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Kummer's Lemma for Some Cyclotomic Fields

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Introduction.

Let p be a prime number and $\zeta_n=\exp(2\pi i/n)$ for each natural number n. Kummer's lemma proved in [2] says that "for regular p, a unit of $\mathbf{Q}(\zeta_p)$ congruent to a rational integer modulo p is a pth power in $\mathbf{Q}(\zeta_p)$". L. C. Washington generalized this theorem as follows:

THEOREM ([3]). *Let $M=\mathrm{Max}\{v_p(L_p(1, \omega^i))\ ;\ i=2, 4, \cdots, p-3\}$ and let ε be a unit of $\mathbf{Z}[\zeta_p]$. If ε is congruent to a rational integer modulo p^{M+1}, then ε is a pth power of a unit of $\mathbf{Z}[\zeta_p]$.*

And further,

THEOREM ([4]). *Let $M=\mathrm{Max}_\chi v_{\pi_n}(\tau(\chi^{-1})L_p(1, \chi))$, where χ runs through the even nontrivial Dirichlet characters of conductor dividing p^n. Here $\tau(\chi)$ is a Gauss sum. If ε is a unit of $\mathbf{Z}[\zeta_{p^n}]$ such that $\varepsilon\equiv 1 \mod p^n\pi_n^{M-1}$, then ε is a pth power in $\mathbf{Z}[\zeta_{p^n}]$.*

In the present paper, following the beautiful method of Washington, we give a proof of a generalization of Kummer's lemma for some other cyclotomic fields. In the following, p denotes an odd prime number. Let m be a natural number such that p does not divide $m\varphi(m)$ and p is congruent to 1 modulo the exponent of the galois group of $\mathbf{Q}(\zeta_m)$ over $\mathbf{Q}$, where φ denotes the Euler function. Let $K=\mathbf{Q}(\zeta_{mp})$, E be the group of units of K, K^+ the maximal real subfield of K and E^+ its group of units. Let G be the galois group of K over $\mathbf{Q}$, $\hat{G}$ its character group and $L_p(s, \chi)$ the p-adic L-function. Let $\mathbf{C}_p$ be the completion of the algebraic closure of the p-adic rational field $\mathbf{Q}_p$. Throughout the paper, we fix an embedding of K in $\mathbf{C}_p$, that is, fix a prime $\wp$ of K over p.

Our main result is the following:

THEOREM. *Let p and m be as above. Let $M_\wp$ be the least natural number such that*

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$$M_{\wp}(p-1) > \mathrm{Max}\left\{ v_{\wp}\left(\frac{p}{\tau(\chi)} L_p(1, \chi)\right) \ ; \ \hat{G} \ni \chi \neq 1, \textit{even} \right\},$$

where $v_{\wp}$ *denotes the* $\wp$*-adic additive valuation normalized by* $v_{\wp}(1-\zeta_p)=1$. *If* $u \in E$ *is congruent to a rational integer modulo* $p^{M_{\wp}}$, *then* u *is a pth power in* E.

This is also a generalization of a result of Washington [3] (corollary to Theorem 2, cf. Theorem 8.22 of [4]) because $v_{\wp}(\tau(\omega^i)) = p-1-i$ $(i=2, \cdots, p-3$, even). The proof is based on the non-vanishing of the p-adic regulator of K proved by Brumer [1].

1. Structure of $E \otimes_{\mathbf{Z}} \mathbf{Z}_p$.

Let Δ be the Galois group of K over $\mathbf{Q}(\zeta_m)$, which is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^{\times}$, and M the Galois group of K over $\mathbf{Q}(\zeta_p)$, which is isomorphic to $(\mathbf{Z}/m\mathbf{Z})^{\times}$. Of course we have $G = \Delta \times M$. Let ω be the Teichmüller character which generates $\hat{\Delta}$. By our assumption on p and m stated in the Introduction, the values of all $\chi \in \hat{G}$ belong to the ring of p-adic integers $\mathbf{Z}_p$ in which $|G|$, the order of G, is invertible. We define the homomorphism ψ from E to $E \otimes_{\mathbf{Z}} \mathbf{Z}_p$ by $\psi : u \mapsto u \otimes 1$ and the logarithm from $E \otimes_{\mathbf{Z}} \mathbf{Z}_p$ to $\mathbf{C}_p$ by

$$\log_p\left(\sum_i (u_i \otimes \alpha_i)\right) = \sum_i \alpha_i \log_p u_i \, .$$

