, a disjoint union of $m$ copies of complete graph $K_n$.
2. $T_{m,mn}$, a complete $m$-partite graph with $n$ vertices in each part.
3. $P(p)$, a Paley graph on $p$ vertices, where $p$ is a prime and $p \equiv 1 \pmod 4$.

The classification is obtained by considering two cases. In the first case the automorphism group is imprimitive and the graphs are shown to be either $mK_n$ or $T_{m,mn}$. In the second case the automorphism group is primitive. The classification is obtained by using a result of Schur on Burnside groups.

As a corollary, we shall classify all self-complementary symmetric circulant graphs.

Similarly the family of circulant digraphs which are edge transitive and whose complement are also edge transitive is classified.

The graphs and digraphs we consider here are assumed to be finite and simple. A graph with $v$ vertices is circulant if its automorphism group contains a $v$-cycle. A graph $G$ is said to be vertex transitive if its automorphism group $A(G)$ is transitive on the vertex set. The automorphism group also induces an action on the edge set. A graph is called edge transitive if the action of the automorphism group on the edge set is transitive. An undirected graph is called strongly edge transitive if for any two edges $ab$ and $cd$ there exists an automorphism $\theta$ such that $\theta a = c$, $\theta b = d$. A graph is symmetric if it is both vertex transitive and edge transitive. The complement $\bar{G}$ of a graph $G$ has the same vertex set as $G$ and the edges of $\bar{G}$ are the non-edges of $G$. A graph $G$ is called

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self-complementary if it is isomorphic to its complement $\bar{G}$.

A permutation group $A$ acting on a set $X$ is said to be doubly transitive if for any two ordered pairs $(a, b)$, $(c, d)$, where $a, b, c, d \in X$, there exists a $\theta \in A$ such that $\theta a = c$, $\theta b = d$.

An abstract group $H$ is called a Burnside group (or B-group) if any primitive group containing the regular representation of $H$ as a transitive subgroup is doubly transitive (see [1] and [6]).

Let $A$ be a transitive permutation group acting on a set $X$. A subset $B$ of $X$ is called a block if for any $\theta \in A$, $\theta B = B$ or $\theta B \cap B = \varnothing$. $\varnothing$, $X$ and the singleton subsets of $X$ are called trivial blocks. $A$ is called primitive if it has no nontrivial blocks. $A$ is imprimitive if it has nontrivial blocks. Let $B$ be a block of $A$. The elements of $A$ that leave $B$ fixed setwise form a subgroup of $A$ which is denoted by $A_{(B)} = \{\theta \in A \mid \theta B = B\}$.

## 2. Constructions.

Consider the disjoint union of $m$ copies of complete graph on $n$ vertices $mK_n$. Clearly the complement of $mK_n$ is the complete $m$-partite graph $T_{m,mn}$, which is also a special type of Turán graphs. We first show that $mK_n$ and $T_{m,mn}$ are examples of circulant and edge transitive graphs.

LEMMA 1. *If G is circulant and edge transitive, then the disjoint union of m isomorphic copies of G, denoted by mG, is also circulant and edge transitive.*

PROOF. Label the vertices of the $i$th copy of $G$ by $v_1^i, v_2^i, \cdots, v_n^i$ such that $(v_1^i, v_2^i, \cdots, v_n^i)$ is an automorphism of $G$. Then it is easy to verify that

$$(v_1^1, v_1^2, \cdots, v_1^m, v_2^1, v_2^2, \cdots, v_2^m, \cdots, v_n^1, v_n^2, \cdots, v_n^m)$$

is an automorphism of $mG$. Hence $mG$ is circulant. $mG$ is clearly edge transitive. □

THEOREM 1. *The graph $mK_n$ and its complement $T_{m,mn}$ are circulant and edge transitive.*

PROOF. $K_n$ is clearly circulant and edge transitive. By Lemma 1, $mK_n$ is also circulant and edge transitive. Since a graph and its complement have the same automorphism group, $T_{m,mn}$, being the complement of $mK_n$, is also circulant. It is easy to see that the automorphism group of $mK_n$ and $T_{m,mn}$ is the wreath product $S_n \wr S_m$ (cf. [4]) and that $T_{m,mn}$ is edge transitive. □

Another family of circulant edge transitive graphs consists of a special type of Paley graphs. Let $p$ be a prime and $p \equiv 1 \pmod 4$. The Paley graph $P(p)$ on $p$ vertices is defined as the graph with the vertex set $Z_p$ and edge set $\{ab \mid b-a \text{ is a nonzero square in } Z_p\}$.

