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On Torus Homeomorphisms of Which Rotation Sets Have No Interior Points

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Abstract. Let us assume that a 2-torus homeomorphism f isotopic to the identity has a segment of irrational slope as its rotation set $\rho(F)$. We prove that if the chain recurrent set $R(f)$ of f is not chain transitive, then $\rho(F)$ has a rational point realized by a periodic point.

1. Introduction.

In [6] rotation sets of torus homeomorphisms are introduced by M. Misiurewicz and K. Ziemian. For a homeomorphism f on a 2-torus T^2 isotopic to the identity, let F be a lift of f to the universal cover $\pi: \mathbf{R}^2 \to T^2$, and set $\Gamma = \{(F^n(x)-x)/n \mid x \in \mathbf{R}^2, n \in \mathbf{Z}_+\}$. Then the rotation set $\rho(F)$ is the set of limit points of Γ, *i.e.* $v \in \rho(F)$ if there exist sequences $x_i \in \mathbf{R}^2$ and $n_i \in \mathbf{Z}_+$ with $\lim_{i\to\infty} n_i = \infty$ such that $\lim_{i\to\infty}(F^{n_i}(x_i)-x_i)/n_i = v$. As a fundamental property of $\rho(F)$, it is known that $\rho(F)$ is compact and convex (see [6]). In this paper, as in [3] and [4] we call $v \in \rho(F)$ an interior point of $\rho(F)$ if there exists an open 2-disk D such that $v \in D \subset \rho(F)$, and let $\mathrm{Int}\,\rho(F)$ denote the set of interior points of $\rho(F)$.

One of the most important problems on rotation sets is how rational points, *i.e.* points with both coordinates rational, in $\rho(F)$ are related to periodic points of f, and J. Franks [3] showed that rational points of $\mathrm{Int}\,\rho(F)$ are realized by periodic points, *i.e.* for any $v \in \mathrm{Int}\,\rho(F) \cap \mathbf{Q}^2$, there exists an f-periodic point $\bar{x}$ of period q such that for any lift x of $\bar{x}$, $(F^q(x)-x)/q = v$. If $\mathrm{Int}\,\rho(F) = \emptyset$, $\rho(F)$ is a single point or a closed segment. In this case, in [4] under the additional assumption that f preserves a Lebesgue measure, he also showed that any $v \in \rho(F) \cap \mathbf{Q}^2$ is realized by an f-periodic point.

In this paper, we will deal with this problem in the case when $\mathrm{Int}\,\rho(F) = \emptyset$, but we do not assume that f preserves a Lebesgue measure. We will show the following.

THEOREM. *Let $f: T^2 \to T^2$ be a homeomorphism isotopic to the identity, and let F be a lift of f to the universal cover. If the rotation set $\rho(F)$ is a closed segment of irrational*

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slope, and the chain recurrent set $R(f)$ of f is not chain transitive, then $\rho(F)$ includes a rational point which is realized by an f-periodic point.

REMARK 1. For an example of f with $\rho(F)$ a closed segment of irrational slope, refer to [5].

2. Proof of Theorem.

To show the theorem, we will use the argument in [3] that uses a complete Lyapounov function. For a homeomorphism of a compact metric space $\varphi: X \to X$ a continuous function $g: X \to \mathbf{R}$ is called a *complete Lyapounov function* if (a) for any $x \notin R(\varphi)$, $g(\varphi(x)) < g(x)$, where $R(\varphi)$ denotes the chain recurrent set of φ, (b) for $x, y \in R(\varphi)$, the necessary and sufficient condition for the equality $g(x) = g(y)$ to hold is that x and y are in the same chain transitive component, (c) $g(R(\varphi))$ is a compact nowhere dense subset of $\mathbf{R}$.

For fundamental results of chain recurrent sets and complete Lyapounov functions, refer to [1] and [2]. Especially for any homeomorphism of a compact metric space, there exists a complete Lyapounov function, and moreover we need the following.

(1.6) THEOREM [3]. *Let $\varphi: X \to X$ be a homeomorphism of a compact metric space, and let $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ be δ-transitive components of $R(\varphi)$ for any $\delta > 0$. Then there are a complete Lyapounov function g for φ and $c_0 < c_1 < \cdots < c_n$ such that $\Lambda_i = R(\varphi) \cap g^{-1}((c_{i-1}, c_i))$ for $1 \leq i \leq n$.*

PROOF OF THEOREM. Since $R(f)$ is not chain transitive, there is $\delta > 0$ such that $R(f)$ is decomposed into two or more δ-transitive components $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$. By (1.6) Theorem, one has a complete Lyapounov function g with values $c_0 < c_1 < \cdots < c_n$ such that $\Lambda_i = R(f) \cap g^{-1}((c_{i-1}, c_i))$. One can choose a smooth approximation g_0 of g such that c_i are regular values of g_0, and $M_i = g_0{}^{-1}((-\infty, c_i])$ satisfy $f(M_i) \subset \operatorname{Int} M_i$ and $\Lambda_i \subset M_i - M_{i-1}$.

