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On the Chern Numbers of Surfaces and 3-Folds of Codimension 2

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In this paper we are interested in the slopes, i.e. the ratios of Chern numbers of surfaces in $\mathbf{P}^4$ and 3-folds in $\mathbf{P}^5$. As these numbers are differential-topological invariants, this gives some clue as to the possible diffeomorphism types of surfaces in $\mathbf{P}^4$ and 3-folds in $\mathbf{P}^5$.

The geography of Chern numbers of surfaces of general type has been studied quite intensively [3], [4], [5], [7], [8], [10], [12], [14], [15]. (See also Persson's article [11].) The Chern numbers satisfy the estimate $(1/5)(c_2-36)\leq c_1^2\leq 3c_2$ and Sommese [14] has shown that any rational number in $[1/5, 3]$ occurs as the slope c_1^2/c_2 of a surface of general type.

For 3-folds of general type it is known that $c_1c_2<0$ and $3c_2-c_1^2$ is pseudo effective (Miyaoka's inequality).

In recent years there has been a lot of interest in the study of codimension 2 subvarieties in $\mathbf{P}^n$. Let X be of codimension 2. It is well known that X can be viewed as a dependency locus of $r-1$ sections of a rank r bundle. Hence its slope is a rational function of the Chern classes of the bundle which are symmetric functions of the Chern roots. (See formulas (1.1)$\sim$(1.6) and (1.8)$\sim$(1.13).) If the bundle is a direct sum of line bundles, then the surface is called *determinantal.* (In this case, the Chern roots are positive integers.)

Here we first study the slopes of determinantal varieties. For surfaces in $\mathbf{P}^4$ the first question to ask is what rational numbers between 1/5 and 3 have the rational expression (1.6) with $s_j=\sum a_i^j$, for some $(a_1, \cdots, a_s)\in\mathbf{N}^s$. We are able to prove the following

THEOREM 1. *The slopes of determinantal surfaces of general type are all rational numbers between* 1 *and* 7/5 *and an infinite sequence accumulating to* 1 *from below.*

Our approach is, for every rational number $a/b\in[1, 7/5]$, assigning each s_j, for $j\leq 3$, a positive integer m_j, then solve s_4 such that (1.6) gives $(a-b)/a$. (This is possible

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because of the special expression we have.) Hence we are looking for conditions such that the system $\{m_j=\sum_{i=1}^{s} n_i^j\}_{j=1}^{l}$ has a solution in $\mathbf{N}^s$ for some s. What we have found is the technical Theorem 2.9. Since this is a multiple version of the Waring's problem, we use the Hardy-Littlewood circle method. After the work had been done, Bombieri pointed out to me that the result in Theorem 2.9 is closely related to some work of Hua's contained in [19]. Since the reference [19] is relatively unaccessible and not generally known, we include the proof in this paper. We give some outline of the idea of the method here for the algebraic geometers. (See Vinogradov's and particularly Vaughan's books [16], [17], for more details.) Hopefully it will arouse enough interest of studying the circle method.

By Cauchy's integral formula ($\int_0^1 e^{2\pi i\alpha h}d\alpha=1$, if $h=0$. Otherwise 0), the system $\{m_j=\sum_{i=1}^{s} n_i^j\}_{j=1}^{l}$ having a solution is equivalent to the nonvanishing of the integral

$$\int\sum=\int\left[\sum_{n=0}^{N} e^{2\pi i(n\alpha_1+\cdots+n^l\alpha_l)}\right]^s e^{-2\pi i(m_1\alpha_1+\cdots+m_l\alpha_l)}d\alpha_1\cdots d\alpha_l$$

over $[0, 1]^l$ for some large integers s and N.

We define the major "box" $M_q(a_1, \cdots, a_l)$ (for the precise definition, see §2) centered at $(a_1/q, \cdots, a_l/q)\in[0, 1]^l$ with q "not too large". The integral $\int\sum$ over the complement of the union of the boxes is negligible. On a major box, $\int\sum$ is approximated by the product of an "algebraic sum" and a "singular integral," the latter being independent of $(a_1/q, \cdots, a_l/q)$. When summing over major boxes with $q=p^k$, $k=0, 1, 2, \cdots$, for p larger enough, the algebraic sums don't have any "ill effect." (Consequence 2 in Proposition 2.7.) If p is small, then we use the fact that the algebraic sum is the number of solutions of the system $\sum_{i=1}^{s} n_i^j\equiv 0$ in $\mathbf{Z}_q^s$ (Lemma 2.8). The implicit function theorem, (2.2) and the second part of (2.3) in Theorem 2.9 imply that the singular integral (which is of a positive function) is over a set of positive measure.

Let X be a 3-fold in $\mathbf{P}^5$. We define the slopes to be $\gamma_1(X)=c_1^3/(c_1c_2)$ and $\gamma_2(X)=c_1^3/c_3$. Then $0<\gamma_1<3$. What we have is

THEOREM 2. *Let the set of slopes be $S_i=\{\gamma_i(X)\mid X$ is a determinantal 3-fold in $\mathbf{P}^5$ of general type$\}$. Then*

(i) *S_1 consists of all rational numbers between* 1 *and* 17/12 *and an infinite sequence accumulating to* 1 *from below.*

(ii) *S_2 consists of all rational numbers between* 1 *and* 17/7 *and an infinite sequence of positive rational numbers accumulating to* 1 *from below.*

The next step is to either prove better bounds for the slopes of codimension 2 varieties or find more nondeterminantal varieties.

One may construct bundles whose Chern classes are symmetric functions of positive integers and a dependency locus of the bundle is not determinantal. However we are unable to get beyond these ranges. What really help are the pull-back's by degree e maps. If the limit, as e goes to infinity, of the slopes of these varieties is l, then pushing our argument in the application of number theory a bit further, we are able to obtain all rational numbers between l and the bounds gotten from the determinantal ones, e.g., by adding the dependency locus of $f_e^*F \bigoplus_i \mathcal{O}(a_i)$, where f_e^*F is the pull back of the Horrocks-Mumford bundle by a degree e map, we have

THEOREM 3. *All rational numbers between* 1 *and* 5/3 *are slopes of surfaces in* $\mathbf{P}^4$.

QUESTION. Is there any surface of general type in $\mathbf{P}^4$ with slope greater than 5/3? Is there any 3-fold in $\mathbf{P}^5$ with slopes $\gamma_1 > 17/12$ or $\gamma_2 > 17/7$?

The paper is organized as follows: In section 1, we derive the slope formula in the desired form. Section 2 is the circle method in which we prove the technical theorem. For the lemmas marked with an A, we, in Appendix to § 2, either remark how to derive our version or give a proof. (This section is put after Section 5 as an appendix.) Section 3 contains some applications of Theorem 2.9. Section 4 has further applications. The proof of Theorem 3 is in Section 5.

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1. The set up.

We do the case of surfaces in $\mathbf{P}^4$ first. Then we give the analogous formulas for 3-folds in $\mathbf{P}^5$. The basic algebraic geometry fact we use is the following:

Let E be a rank s vector bundle on $\mathbf{P}^4$ with Chern classes $c_1, \cdots, c_4$. If E is generated by global sections, then the dependency locus of $s-1$ general sections is a smooth surface Y with

$$0 \longrightarrow \mathcal{O}^{s-1} \longrightarrow E \longrightarrow \mathcal{I}_Y(c_1) \longrightarrow 0$$

and the numerical relations

$$\text{(1.1)} \qquad \begin{aligned} &H^2 = d = c_2 , \qquad HK = c_3 + (c_1 - 5)c_2 , \\ &K^2 = c_4 + 2(c_1 - 5)c_3 + (c_1 - 5)^2 c_2 . \end{aligned}$$

Here H is the hyperplane class, K is the canonical class, and d is the degree of Y.