It is well-known that the Paley graph $P(p)$ is edge transitive and self-complementary (see [7]). It is also clear that $P(p)$ is circulant when $p$ is a prime.

## 3. Classification.

We shall prove that the graphs constructed in Section 2 are the only edge transitive circulant graphs whose complements are also edge transitive by studying the automorphism groups. We first show that for circulant graphs edge transitivity and strong edge transitivity are equivalent.

LEMMA 2. *If a graph $G$ is circulant and edge transitive, then $G$ is strongly edge transitive.*

PROOF. Identify the vertices of $G$ as the elements of $\mathbf{Z}_v$ in such a way that $\rho(x)=x+1$ is an automorphism. Let $a$ be a vertex adjacent to the vertex 0. It suffices to show that there is an automorphism $\theta$ such that $\theta(0)=a$ and $\theta(a)=0$. The permutation $\sigma(x)=-x$ is an automorphism of $G$ since any two vertices $s$ and $t$ of $G$ are adjacent if and only if the vertices $\rho^{-s-t}(s)=-t$ and $\rho^{-s-t}(t)=-s$ are adjacent. Let $\theta=\sigma\rho^{-a}$. Then $\theta(0)=-(0-a)=a$ and $\theta(a)=-(a-a)=0$. □

The following theorem gives a classification for the case of imprimitive automorphism groups. The graphs are only assumed to be vertex transitive.

THEOREM 2. *Let $G$ be a vertex transitive graph. If both $G$ and $\bar{G}$ are strongly edge transitive and the automorphism group $A(G)$ is imprimitive, then either $G$ or $\bar{G}$ is isomorphic to $mK_n$ for some positive integers $m$ and $n$.*

PROOF. Let $B$ be a nontrivial block of $A(G)$ and $a$ a fixed element of $B$. Let $S=\{s \mid as \in E(G)\}$ and $S'=\{s \mid as \in E(\bar{G})\}$. Since $|B| \geq 2$, $B$ intersects either $S$ or $S'$. Suppose that $S$ intersects $B$ and $b \in B \cap S$. Since $G$ is strongly edge transitive, for each $s \in S$, there exists an automorphism $\theta$ such that $\theta a=a$ and $\theta b=s$. This implies $s \in B$ since $a, b \in B$ and $B$ is a block. Hence $S \subset B$. Similarly, if $S'$ intersects $B$, then $S' \subset B$. Therefore, we have either $B=\{a\} \cup S$ or $B=\{a\} \cup S'$. If $B=\{a\} \cup S$, consider the action of $A_{(B)}$ on $B$. Since $A$ is transitive and $B$ is a block, $A_{(B)}$ is transitive on $B$. By the strong edge transitivity of $G$, the stabilizer of $A_{(B)}$ at $a$ is transitive on $S$. Hence the action of $A_{(B)}$ on $B$ is doubly transitive. Therefore $B$ induces a complete graph $K_n$. Since $G$ is regular, $B$ must be a component of $G$. By the vertex transitivity, each component of $G$ is a complete graph $K_n$ and $G=mK_n$. If $B=\{a\} \cup S'$, by a similar argument, we have $\bar{G}=mK_n$. □

For a nontrivial circulant graph with a primitive automorphism group, the following theorem shows that it has a prime number of vertices.

THEOREM 3. *Let $G$ be a circulant graph which is neither complete nor null. If the automorphism group $A(G)$ is primitive, then $G$ has $p$ vertices, where $p$ is a prime.*

PROOF. By a theorem of Schur [5] (see [6], Theorem 25.3), a cyclic group of composite order is a B-group. If the number of vertices $v$ of $G$ is not a prime, then the cyclic group of order $v$ is a B-group. Since $G$ is circulant, the primitive group $A(G)$ contains the regular representation of the cyclic group as a transitive subgroup. Hence $A(G)$ is doubly transitive and $G$ is either complete or null, which is a contradiction. □

We now give a classification of the edge transitive circulant graphs whose complements are also edge transitive.