We may consider $E \otimes_{\mathbf{Z}} \mathbf{Z}_p$ as a $\mathbf{Z}_p[G]$-module and the logarithm is a $\mathbf{Z}_p$-homomorphism.

LEMMA 1. *For any unit* $u \in E$, $u \in E^p$ *if and only if* $\psi(u) \in E \otimes_{\mathbf{Z}} p\mathbf{Z}_p$.

PROOF. Clearly, $u \in E^p$ implies $\psi(u) \in E \otimes_{\mathbf{Z}} p\mathbf{Z}_p$. For a subset S of $E \otimes_{\mathbf{Z}} \mathbf{Z}_p$, we denote by $\langle S \rangle$ the subgroup generated by S over $\mathbf{Z}_p$. Now assume $\psi(u) \in E \otimes_{\mathbf{Z}} p\mathbf{Z}_p$ and $u \notin E^p$, then we can choose $\varepsilon_1, \cdots, \varepsilon_{r-1}$ such that the subgroup of E generated by $\{\varepsilon_1, \cdots, \varepsilon_{r-1}, u, \zeta_p\}$, say E', has the index in E which is prime to p, where r is the $\mathbf{Z}$-rank of E. So we have $E \otimes_{\mathbf{Z}} \mathbf{Z}_p = \langle \psi(E) \rangle = \langle \psi(E') \rangle$. It holds that

$$\dim_{\mathbf{F}_p}(E \otimes_{\mathbf{Z}} \mathbf{Z}_p / E \otimes_{\mathbf{Z}} p\mathbf{Z}_p) = r+1 \, .$$

But from our assumption that $\psi(u) \in E \otimes_{\mathbf{Z}} p\mathbf{Z}_p$, we have

$$\dim_{\mathbf{F}_p}(E \otimes_{\mathbf{Z}} \mathbf{Z}_p / E \otimes_{\mathbf{Z}} p\mathbf{Z}_p) \leq r \, .$$

That is a contradiction and the lemma is proved.

Next, we study the $\mathbf{Z}_p[G]$-module structure of $E \otimes_{\mathbf{Z}} \mathbf{Z}_p$. Define

$$e_{\chi} = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma) \sigma^{-1} \in \mathbf{Z}_p[G] \, .$$

The e_{χ}'s are orthogonal idempotents of $\mathbf{Z}_p[G]$.

LEMMA 2. *Let p and m be the same as before. Then we have:*

$$E\otimes_{\mathbf{Z}}\mathbf{Z}_p\simeq\left(\bigoplus_{1\neq\chi:even} e_\chi(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)\right)\oplus e_\omega(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)\,,$$

$$e_\omega(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)\simeq\langle\zeta_p\otimes 1\rangle\simeq\mathbf{Z}/p\mathbf{Z}\,.$$

PROOF. Let W_K be the group of roots of unity in K. Then since p does not divide $[E:E^+W_K]$, we have

$$E\otimes_{\mathbf{Z}}\mathbf{Z}_p=E^+W_K\otimes_{\mathbf{Z}}\mathbf{Z}_p=E^+\langle\zeta_p\rangle\otimes_{\mathbf{Z}}\mathbf{Z}_p\,.$$