Let $a_1, \cdots, a_s$ be the Chern roots of E (see [2] Appendix A) and $s_j = \sum_{i=1}^{s} a_i^j$ for $j = 1, \cdots, 4$ be the sum of the jth power of a_i (i.e. s_j is $j!$ times the coefficient of t^j in the Chern character $\sum e^{a_i t}$ of E). Then we have

$$\text{(1.2)} \qquad \begin{aligned} &c_1 = s_1 , \quad c_2 = \frac{1}{2}(s_1^2 - s_2) , \quad c_3 = \frac{1}{6}(s_1^3 - 3s_1 s_2 + 2s_3) , \\ &c_4 = \frac{1}{24}(s_1^4 - 6s_4 + 8s_1 s_3 + 3s_2^2 - 6s_1^2 s_2) . \end{aligned}$$

By definition, the slope of a surface Y is

(1.3) $$s(Y):=\frac{K^2}{c_2(Y)}.$$

If the surface Y lies in $\mathbf{P}^4$, then we have the double point formula (see [2, p. 434]).

(1.4) $$c_2(Y)=K^2-(d^2-10d-5HK).$$

It is more convenient to work on the following formula

(1.5) $$t:=1-(s(Y))^{-1}=\frac{d^2-10d-5HK}{K^2}.$$

Combining formulas (1.1), (1.2) and (1.5), we see that

(1.6) $$t=\frac{(s_1^2-s_2)^2+[30s_1^2-\frac{40}{3}s_1^3-30s_2+20s_1s_2-\frac{20}{3}s_3]}{\frac{7}{2}(s_1^2-s_2)^2+4s_1s_3-3s_2^2-s_4+[50s_1^2-\frac{80}{3}s_1^3-50s_2+40s_1s_2-\frac{40}{3}s_3]}.$$

REMARK 1.1. If Y is of general type, then $1/5\le s(Y)$. A result by Ellisgrud and Peskine says that for any $b<1$, there are only finitely many surfaces of slope $s(Y)<b$. So we concentrate to the case $1\le s(Y)\le 3$ which is equivalent to $0\le 1-(s(Y))^{-1}\le 2/3$. We want to know *if all rational numbers between* 0 *and* 2/3 *can be expressed as* (1.6).

REMARK 1.2. Let $Y\subset\mathbf{P}^4$ be a smooth surface of general type, then $\omega_Y(1)$ is generated by three sections. A theorem of Vogolaar [9] implies that Y is the dependency locus of a rank 4 bundle E with $c_1(E)=4$.

For a 3-fold X in $\mathbf{P}^5$, we have the following formula analogous to (1.1)–(1.6).

(i) The relations between the intersection numbers of X and the Chern classes c_i of the bundle:

(1.8) $$\begin{aligned}&H^3=d=c_2,\quad H^2K=c_3+(c_1-6)c_2,\\&HK^2=c_4+2(c_1-6)c_3+(c_1-6)^2c_2,\\&K^3=c_5+3(c_1-6)c_4+3(c_1-6)^2c_3+(c_1-6)^3c_3.\end{aligned}$$

(ii) The relation between the two sets of symmetric functions:

(1.9) $$c_5=\frac{1}{120}(s_1^5-10s_1^3s_2+15s_1s_2^2+20s_1^2s_3-20s_2s_3-30s_1s_4+24s_5)$$

and (1.2).

(iii) The slopes:

(1.10) $$\gamma_1(X)=\frac{c_1^3(X)}{c_1(X)c_2(X)}=\frac{K^3}{Kc_2(X)},\qquad \gamma_2(X)=\frac{c_1^3(X)}{c_3(X)}=\frac{-K^3}{c_3(X)}.$$

(iv) $c_i(X)$ in terms of the intersection numbers in (1.8):

$$(1.11)\qquad \begin{aligned} c_2(X)&=(15-d)H^2+6HK+K^2\,,\\ c_3(X)&=(6d-70)d+(2d-51)H^2K-12HK^2-K^3\,. \end{aligned}$$

(v) The formulas we work on:

$$(1.12)\qquad \begin{aligned} t_1&:=1-(\gamma_1(X))^{-1}=\frac{(d-15)H^2K-6HK^2}{K^3}\,,\\ t_2&:=1-(\gamma_2(X))^{-1}=\frac{(6d-70)d+(2d-51)H^2K-12HK^2}{K^3}\,. \end{aligned}$$

In terms of the s_i's, these are

$$(1.13)\qquad \begin{aligned} t_1&=\frac{[4s_1^5-10s_1^3s_2+2s_1^3s_3-2s_2s_3+6s_1s_2^2]+*/12}{(68s_1^5-170s_1^3s_2+130s_1^2s_3-10s_2s_3+30s_1s_2^2-60s_1s_4+12s_5)+*/60}\,,\\ t_2&=\frac{[2s_1^5-5s_1^3s_2+s_1^2s_3-s_2s_3+3s_1s_2^2]+*/3}{(68s_1^5-170s_1^3s_2+130s_1^2s_3-10s_2s_3+30s_1s_2^2-60s_1s_4+12s_5)+*/60}\,, \end{aligned}$$

where *'s are polynomials of degree (in a_i) less than 5.

DEFINITION. Let $X\subset \mathbf{P}^n$ be of codimension 2 and a dependency locus of E. If $E=\bigoplus \mathcal{O}(a_i)$, then X is *determinantal.*

2. Some number theory.

In this section we study the following problem.

Given integers $m_1\cdots, m_l$, are there positive integers $n_1,\cdots, n_s$ such that $m_j=\sum_{i=1}^s n_i^j$ for $j=1,\cdots, l$?

The existence of the n_i's is equivalent to the nonvanishing of the integral

$$\int\sum=\int\left[\sum_{n=0}^{N} e^{2\pi i(n\alpha_1+\cdots+n^l\alpha_l)}\right]^s e^{-2\pi i(m_1\alpha_1+\cdots+m_l\alpha_l)}d\alpha_1\cdots d\alpha_l$$

over $[0,1]^l$ for some large integers s and N.

In Theorem 2.9 we will give some conditions on the m_j's for this to be true.

Let δ denote (various) small constants (may depend on l) and N be a large integer specified later. For any $q, a_1,\cdots, a_l\in\mathbf{N}$ such that $a_j\le q<N^\delta$, we define the *major box*

$$M_q(a_1,\cdots,a_l)=\{(\alpha_1,\cdots,\alpha_l)\in[0,1]^l \mid |\alpha_j-a_j/q|<N^{-j+\delta},\ \text{for } j=1,\cdots,l\}\,.$$

REMARK. The major boxes are mutually disjoint.

Let $M=\bigcup M_q(a_1,\cdots,a_l)$. We will show [Proposition 2.2] that the integral $\int\sum$ over the complement of M is negligible.

Let $f(x)=\alpha_l x^l+\cdot+\alpha_1 x$. First we give a bound of the exponential sum $\sum_{n=0}^N e^{2\pi i f(n)}$.

CONVENTION. Let f, g real valued functions. $f=O(g)$ means $|f|<cg$ for some constants $c>0$.

In our estimate we will freely use the following

FACT 1. For $\alpha, \beta \in \mathbf{R}$, $|e^{i\alpha}-e^{i\beta}|=O(|\alpha-\beta|)$.

LEMMA 2.1.A. *If* $\bar{\alpha}=(\alpha_1, \cdots, \alpha_l)\notin M$, *then* $\sum_{n=0}^{N} e^{2\pi i f(n)}=O(N^{1-\delta})$.

This immediately implies

PROPOSITION 2.2. $\int_{[0,1]^l\setminus M}\sum=O(N^{s(1-\delta)})$.