THEOREM 4. *Let $G$ be a circulant graph and both $G$ and $\bar{G}$ be edge transitive. Then $G$ is isomorphic to one of the following.*

1. *$mK_n$, a disjoint union of $m$ copies of complete graph $K_n$.*
2. *$T_{m,mn}$, a complete $m$-partite graph with $n$ vertices in each part.*
3. *$P(p)$, a Paley graph on $p$ vertices, where $p$ is a prime and $p \equiv 1 \pmod 4$.*

PROOF. By Lemma 2, $G$ and $\bar{G}$ are strongly edge transitive. If the automorphism group $A(G)$ is imprimitive, then by Theorem 2, $G$ is isomorphic to $mK_n$ or $T_{m,mn}$ for some $m$ and $n$. If $G$ is complete or null, then $G$ is also isomorphic to $mK_n$ for $m=1$ or $n=1$. Otherwise, $A(G)$ is primitive and $G$ is neither complete nor null. By Theorem 3, the number of vertices of $G$ is a prime $p$. By a theorem of Chao [2], a non-complete non-null symmetric graph with $p$ vertices has an even degree $d$ with $d \mid (p-1)$, and for any even integer $d$ with $d \mid (p-1)$ there exists a unique symmetric graph with $p$ vertices and degree $d$. Consequently, let the degree of $G$ be $d$. Then $d \mid (p-1)$ and $(p-1-d) \mid (p-1)$ since both $G$ and $\bar{G}$ are symmetric. Therefore, $d=(p-1)/2$ is even and $p \equiv 1 \pmod 4$. By the uniqueness of the symmetric graph, $G$ must be the Paley graph $P(p)$. □

In [7], it is shown that the number of vertices of a self-complementary symmetric graph must be a prime power. By the above theorem, we obtain a classification of self-complementary symmetric circulant graphs.

COROLLARY 1. *A graph with at least two vertices is self-complementary, symmetric, and circulant if and only if it is isomorphic to the Paley graph $P(p)$, where $p$ is a prime and $p \equiv 1 \pmod 4$.*

## 4. Digraphs.

The results may be generalized to digraphs. The definitions of circulant graphs, vertex transitive graphs, edge transitive graphs, symmetric graphs, and Paley graphs can be naturally extended to digraphs.

THEOREM 5. *Let $G$ be a circulant digraph and both $G$ and $\bar{G}$ be edge transitive. Then $G$ is isomorphic to one of the following.*

1. *$mK_n$, a disjoint union of $m$ copies of complete digraph $K_n$.*
2. *$T_{m,mn}$, a complete $m$-partite digraph with $n$ vertices in each part.*

3. *A Paley digraph $P(p)$, where $p$ is an odd prime.*

PROOF. The proofs for Theorem 2 and Theorem 3 are still valid for digraphs. If the automorphism group $A(G)$ is imprimitive, similar to the proof of Theorem 2, we can show that $G$ is either a disjoint union of complete digraphs of the same size or its complement. If $A(G)$ is primitive it can be shown, as in Theorem 3, that the number of vertices of $G$ is a prime $p$. By a theorem of Chao and Wells [3], a non-complete non-null symmetric digraph with $p$ vertices has an outdegree $d$ with $d \mid (p-1)$, and for any integer $d$ with $d \mid (p-1)$ there exists a unique symmetric digraph with $p$ vertices and outdegree $d$. Consequently, let the outdegree of $G$ be $d$. Then $d \mid (p-1)$ and $(p-1-d) \mid (p-1)$ since both $G$ and $\bar{G}$ are symmetric. Therefore, $d=(p-1)/2$ and $p$ is odd. By the uniqueness of the symmetric digraphs, $G$ must be the Paley digraph $P(p)$. □

Similarly, we have the following corollary that classifies the self-complementary circulant symmetric digraphs.

COROLLARY 2. *A digraph with at least two vertices is self-complementary, symmetric, and circulant if and only if it is isomorphic to the Paley digraph $P(p)$, where $p$ is an odd prime.*

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