As the proof of (3.1) Proposition [3], we will also show that all circles of $g_0^{-1}(\{c_0, c_1, \cdots, c_n\})$ are inessential. Suppose by contradiction that there exists an essential circle γ. Then γ is a boundary component of M_j for some j, and M_j is the disjoint union of essential annuli A_r, $1 \leq r \leq \bar{r}$, possibly with holes and, possibly, components D_s, $1 \leq s \leq \bar{s}$, included in disks in T^2. Since any A_r cannot be mapped into D_s by f, one can find A_r and a positive integer k with $f^k(A_r) \subset A_r$. Let us take a lift F_k of f^k such that for each component $\tilde{A}_r$ of $\pi^{-1}(A_r)$, $F_k(\tilde{A}_r) \subset \tilde{A}_r$. Since $\rho(F_k)$ is obtained by translating $\rho(F^k)$ by a rational vector and by Proposition 2.1[6] $\rho(F^k) = k\rho(F)$, $\rho(F_k)$ is a segment of irrational slope, too.

Let v_0, v_1 be the end points of $\rho(F_k)$. Then by Theorem 3.5[6] and the ergodic theorem, there exist points x_0, $x_1 \in \mathbf{R}^2$ with $\lim_{m \to \infty}(F_k^m(x_0) - x_0)/m = v_0$, $\lim_{m \to \infty}(F_k^m(x_1) - x_1)/m = v_1$. Since the slope of $\tilde{A}_r$ is rational and F_k preserves each of them, v_0 and v_1 are in a line of the same rational slope as $\tilde{A}_r$, but this contradicts that the slope of $\rho(F_k)$ is irrational.

By the result of the previous paragraph, $g_0^{-1}(c_{n-1})$ is the disjoint union of finite inessential circles, and note that $g_0^{-1}(c_{n-1}) \neq \emptyset$ because $g_0^{-1}(c_{n-1})$ separates Λ_n from $\Lambda_j, j \leq n-1$. From disks bounded by circles $\subset g_0^{-1}(c_{n-1})$, let us choose the ones that are not included in others, and let us denote them by $D_1, D_2, \cdots, D_k$. Set $N_0 = T^2 - \bigcup_{i=1}^k \mathrm{Int} D_i$. Assume that N_0 is a connected component of M_{n-1}. Then M_{n-1} is the disjoint union of N_0 and, possibly, components which lie in $\bigcup_{i=1}^k \mathrm{Int} D_i$. Since $f(M_{n-1}) \subset M_{n-1}$ and since N_0 cannot be mapped into a disk by f, one has that $f(N_0) \subset N_0$. This implies that there exist a disk D_i and a positive integer q with $f^{-q}(D_i) \subset D_i$, and thus one obtains a periodic point $\bar{x}_*$ of period q. Obviously for any lift x_* of $\bar{x}_*$ we have that $v = \lim_{m \to \infty}(F^m(x_*) - x_*)/m = (F^q(x_*) - x_*)/q \in \rho(F) \cap \mathbf{Q}^2$. This shows the theorem in this case.

Let us investigate the rest case, *i.e.* the case when N_0 is not a connected component of M_{n-1}. In this case, obviously $f(\bigcup_{i=1}^k D_i) \subset \bigcup_{i=1}^k D_i$, and one can find a disk D_i and $q > 0$ such that $f^q(D_i) \subset D_i$. Therefore we obtain a periodic point again, and this finishes the proof of the theorem. □

REMARK 2. If $R(f)$ is chain transitive, we can only show that $R(f) = T^2$ as a special case of the following general result.

PROPOSITION. *Let φ be a homeomorphism of a compact metric space X. If $R(\varphi)$ is chain transitive, then $R(\varphi) = X$.*

PROOF. Suppose by contradiction that $R(\varphi) \neq X$. Let g be a complete Lyapounov function for φ. Note that by the assumption that $R(\varphi)$ is chain transitive, g is constant on $R(\varphi)$, and let $a_0 = g(R(\varphi))$. On all of X, however, g is not constant because $g(\varphi(x)) < g(x)$ for $x \notin R(\varphi)$. Since $R(\varphi) = R(\varphi^{-1})$, passing to φ^{-1}, if necessary, we may assume that g has the minimum value a_1 not equal to a_0. Let us choose $x_1 \in X$ with $g(x_1) = a_1$. Then since $x_1 \notin R(\varphi)$, we have that $g(\varphi(x_1)) < g(x_1)$, but this is impossible. This completes the proof. □

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