Next we describe the exponential sum $A=\sum_{n=0}^{N} e^{2\pi i f(n)}$ on the major box $M_q(a_1, \cdots, a_l)$.

Let $\bar{\alpha}=(\alpha_1, \cdots, \alpha_l)\in M_q(a_1, \cdots, a_l)$, where $\alpha_j=a_j/q+\beta_j$ with $|\beta_j|<N^{-j+\delta}$. Write $n=qd+r$ with $0\le d<N/q$ and $0\le r\le q-1$. Then

$$\alpha_j n^j=\left(\frac{a_j}{q}+\beta_j\right)(qd+r)^j=\mathbf{Z}+\frac{a_j}{q}r^j+\beta_j q^j d^j+O(N^{\delta-1}q)\,.$$

So we have the factorization $A=S(q, a_1, \cdots, a_l)v(\bar{\beta})+O(N^{2\delta})$ where

$$S(q, a_1, \cdots, a_l)=\sum_{r=0}^{q-1} e^{2\pi i(a_1 r+\cdots+a_l r^l)/q} \quad \text{and}$$

$$v(\bar{\beta})=\sum_{d=0}^{N/q} e^{2\pi i(\beta_1 q d+\cdots+\beta_l q^l d^l)}\,.$$

Since

$$\max_{d,d'\in[k,k+1]} |e^{2\pi i(\beta_1 q d+\cdots+\beta_l q^l d^l)}-e^{2\pi i(\beta_1 q d'+\cdots+\beta_l q^l d'^l)}|$$
$$=O(\sum|\beta_j q^j(d^j-d'^j)|)=O(N^{-j+\delta}q^j(N/q)^{j-1})=O(N^{\delta})\,,$$

we have $v(\bar{\beta})=\int_0^{N/q} e^{2\pi i(\beta_1 q y+\cdots+\beta_l q^l y^l)}dy+O(N^{\delta})$. Changing variable gives $v(\bar{\beta})=(1/q)\int_0^N e^{2\pi i(\beta_1 y+\cdots+\beta_l y^l)}dy+O(N^{\delta})$. Hence

$$A=\frac{1}{q}S(q, a_1, \cdots, a_l)\int_0^N e^{2\pi i(\beta_1 y+\cdots+\beta_l y^l)}dy+O(N^{\delta})\,. \tag{2.1}$$

The algebraic sums are multiplicative:

LEMMA 2.3.A [16, p. 46]. $S(q', a_1', \cdots, a_l')S(q'', a_1'', \cdots, a_l'')=S(q'q'', a_1'q''+a_1''q', \cdots, a_l'q''+a_l''q')$, *if* $(q', q'')=1$.

We need the following two lemmas to approximate the integral $\int\sum$ over a major box.

LEMMA 2.4.A. $|S(q, a_1, \cdots, a_l)|<c(l)q^{1-1/l}$.

LEMMA 2.5.A. *Let* $V=\int_0^N e^{2\pi i(\beta_1 y+\cdots+\beta_l y^l)}dy$. *Then*

$$|V|<c'(l)N\left[1+\sum_{j\le l}N^j|\beta_j|\right]^{-1/l}.$$

LEMMA 2.6. *For s sufficiently large,* $\int_{M_q(a_1,\cdots,a_l)}\sum=BC+O(N^{s-1+\delta-(1+2+\cdots+l)})$, *where*

$$B=\left[\frac{1}{q}S(q,a_1,\cdots,a_l)\right]^s e^{-2\pi i(m_1a_1+\cdots+m_la_l)/q},$$

$$C=\int_{|\beta_j|<N^{-j+\delta}}\left[\int_0^N e^{2\pi i(\beta_1 y+\cdots+\beta_l y^l)}dy\right]^s e^{-2\pi i(m_1\beta_1+\cdots+m_l\beta_l)}d\bar\beta.$$

PROOF. In (2.1), let $S=S(q,a_1,\cdots,a_l)$ and $V=\int_0^N e^{2\pi i(\beta_1 y+\cdots+\beta_l y^l)}dy$. Then

$$\int_{M_q(a_1,\cdots,a_l)}\sum=\int_{M_q(a_1,\cdots,a_l)}A^s e^{-2\pi i(m_1\alpha_1+\cdots)}d\bar\alpha$$

$$=\int_{|\beta_j|<N^{-j+\delta}}\left[\frac{1}{q}SV+O(N^\delta)\right]^s e^{-2\pi i(m_1\beta_1+\cdots)}e^{-2\pi i(m_1a_1+\cdots)/q}d\bar\beta$$

$$=\left[\frac{S}{q}\right]^s e^{-2\pi i(m_1a_1+\cdots)/q}\int_{|\beta_j|<N^{-j+\delta}}V^s e^{-2\pi i(m_1\beta_1+\cdots)}d\bar\beta$$

$$+\max_{e+f=s,f\ge1}O\left(c(l)^eN^{f\delta}\int|V|^e d\bar\beta\right).$$

Clearly the maximum occurs when $e=s-1$. Lemma 2.5.A implies that the error is

$$O\left(N^{\delta+s-1}\int_{|\beta_j|<N^{-j+\delta}}\left[1+\sum_{j\le l}N^j|\beta_j|\right]^{-(s-1)/l}d\bar\beta\right).$$

Integrating gives the expected error (assuming $(s-1)/l>l$). □

PROPOSITION 2.7. *For s sufficiently large, there is an integer* $Q=Q(l)$ *such that if* $Q\mid m_j$ *for all j, then*

$$\int_{[0,1]^l}\sum=cC+O(N^{s-1+(l+2)\delta-(1+2\cdots+l)}),$$

where C is as in Lemma 2.6 *and* $c=c(m_1,\cdots,m_l,s)>0$.

PROOF. Since s is sufficiently large, the integral $\int\sum$ over $[0,1]^l\backslash M$ can be ignored. We sum up the right hand side of Lemma 2.6 over all major boxes and denote B by $B(q,a_1,\cdots,a_l)$ for the major box $M_q(a_1,\cdots,a_l)$. (Note that C is independent of the major box.) Therefore,

$$\int_{[0,1]^l}\sum = C\sum{}' B(q, a_1, \cdots, a_l) + O(N^{s-1+\delta(l+2)-(1+2+\cdots+l)}),$$

where the summation $\sum'$ is over all $0 \le a_j < q < N^\delta$ with $(q, a_1, \cdots, a_l) = 1$. From Lemma 2.4.A we have the following useful

CLAIM. *Let q be fixed. Then $\sum_{(q,\bar{a})=1} |B(q, a_1, \cdots, a_l)| < c^s q^{-s/(2l)}$ if $s/(2l) > l$.*

PROOF OF THE CLAIM. The left hand side is bounded by $(cq^{-1/l})^s q^l$..

The easy consequences are

1. $\sum_{q>Q}\sum_{(q,\bar{a})=1} |B(q, a_1, \cdots, a_l)| < c^s Q^{1-s/(2l)}$, i.e., the expression (given by Lemma 2.3.A)

$$\sum_q \sum_{(q,\bar{a})=1} B(q, a_1, \cdots, a_l) = \prod_p \sum_{k=0}^{\infty} \sum_{p \nmid a_j} B(p^k, a_1, \cdots, a_l)$$

makes sense.