Now we have

$$|G|e_\chi=\sum_{1\leq a\leq mp}{}'\chi(\sigma_a)\sigma_a^{-1}=\sum_{1\leq a\leq mp/2}{}'\{\chi(\sigma_a)\sigma_a^{-1}+\chi(\sigma_{-a})\sigma_{-a}^{-1}\}\,,$$

here $\sum'$ denotes the sum over all a which are prime to mp and σ_a denotes the automorphism sending ζ_{mp} to ζ_{mp}^a. And we have $\sigma_{-a}=\sigma_{-1}\sigma_a$, $\sigma_{-a}^{-1}=(\sigma_{-1}\sigma_a)^{-1}=\sigma_{-1}\sigma_a^{-1}$. Therefore, for each $u\in E^+$,

$$\begin{aligned}(\chi(\sigma_a)\sigma_a^{-1}+\chi(\sigma_{-a})\sigma_{-a}^{-1})(u\otimes 1)&=(\chi(\sigma_a)\sigma_a^{-1}+\chi(\sigma_{-1})\chi(\sigma_a)\sigma_{-1}\sigma_a^{-1})(u\otimes 1)\\&=(\chi(\sigma_a)\sigma_a^{-1}+\chi(\sigma_{-1})\chi(\sigma_a)\sigma_a^{-1})(u\otimes 1)\,.\end{aligned}$$

Consequently, if χ is odd, we have $(\chi(\sigma_a)\sigma_a^{-1}+\chi(\sigma_{-a})\sigma_{-a}^{-1})(u\otimes 1)=0$. Since $\{u\otimes 1\,;\,u\in E^+\}$ generates $E^+\otimes_{\mathbf{Z}}\mathbf{Z}_p$,

$$e_\chi(E^+\otimes_{\mathbf{Z}}\mathbf{Z}_p)=0\qquad\text{for all odd }\chi\,.$$

Corresponding to the decomposition $G\simeq M\times\Delta$ we may write any character $\chi\in\hat{G}$ as $\chi=\psi\omega^i$ $(0\leq i\leq p-2,\ \psi\in\hat{M})$.

$$\begin{aligned}|G|e_\chi&=\sum_{\tau\in M}\sum_{\sigma\in\Delta}\psi(\tau\sigma)\omega^i(\tau\sigma)\tau^{-1}\sigma^{-1}=\sum_{\tau\in M}\sum_{\sigma\in\Delta}\psi(\tau)\omega^i(\sigma)\tau^{-1}\sigma^{-1}\\&=\left(\sum_{\tau\in M}\psi(\tau)\tau^{-1}\right)\left(\sum_{\sigma\in\Delta}\omega^i(\sigma)\sigma^{-1}\right).\end{aligned}$$

Now we have

$$\left(\sum_{\tau\in M}\psi(\tau)\tau^{-1}\right)(\zeta_p\otimes 1)=\left(\sum_{\tau\in M}\psi(\tau)\right)(\zeta_p\otimes 1)=0$$

for $\psi\neq 1_M$ (the principal character of $\hat{M}$) and $(\sum_{\tau\in M}1_M(\tau)\tau^{-1})(\zeta_p\otimes 1)=\zeta_p^{|M|}\otimes 1$. Therefore,

$$\left\langle\left(\sum_{\tau\in M}\psi(\tau)\tau^{-1}\right)(\zeta_p\otimes 1)\right\rangle\simeq\begin{cases}0 & \text{if}\quad\psi\neq 1_M\,,\\ \langle\zeta_p\otimes 1\rangle & \text{if}\quad\psi=1_M\,.\end{cases}$$

On the other hand, from the fact that $(\sum_{\sigma\in\Delta}\omega^i(\sigma)\sigma^{-1})\zeta_p=(\sum_{a=1}^{p-1}a^{i-1})\zeta_p$, we have

$$\left\langle\left(\sum_{\sigma\in\Delta}\omega^i(\sigma)\sigma^{-1}\right)(\zeta_p\otimes 1)\right\rangle\simeq\begin{cases}0 & \text{if } i\neq 1\,.\\ \langle\zeta_p\otimes 1\rangle & \text{if } i=1\,.\end{cases}$$

Therefore, we have $e_\chi(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)=0$ for all odd $\chi\neq\omega$ and $e_\omega(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)\simeq\langle\zeta_p\otimes 1\rangle$. In addition, when $\chi=1$, $|G|e_\chi$ is the norm map from K to $\mathbf{Q}$, so $e_\chi(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)=0$. This completes the proof of our lemma.