2. Let p be fixed, then

$$\sum_{k=1}^{\infty} \sum_{p \nmid a_j} |B(p^k, a_1, \cdots, a_l)| < \frac{c^s}{p^{s/(2l)-1}}$$

which is less than 1 for all $p > P$. Moreover for some $P' \ge P$, $\varepsilon := \sum_{p>P'} c^s/(p^{s/(2l)-1}) \to 0$. Therefore

$$\prod_{p>P'} \sum_{k=0}^{\infty} \sum_{p \nmid a_j} B(p^k, a_1, \cdots, a_l) > \prod_{p>P'} \left(1 - \frac{c^s}{p^{s/(2l)-1}}\right) \sim e^{\varepsilon} > 0.$$

The following lemma applied to

$$\sum_{k=0}^{K} \sum_{p \nmid a_j} B(p^k, a_1, \cdots, a_l) = \sum_{1 \le a_j \le p^K} B(p^K, a_1, \cdots, a_l) \qquad \text{for} \quad K \gg 0$$

gives $\sum_{k=0}^{\infty} \sum_{p \nmid a_j} B(p^k, a_1, \cdots, a_l) > 0$. This takes care of the factors with $p < P'$. So

$$c = \sum_q \sum_{(q,\bar{a})=1} B(q, a_1, \cdots, a_l) = \prod_p \sum_{k=0}^{\infty} \sum_{p \nmid a_j} B(p^k, a_1, \cdots, a_l) > 0. \qquad \square$$

LEMMA 2.8. *If $q \mid m_j$ for all j, then $\sum_{1 \le a_j \le q} B(q, a_1, \cdots, a_l) > q^{(l-s_1)}$ for some s_1.*

PROOF. Writing

$$\sum_{1 \le a_j \le q} B(q, a_1, \cdots, a_l) = q^{(l-s)} \left(\frac{1}{q} \sum_{a_1=1}^{q} e^{2\pi i(r_1 + \cdots + r_s - m_1)a_1/q}\right) \cdots$$

$$\cdots \left(\frac{1}{q} \sum_{a_l=1}^{q} e^{2\pi i(r_1^l + \cdots + r_s^l - m_l)a_l/q}\right)$$

and using the fact that

$$\frac{1}{q}\sum_{r=1}^{q} e^{2\pi i h r/q} = \begin{cases} 1 & \text{if } q \mid h \\ 0 & \text{otherwise}, \end{cases}$$

we see that the left hand side of the inequality is $q^{l-s}n$, where n is the number of the solutions to the system $\{\sum_{i=1}^{s} r_i^j \equiv m_j \pmod q\}_{j=1}^{l}$.

CLAIM. *$n \geq q^{s-s_1}$ for some $s_1 < s$, if s is large.*

PROOF OF THE CLAIM. Let $\mathscr{U} = \{\bar{r} = (r, r^2, \cdots, r^l) \in \mathbf{Z}_q^l\}$ and denote $\mathscr{U} + \cdots + \mathscr{U}$ (k times) by $k\mathscr{U}$. Then there is s_1 such that $s_1\mathscr{U} = s\mathscr{U}$, $\forall s > s_1$. i.e. $s_1\mathscr{U}$ is a closed subset of the finite group $(Z_q^l, +)$. So $s_1\mathscr{U}$ is a subgroup. Now for any $r_{s_1+1}, \cdots, r_s$, there is an inverse of $\bar{r}_{s_1+1} + \cdots + \bar{r}_s$ in $s_1\mathscr{U}$; i.e. there are $r_1, \cdots, r_{s_1} \in \mathbf{Z}_q$ such that

$$r_1^j + \cdots + r_{s_1}^j \equiv -(r_{s_1+1}^j + \cdots + r_s^j) \pmod q . \qquad \square$$

THEOREM 2.9. *Given $l \in \mathbf{N}$, and for $s = s(l)$ large, if $(z_1, \cdots, z_s) \in \mathbf{R}_+^s$ satisfies*

$$\#\{z_j \neq 0\} \geq l, \tag{2.2}$$

then for sufficiently large $K = K(z_1, \cdots, z_s)$ and integers $m_1, \cdots, m_l$ satisfying

$$m_j \equiv 0 \pmod Q \quad \text{and} \quad |K^j(z_1^j + \cdots + z_s^j) - m_j| < K^{j-\varepsilon} \tag{2.3}$$

where $1 \leq j \leq l$ and $Q = Q(l)$, the system $\{m_j = n_1^j + \cdots + n_s^j\}_{j=1}^{l}$ has a solution in $\mathbf{N}^s$.

PROOF. We will show that under the assumptions (2.2) and (2.3), the integral $\int\sum$ in Proposition 2.7 is positive, i.e.

$$C = \int_{\mathbf{R}^l}\left[\int_0^N e^{2\pi i(\beta_1 y + \cdots + \beta_l y^l)}dy\right]^s e^{-2\pi i(m_1\beta_1 + \cdots + m_l\beta_l)}\varphi(\bar{\beta})d\bar{\beta} > O(N^{s-1+(l+2)\delta-(1+2+\cdots+l)}),$$

where $\varphi(\bar{\beta})$ is the bump function centered at $(0, \cdots, 0)$ with $\varphi(\bar{\beta}) = 1$ if $|\beta_j| < N^{-j+\delta}$.

Let $\hat{\varphi}_\beta(\lambda) = \int e^{2\pi i\lambda\beta}\varphi(\beta)d\beta$, where $\varphi(\beta)$ is the bump function centered at 0 with $\varphi(\beta) = 1$ for $|\beta| < b$. Then $\hat{\varphi}_\beta(\lambda) = \sin(2\pi b\lambda)/(\pi\lambda) > 0$ if $|\lambda| \leq 1$ and $b > 0$ small. So

$$C = \int_0^N \cdots \int_0^N \hat{\varphi}_{\beta_1}(y_1 + \cdots + y_s - m_1) \cdots \hat{\varphi}_{\beta_l}(y_1^l + \cdots + y_s^l - m_l)dy_1 \cdots dy_s$$

and $C \geq$ measure of $T := \{(y_1, \cdots, y_s) \mid 0 \leq y_i \leq N \text{ and } |\sum_{i=1}^{s} y_i^j - m_j| \leq 1\}$.

CLAIM. *The measure of T is $\geq O(N^{s-(1+2+\cdots+l)})$.*

PROOF OF THE CLAIM. We consider the map $(w_1, \cdots, w_s) \xrightarrow{\varphi} (\sum w_i, \cdots, \sum w_i^l)$. (2.2) implies that $[\partial\varphi/\partial w_j]_j$ has maximal rank at $(z_1, \cdots, z_s)$ and the implicit function theorem implies that for K large, $(m_1/K, \cdots, m_l/K^l) \in \operatorname{Im}\varphi$, i.e. there are $w_1, \cdots, w_s \in \mathbf{R}_+$ such that

$$w_1^j + \cdots + w_s^j = \frac{m_j}{K^j}, \qquad 1 \le j \le l. \tag{2.3}$$

Now, $T = K\{(u_1, \cdots, u_s) \mid 0 \le u_i \le N/K \text{ and } |\varphi_j(\bar{u}) - \varphi_j(\bar{w})| \le K^{-j}\}$ and observe that measure$\{(v_1, \cdots, v_l) \mid |v_j| \le K^{-j}\} = K^{-(1+2+\cdots+l)}$. □

3. Applications of Theorem 2.9.

In this section we will apply Theorem 2.9 to see what rational numbers can be certain rational expressions of some Newton functions of *integers*.

Let $\bar{z} = (z_1, \cdots, z_s) \in \mathbf{R}^s$ and the jth Newton functions $s_j = \sum_{i=1}^s z_i^j$. Suppose $\Phi = \Phi_0 + \Phi_1$ and $\Psi = \Psi_0 + \Psi_1$ are polynomials in $s_1, \cdots, s_l$, where Φ_0 (resp. Ψ_0) is the form of maximal degree D (in z_i).

PROPOSITION 3.1. *For s sufficiently large, the image $(\Phi/\Psi)(\mathbf{N}^s)$ is a dense subset in $(\Phi_0/\Psi_0)(\mathbf{R}_+^s)$.*

PROOF. Given $(\Phi_0/\Psi_0)(\bar{z}) \in (\Phi_0/\Psi_0)(\mathbf{R}_+^s)$, we may assume (2.2) holds for $\bar{z}$. Now choose $K \gg 0$ as in Theorem 2.9 and let m_j be a multiple of Q so that

$$|K^j(z_1^j + \cdots + z_s^j) - m_j| < Q \tag{3.1}$$

and in particular (2.3) holds. Therefore $m_j = s_j(\bar{n})$ for some $\bar{n} \in \mathbf{N}^s$, and (3.1) implies $|K^j s_j(\bar{z}) - s_j(\bar{n})| < Q$. Thus

$$\begin{aligned}\Phi(s_1(\bar{n}), \cdots, s_l(\bar{n})) &= K^D \Phi_0(s_1(\bar{z}), \cdots, s_l(\bar{z})) + O(QK^{D-1}) \\ &= K^D(\Phi_0(s(\bar{z})) + O(QK^{D-1}))\,.\end{aligned}$$

The same holds for $\Psi(s(\bar{n}))$ and thus

$$\frac{\Phi(s(\bar{n}))}{\Psi(s(\bar{n}))} = \frac{\Phi_0(s(\bar{z})) + O(QK^{-1})}{\Psi_0(s(\bar{z})) + O(QK^{-1})} = \frac{\Phi_0(s(\bar{z}))}{\Psi_0(s(\bar{z}))} + \varepsilon_K\,,$$

where $\varepsilon_K \to 0$ as $K \to \infty$. □

PROPOSITION 3.2. *Let $\Phi = \Phi_0 + \Phi_1$ and $\Psi = (s_l + \Psi_0) + \Psi_1$ be polynomials over $\mathbf{Q}$ such that $\Phi, \Psi_0, \Psi_1 \in \mathbf{Q}[s_1, \cdots, s_{l-1}]$, and $\Phi_0, s_l + \Psi_0$ are forms of maximal degree l in z_i. Then*

$$\frac{\Phi}{\Psi}(\mathbf{N}^s) \supset \frac{\Phi_0}{s_l + \Psi_0}(\tilde{\mathbf{R}}_+^s) \cap \mathbf{Q}\,,$$

where $\tilde{\mathbf{R}}_+^s$ consists of elements in $\mathbf{R}_+^s$ satisfying (2.2), *i.e.* $\#\{z_j \ne 0\} \ge l$.

PROOF. Let $a/b = (\Phi_0/(s_l + \Psi_0))(\bar{z})$, where $\bar{z}$ satisfies (2.2). Let $Q_1 = Q_1(\Phi, \Psi, a/b) \in \mathbf{N}$ to be specified later, and K be a large integer (according to Theorem 2.9). Choose multiples $m_1, \cdots, m_{l-1}$ of Q_1 such that

(3.2) $$|K^j s_j(\bar{z}) - m_j| < Q_1\,, \qquad j=1, \cdots, l-1$$

and define m_l by the equation

(3.3) $$\frac{\Phi(m_1, \cdots, m_{l-1})}{m_l + (\Psi_0 + \Psi_1)(m_1, \cdots, m_{l-1})} = \frac{a}{b}\,.$$

In order to use the Theorem 2.9, we need to verify condition (2.3) for m_l.

CLAIM 1. $Q_1 \mid m_l$. This is clear since $m_l = (b/a)\Phi(m_1, \cdots, m_{l-1}) - (\Psi_0 + \Psi_1)(m_1, \cdots, m_{l-1})$, $Q_1 \mid m_j$ for $j=1, \cdots, l-1$, and we can choose the appropriate Q_1.

CLAIM 2. $|K^l s_l(\bar{z}) - m_l| < O(Q_1 K^{l-1})$. This follows from the construction

$$\begin{aligned} m_l - K^l s_l(\bar{z}) &= \left(\frac{b}{a}\Phi - (\Psi_0 + \Psi_1)\right)(m_1, \cdots, m_{l-1}) - K^l s_l(\bar{z}) \\ &= \left(\frac{b}{a}\Phi_0 - \Psi_0\right)(K s_1(\bar{z}), \cdots, K^{l-1} s_{l-1}(\bar{z})) - K^l s_l(\bar{z}) + O(Q_1 K^{l-1}) \\ &= O(Q_1 K^{l-1})\,. \end{aligned}$$

(The first equality is by (3.3), the second is by (3.2), and the last is by the definition of b/a).

Theorem 2.9 implies that there is $\bar{n} \in \mathbf{N}^s$ such that $m_j = s_j(\bar{n})$, $j=1, \cdots, l$. It follows from (3.3) that $(\Phi/\Psi)(s(\bar{n})) = (a/b)$. □

PROPOSITION 3.3. *Let Φ and Ψ be as in Proposition 3.2. For $j=1, \cdots, l$, let $\tilde{s}_j \in \mathbf{Z}$ be given and let $s(\bar{z}, e) = (e\tilde{s}_1 + s_1', \cdots, e^l \tilde{s}_l + s_l')$, where $e \in \mathbf{N}$ and $s_j' = \sum z_i^j$. If $\Phi_0(s(\bar{z}, e))/(s_l + \Psi_0(s(\bar{z}, e))) = a/b$ for some $\bar{z} \in \mathbf{R}_+^s$, and $e \in \mathbf{N}$, then $\Phi(s(\bar{n}, e))/\Psi(s(\bar{n}, e)) = a/b$ for some $\bar{n} \in \mathbf{N}^s$.*

PROOF. We choose $Q_1, K, m_1, \cdots, m_{l-1}$ as in Proposition 3.2, and define

$$m_l = \frac{b}{a}\Phi(s_0) - (\psi_0 + \psi_1)(s_0) - K^l e^l \tilde{s}_l\,,$$

where $s_0 = (Ke\tilde{s}_1 + m_1, \cdots, K^{l-1} e^{l-1} \tilde{s}_{l-1} + m_{l-1})$.

The same argument as in Proposition 3.2 works here.

$$\begin{aligned} m_l - K^l s_l'(\bar{z}) &= \left(\frac{b}{a}\Phi_0 - \psi_0\right)(Ke\tilde{s}_1 + m_1, \cdots, K^{l-1} e^{l-1} \tilde{s}_{l-1} + m_{l-1}) \\ &\quad - K^l s_l'(\bar{z}) + O(Q_1 K^{l-1}) \\ &= \left(\frac{b}{a}\Phi_0 - \psi_0\right)(Ke\tilde{s}_1 + K s_1'(\bar{z}), \cdots, K^{l-1} e^{l-1} \tilde{s}_{l-1} + K^{l-1} s_{l-1}'(\bar{z})) \\ &\quad - K^l s_l'(\bar{z}) + O(Q_1 K^{l-1}) \end{aligned}$$

$$=K^l\left[\left(\frac{b}{a}\Phi_0-\psi_0\right)(s(\bar{z},e))\right]+O(Q_1K^{l-1})=O(Q_1K^{l-1})\,. \qquad \square$$

4. The slopes of determinantal varieties.

In this section we will prove the theorems. First we prove that the slopes of a determinantal variety have the upper bounds, then we show the existence of sequences accumulating to the lower bounds. Finally we use Proposition 3.2 to show all rational numbers in between occur.

(i) Surfaces. We will work on $t=1-s^{-1}$ and use formula (1.6).

LEMMA 4.1. *If* $4s_1s_3-3s_2^2-s_4\geq 0$, *then* $t<2/7$.

PROOF. Let $A=(s_1^2-s_2)^2$, $B=30s_1^2-(40/3)s_1^3-30s_2+20s_1s_2-(20/3)s_3$, $C=4s_1s_3-3s_2^2-s_4$ and $D=10(s_2-s_1^2)$. Then

$$t=\frac{A+B}{(7/2)A+C+2B+D}<\frac{2}{7}$$

is equivalent to $(3/2)B-D<C$. Using the inverse of formulas (1.1) and (1.2), we have the left hand side

$$(3/2)B-D=-40d-30HK<0\,,$$

while the right hand side is nonnegative by assumption. $\square$

LEMMA 4.2. *Let* $s_j=\sum_{i=1}^{s}a_i^j$. *If* $a_i\geq 0$, *then* $4s_1s_3-3s_2^2-s_4\geq 0$.

PROOF. Hölder's inequality gives

$$s_2=\sum a_i^2=\sum a_i^{1/2}a_i^{3/2}\leq\left(\sum a_i\right)^{1/2}\left(\sum a_i^3\right)^{1/2}=(s_1s_3)^{1/2}\,.$$

Similarly $s_4\leq s_1s_3$. $\square$

Next, we work on the sequence accumulating to the lower bound.

LEMMA 4.3. *Let* $Y\subset\mathbf{P}^4$ *be a determinantal surface which comes from a bundle* $E=\bigoplus\mathcal{O}(a_i)$ *with Chern classes* $c_1,\cdots,c_4$. *If* $c_2\geq 7c_1$, *then* $d^2-10d-5HK>0$.

PROOF. By (1.1), we have

$$d^2-10d-5HK=c_2(c_2-5c_1+15)-5c_3\,.$$

This is at least $2c_1c_2-5c_3=c_3+\sum_{i<j}a_ia_j(a_i+a_j)$. $\square$

LEMMA 4.4. *Let* $E=\bigoplus\mathcal{O}(a_i)$ *with* $a_1\geq a_2\geq\cdots\geq a_s$. *If* $a_1>k$ *for some* k, *then* $c_2-kc_1\geq -k^2$ *unless* $a_2<k$.

PROOF. $c_2-kc_1\geq a_1a_2-k(a_1+a_2)+a_1\sum_{i>2}a_i-k\sum_{i>2}a_i\geq(a_1-k)(a_2-k)-k^2$. □

PROPOSITION 4.5. *t has exactly one accumulating point coming from a sequence outside of* $[0, 2/7]$.

PROOF. It follows from Lemmas 4.3 and 4.4 that this sequence comes from bundles with $a_1\gg 0$ and the other a_i's bounded by 7. So we write t in (1.6) as quotient of polynomials in a_1, and we have $t=O(a_1^2)/O(a_1^3)$. This is 0 when $a_1\to\infty$. □

PROOF OF THEOREM 1. To use Proposition 3.2, we put $\Phi_0=(s_1^2-s_2)^2$ and $\Psi_0=(7/2)(s_1^2-s_2)^2+4s_1s_3-3s_2^2$, and check the range of $T:=\Phi_0/(\Psi_0-s_4)$ for Newton functions evaluated on $\bar{z}=(z_1, \cdots, z_s)\in\mathbf{R}_+^s$ with at least four different nonzero coordinates for s sufficiently large.

Take $s-1$ numbers $1>\lambda_2>\cdots>\lambda_s>1-\varepsilon$, and let $\bar{z}=(1, t\lambda_2, \cdots, t\lambda_s)\in\widetilde{\mathbf{R}_+^s}$. Then $T=O(t)$. Hence $\lim_{t\to 0}T=0$. On the other hand, when all the z_i's are very near 1, then $s_j\sim s$ for all j. So $\lim_{s\to\infty}T=2/7$. So T evaluated on the line segment $L=\{(1, t\lambda_2, \cdots, t\lambda_s)\mid 0<t\leq 1\}$ gives $(0, 2/7)$. Note that $L\subset\widetilde{\mathbf{R}_+^s}$. □

(ii) 3-fold. Now we work on formulas (1.12)–(1.13):

$$t_1=\frac{[4s_2^5-10s_1^3s_2+2s_1^2s_3-2s_2s_3+6s_1s_2^2]+**/12}{(68s_1^5-170s_1^3s_2+130s_1^2s_3-10s_2s_3+30s_1s_2^2-60s_1s_1s_4+12s_5)*/60},$$

$$t_2=\frac{[2s_1^5-5s_1^3s_2+s_1^2s_3-s_2s_3+3s_1s_2^2]+***/3}{(68s_1^5-170s_1^3s_2+130s_1^2s_3-10s_2s_3+30s_1s_2^2-60s_1s_4+12s_5)+*/60},$$

where

$$*=270s_4-1080s_1s_3-135s_2^2+1890s_1^2s_2-945s_1^4-6480(s_1s_2+s_1^2-s_2)+4320s_1^3$$

$$**=18s_4-72s_1s_3-27s_2^2+162s_1^2s_2-81s_1^4+228s_3-684s_1s_2+456s_1^3-756(s_1^2-s_2)$$

$$***=9s_4-36s_1s_3-9s_2^2+72s_1^2s_2-36s_1^4+93s_3-279s_1s_2+186s_1^3-294(s_1^2-s_2).$$

PROPOSITION 4.6. *If* $c_1\geq 6$, *then* $t_1\leq 5/17$ *and* $t_2\leq 10/17$.

PROOF. The two inequalities $5/17-t_1>0$ and $10/17-t_2>0$ are equivalent to

$$F+24c_1c_3+12c_1^2c_2+102c_2^2+12c_4-429(c_3+c_1c_2)+1062c_2>0 \quad\text{and}$$

$$4F+48c_1c_3+24c_1^2c_2+102c_2^2+24c_4-501(c_3+c_1c_2)+1172c_2>0,$$

where $F=8s_1^2s_3-6s_1s_2^2-5s_1s_4+2s_2s_3+s_5$ is the degree 5 part.

We show that $F>0$ in Lemma 4.7. It is easy to see that when c_1 is large $(24c_1-429)c_3+(12c_1-429)c_1c_2>0$ and $(48c_1-501)c_3+(24c_1-501)c_1c_2>0$. The cases with c_1 small, one can check directly. □

LEMMA 4.7. $8s_1^2s_3-6s_1s_2^2-5s_1s_4+2s_2s_3+s_5>0$.

PROOF. First, we have $s_1^2s_3+s_5>2s_1s_4$, because

$$s_1^2 s_3 = \left(s_2 + 2\sum_{i<j} a_i a_j\right)\sum a_k^3 > s_5 + 2\sum_{i\neq j} a_i^4 a_j \quad \text{and} \quad s_1 s_4 = s_5 + \sum_{i\neq j} a_i^4 a_j .$$

It suffices to show that $6s_1(s_1s_3 - s_2^2) + s_1^2 s_3 - 3s_1s_4 + 2s_2s_3 > 0$. Assuming $a_1 \geq a_2 \cdots$, and dehomogenizing the expression by dividing a_1^5, we can replace s_i by $1+\tilde{s}_i$, where $\tilde{s}_i$ is the sum of ith powers of $\tilde{a}_i = a_i/a_1$. Note that $0 < \tilde{a}_i \leq 1$ and $0 < \tilde{s}_{i+1} \leq \tilde{s}_i$. We have

(i) $$s_1^2 s_3 - 3s_1s_4 + 2s_2s_3 > (1+\tilde{s}_1)^2(1+\tilde{s}_3) - 3(1+\tilde{s}_1)(1+\tilde{s}_3) + 2(1+\tilde{s}_2)(1+\tilde{s}_3)$$
$$= \tilde{s}_1^2 + \tilde{s}_1^2\tilde{s}_3 + 2\tilde{s}_2 + 2\tilde{s}_2\tilde{s}_3 - \tilde{s}_1 - \tilde{s}_3 - \tilde{s}_1\tilde{s}_3,$$

(ii) $$6s_1(s_1s_3 - s_2^2) > (1+\tilde{s}_1)(1+\tilde{s}_3) - (1+\tilde{s}_2)^2 = \tilde{s}_1 + \tilde{s}_3 + \tilde{s}_1\tilde{s}_3 - 2\tilde{s}_2 - \tilde{s}_2^2,$$

$$\text{(i)} + \text{(ii)} = \tilde{s}_1^2\tilde{s}_3 + 2\tilde{s}_2\tilde{s}_3 + (\tilde{s}_1^2 - \tilde{s}_2^2) > 0.$$ □

PROPOSITION 4.8. *t_1 (respectively t_2) has exactly one accumulating point coming from a sequence outside of* [0, 5/17] (*resp.* [0, 10/17]).

PROOF. By the same reasoning as in Proposition 4.5, we use $t_i = O(a_1^3)/O(a_1^4)$, $i = 1, 2$. In (1.12), for $t_1 > 0$ we need $(d-15)H^2K - 6HK^2 > 0$. This is $c_2(c_1-7)(c_2-6c_1) + (c_2^2 - 6c_4) + (15c_1c_2 - 126c_2) + c_3(c_2 - 12c_1 + 57)$. For $t_2 > 0$, we use

$$(6d-70)d + (2d-51)H^2K - 12HK^2$$
$$= \left(c_3 + \frac{c_1c_2}{2}\right)(2c_2 - 24c_1 + 93) + c_1c_2^2 - 6c_2^2 - 12c_4 .$$

When $c_2 - 12c_1 > 0$, these expressions are positive. □

LEMMA 4.9. *Let X be a determinantal 3-fold. Then $c_3(X) < 0$.*

PROOF. This is because $rk\,\mathrm{Pic}(X) \leq 2$ and $h^4(\mathcal{I}_Y) \gg 0$.

PROOF OF THEOREM 2. The line segment in the proof of Theorem 1 works here.

REMARK. A straight forward generalization of the determinantal surfaces is the dependency locus of $\bigoplus E(a_i)$, where E is a bundle generated by sections and $a_i \in \mathbf{N}$. However none of the constructions (sum, or tensor of known bundles) gives any new slope.

5. Some slopes of nondeterminantal varieties.

In this section we show how to use Proposition 3.3 to get the rational numbers between the slopes of the determinantal ones and the limit of pull-backs's by finite maps of a fixed variety.

PROPOSITION 5.1. *Let F be a bundle on $\mathbf{P}^4$ with Chern classes $c_1, \cdots, c_4$. If for all $a_i \in \mathbf{N}$, $f^*F \bigoplus_i \mathcal{O}(a_i)$ has a nonsingular surface as a dependency locus, where f is any degree* *e map. Then any rational number between* 1 *and* $(c_4+2c_1c_3+c_1^2c_2)/(c_4+2c_1c_3+c_1^2c_2-c_2^2)$ *is a slope.*

PROOF. If the Chern character of F is $ch(F)=1+\tilde{s}_1+\frac{1}{2}\tilde{s}_2+\frac{1}{6}\tilde{s}_3+\cdots$, then $ch(f^*F)=1+e\tilde{s}_1+\frac{1}{2}e^2\tilde{s}_2+\frac{1}{6}e^3\tilde{s}_3+\cdots$. Also note that Chern character is additive.

The proposition follows from Proposition 3.3 and our assumption. To see the range of the slopes gotten from f^*F, we use formulas (1.1) and (1.5), so that

$$t(f^*F)=\frac{e^4(c_2^2)+\cdots}{e^4(c_4+2c_1c_3+c_1^2c_2)+\cdots}. \qquad \square$$

Similarly, we have the following for 3-folds in $\mathbf{P}^5$.

PROPOSITION 5.2. *Let F be a bundle on $\mathbf{P}^5$ with Chern classes $c_1, \cdots, c_5$. If for all $a_i\in\mathbf{N}$, $f^*F\bigoplus_i\mathcal{O}(a_i)$ has a nonsingular* 3*-fold as a dependency locus, where f is any degree e map. Then*

(i) *Any rational number between* [1, 17/12] *and* $(c_5+3c_1c_4+3c_1^2c_3+c_1^3c_2)/(c_5+3c_1c_4+3c_1^2c_3+c_1^3c_2-c_2c_3)$ *is the slope $c_2(X)$ for some* 3*-fold X.*

(ii) *Any rational number between* [1, 17/7] *and* $(c_5+3c_1c_4+3c_1^2c_3+c_1^3c_2)/(c_5+3c_1c_4+3c_1^2c_3+c_1^3c_2-2c_2c_3)$ *is the slope $c_3(X)$ for some* 3*-fold X.*

PROOF OF THEOREM 3. Let F be the Horrocks-Mumford bundle with $c_1=5$ and $c_2=10$.

CLAIM. $f^*F\bigoplus_{i=1}^l\mathcal{O}(a_i)$ *has nonsingular surfaces as dependency loci.*

PROOF. This is already in the argument given by Horrocks and Mumford [20]. It is sufficient to see the case of $E=F\bigoplus_{i=1}^l\mathcal{O}(a_i)$. A point $x\in\mathbf{P}^4$ is in the dependency locus of $l+1$ general sections $s_1, \cdots, s_{l+1}$, if and only if there are scalars $c_1, \cdots, c_{l+1}$ such that $c_1s_1+\cdots+c_{l+1}s_{l+1}(x)=0$. As in [20], E is generated by global sections outside of a set of 25 skew lines L. So the dependency locus is nonsingular except possibly at points of L. Let $x\in L$ and e_1, e_2 be basis of the $\mathcal{O}_x$-module F_x. Then a section $s=c_1s_1+\cdots+c_{l+1}s_{l+1}$ can be written as $as+bt+ct'+dt''+\sum_{i=1}^l\alpha_i$. Here $\alpha_i\in\mathcal{O}(\alpha_i)$, $(a, b, c, d)\in\mathbf{P}(\Gamma(F))$, $s=e_1$, $t=fe_1+ue_2$, $t'=f'e_1+u'e_2$, $t''=f''e_1+u''e_2$, where f, f', f'', $u, u', u''\in\mathcal{O}_x$ and u, u', u'' generate the ideal of L at x.

Let Q be the projective space associated to $\Gamma(E)$ and let $Z\subset Q\times\mathbf{P}^4$ be represented by (s, x) such that $s(x)=0$. Then Z is described near x by equations

$$a+bf+cf'+df''=0\,,$$
$$bu+cu'+du''=0\,,$$
$$\alpha_1=0, \quad \cdots, \quad \alpha_l=0\,.$$

Hence Z is everywhere nonsingular. So is the dependency locus of $s_1, \cdots, s_{l+1}$. $\square$

Appendix to §2. Notations as in §2. In particular, δ denotes various constants. To prove Lemma 2.1.A, we need

DIRICHLET PRINCIPLE [16, p. 9]. Given $\theta \in \mathbf{R}$, $\forall Q \geq 1$, there exists $a/q \in \mathbf{Q}$, with $(a, q)=1$ and $1 \leq q \leq Q$ such that $|\theta - a/q| < 1/qQ$, and

WEYL'S INEQUALITY [16, p. 11]. Let $f(x)=\alpha x^l + \cdots + \alpha_1 x \in Q[x]$ with $|\alpha - a/q| \leq q^{-2}$. Then $\forall \varepsilon > 0$, there exists $c=c(\varepsilon)$ such that

$$\left|\sum_{n=0}^{N} e^{2\pi i f(n)}\right| < cN^{1+\varepsilon}(q^{-1}+N^{-1}+qN^{-l})^{1/(2^{l-1})}.$$

REMARK. If $N^\delta < q < N^{l-\delta}$, then $q^{-1}+N^{-1}+qN^{-l} < 3N^{-\delta}$ and the right hand side of Weyl's inequality is bounded by $cN^{1-(\delta/2^{l-1}-\varepsilon)}$.

PROOF OF LEMMA 2.1.A. Let $A=\sum_{n=0}^{N} e^{2\pi i(\alpha_l n^l + \cdots + \alpha_1 n)}$. Use Dirichlet Principle to find a/q with $(a, q)=1$ such that

$$\left|\alpha_l - \frac{a}{q}\right| < \frac{1}{qN^{l-\delta_1}} \quad \text{and} \quad q < N^{l-\delta_1}. \tag{A.1}$$

We will approximate A by an exponential sum with lower degree in n inductively until the Remark above is applicable.

Now let $M \in \mathbf{Z}$ with

$$M \sim N^{1-\delta_2} \qquad \text{where } \delta_2 > \delta_1. \tag{A.2}$$

We break A in subsummations of M terms each, i.e. $A=\sum_{j=1}^{N/M} S_j$ and $S_j=\sum_{n=n_j}^{n_{j+1}-1} e^{2\pi i(\alpha_l n^l + g(n))}$, where $g(n)=\alpha_{l-1}n^{l-1}+\cdots+\alpha_1 n$ and

$$n_j=(j-1)M. \tag{A.3}$$

CLAIM. $|S_j| \leq \left|\sum_{n=n_j}^{n_{j+1}-1} e^{2\pi i(g(n)+(a/q)n^l)}\right| + O(N^{\delta_1-\delta_2}M)$.

PROOF OF CLAIM. (A.1) implies $\alpha_l n^l \leq \alpha_l n_j^l + (a/q)(n^l - n_j^l) + N^{\delta_1 - l}(n^l - n_j^l)$. (A.2) and (A.3) imply $N^{\delta_1-l}(n^l-n_j^l)=O(N^{\delta_1-l}MN^{l-1})=O(N^{\delta_1-\delta_2})$. Now factoring out $e^{2\pi i(\alpha_l - a/q)n_j^l}$ from each summand and Fact 1 in §2 gives the claim.

Denote $\sum_{n=n_j}^{n_{j+1}-1} e^{2\pi i(g(n))+(a/q)n^l)}$ in the Claim by T_j. We have

$$|A| \leq \sum_{j=1}^{N/M} |T_j| + O(N^{1+\delta_1-\delta_2}) \leq N^{\delta_2} \max_j |T_j| + O(N^{1+\delta_1-\delta_2}), \quad \text{and}$$

$$\begin{aligned}
\max_j |T_j| &\leq 2 \max_{N_1 \leq N} \left|\sum_{n=0}^{N_1} e^{2\pi i(g(n)+(a/q)n^l)}\right| \\
&= 2 \max_{N_1 \leq N} \left|\sum_{r=0}^{q-1} \sum_{\substack{n \equiv r(q) \\ n \leq N_1}} e^{2\pi i(g(n)+(a/q)n^l)}\right| \\
&\leq 2N^\delta \max_{0 \leq r < q} \left|\sum_{\substack{n \equiv r(q) \\ n \leq N_1}} e^{2\pi i g(n)}\right|.
\end{aligned}$$

The last inequality is because $q<N^{\delta}$ (otherwise the Remark is applicable.).

Let r be fixed and $n=qd+r$. Then $g(n)=\alpha_{l-1}(qd)^{l-1}+\cdots+\alpha_1(qd)+\cdots$ is a polynomial of degree $l-1$ in d with leading coefficient $\beta=q^{l-1}\alpha_{l-1}$. Let $S=\sum_{d\le N_1/q}e^{2\pi i g(n)}$. Use Dirichlet Principle to find a'/q' with $(a',q')=1$, such that $|q^{l-1}\alpha_{l-1}-a'/q'|<1/(q'N^{l-1-\delta_3})$ and $q'<N^{l-1-\delta_3}$.

Case 1. If $q'>N^{\delta_3}$, then the Remark implies $S=O(N^{1-\delta_3})$. Hence $A=O(N^{1-(\delta_3-\delta_2-\delta)})$.

Case 2. If $q'<N^{\delta_3}$, then we continue the process. This process will terminate, because the property of the coefficients $\alpha_1,\cdots,\alpha_l$ that $\bar{\alpha}\notin M$ is carried to the coefficients of the new polynomials, e.g. if $q'<N^{\delta_3}$, then $q'q^{l-1}<N^{l-1-\delta_1'}$ and $q'q^{l-1}<N^{\delta_1'}$. On the other hand the coefficient of d^{l-2} is $\beta_{l-2}=q^{l-2}(\alpha_{l-1}(l-1)r+\alpha_{l-2})$. If $\beta_{l-2}=a''/q''+O(N^{-(l-2+\delta_4)})$ with $q''<N^{\delta_4}$ and $\alpha_{l-1}=a_1/q_1+O(N^{-(l-1+\delta_1)})$ with $q_1<N^{\delta_1}$, then $\alpha_{l-2}=a''/(q''q^{l-2})+(l-1)ra_1/q_1+O(N^{-(l-2+\delta_4)})$ with $q_1q''q^{l-2}<N^{\delta_1(l-2)+\delta_1'+\delta_4}$. □

PROOF OF LEMMA 2.3.A. The left hand side is

$$P=\sum_{r''=1}^{q''}\sum_{r'=1}^{q'}e^{2\pi i(((a_1'/q')r'+(a_1''/q'')r'')+\cdots+((a_j'/q')r'^j+(a_j''/q'')r''^j)+\cdots)}.$$

Since $(q',q'')=1$, $\{r'^j\}_{r'=1}^{q'}=\{(r'q'')^j\}_{r'=1}^{q'}$(modulo q'). Also note that

$$\frac{a_j'}{q'}(q''r')^j+\frac{a_j''}{q''}(q'r'')^j=\mathbf{Z}+\left(\frac{a_j'}{q'}+\frac{a_j''}{q''}\right)(q''^jr'^j+q'^jr''^j)$$
$$=\mathbf{Z}+\left(\frac{a_j'}{q'}+\frac{a_j'}{q''}\right)(q''r'+q'r'')^j.$$

Since $(q',q'')=1$, $y=q''r'+q'r''$ represents all integers between 1 and $q'q''$ when r' and r'' run through all possible integers. Therefore

$$P=\sum_{y=1}^{q'q''}e^{2\pi i((a_1'/q'+a_1''/q'')y+\cdots+(a_j'/q'+a_j''/q'')y^j+\cdots)}.$$ □

REMARK ON LEMMA 2.4.A. This is Weil's Theorem, see e.g. [6] p. 223: Let $f\in\mathbf{F}_q[x]$ of degree $n\ge1$ and g.c.d.$(n,q)=1$. Then $|\sum_{c\in\mathbf{F}_q}\chi(f(c))|\le(n-1)q^{1/2}$.

We take a larger constant $c(l)$ to cover the cases with $(l,q)\neq1$.

REMARK ON LEMMA 2.5.A. This is Van den Corput. See, e.g. [13] p. 309: Let $\phi(x)\in\mathbf{R}$ be smooth in $[a,b]$. If $|\phi^{(k)}(x)|\ge1$, then $|\int_a^b e^{i\lambda\phi(x)}dx|\le c_k\lambda^{-1/k}$ holds when (i) $k\ge2$, (ii) or $k=1$ and if $\phi'(x)$ is monotonic. We need to normalize our polynomial.

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