Let f be the order of p modulo m. In the following, we shall regard every unit in E as its image by ψ.

LEMMA 3. *Let $\chi\in\hat{G}$ be an even nontrivial character of conductor f_χ which is not a prime power. Let*

$$U(\chi)=(1-\zeta_{f_\chi})^{(p^f-1)e_\chi}\,.$$

Then, we have

$$\log_p U(\chi)=(p^f-1)\frac{1}{\varphi(f_\chi)}\frac{p}{\chi(p)-p}\frac{f_\chi}{\tau(\chi)}L_p(1,\chi)\,.$$

PROOF. Let H be the subgroup of G corresponding to $\mathbf{Q}(\zeta_{f_\chi})$ and G_1 a subset of G representing G/H. Since χ has conductor f_χ, $\chi(H)=1$.

$$\begin{aligned}\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1}&=\sum_{\sigma\in G_1}\sum_{\tau\in H}\chi(\sigma\tau)\sigma^{-1}\tau^{-1}=\sum_{\sigma\in G_1}\sum_{\tau\in H}\chi(\sigma\tau)\sigma^{-1}\tau^{-1}\\&=\sum_{\sigma\in G_1}\sum_{\tau\in H}\chi(\sigma)\sigma^{-1}\tau^{-1}=\left(\sum_{\sigma\in G_1}\chi(\sigma)\sigma^{-1}\right)\left(\sum_{\tau\in H}\tau^{-1}\right).\end{aligned}$$

Consequently, it holds that

$$\begin{aligned}U(\chi)&=(1-\zeta_{f_\chi})^{(p^f-1)\frac{1}{|G|}(\sum_{\sigma\in G_1}\chi(\sigma)\sigma^{-1})(\sum_{\tau\in H}\tau^{-1})}\\&=(1-\zeta_{f_\chi})^{(p^f-1)\frac{1}{\varphi(f_\chi)}\sum_{\sigma\in G_1}\chi(\sigma)\sigma^{-1}}\,.\end{aligned}$$

So, we have

$$\log_p U(\chi)=(p^f-1)\frac{1}{\varphi(f_\chi)}\sum_{\sigma\in G_1}\chi(\sigma)\log_p(1-\zeta_{f_\chi}^{\sigma^{-1}})\,.$$

Our lemma is now clear from the well known fact:

$$L_p(1,\chi)=-\left(1-\frac{\chi(p)}{p}\right)\frac{\tau(\chi)}{f}\sum_{l=1}^{f}\chi^{-1}(l)\log_p(1-\zeta_f^l)\,,$$

where χ is an even nontrivial Dirichlet character of conductor f.

By means of a discussion similar to the above, we have the following:

LEMMA 4. *Let $\chi \in \hat{G}$ be an even nontrivial character of conductor f_χ which is a prime power and α a primitive root modulo f_χ. Let*

$$V(\chi)=\left(\frac{1-\zeta_{f_\chi}^{\alpha}}{1-\zeta_{f_\chi}}\right)^{(p^f-1)e(\chi)}.$$

Then we have:

$$\log_p V(\chi)=(p^f-1)\frac{1}{\varphi(f_\chi)}(\chi(\sigma_\alpha)-1)\frac{p}{\chi(p)-p}\frac{f_\chi}{\tau(\chi)}L_p(1,\chi).$$

Note that $\chi(\sigma_\alpha)$ is a $(p-1)$-st root of unity and different from 1. So, we have $\chi(\sigma_\alpha)-1 \not\equiv 0 \bmod p$.

PROPOSITION 5. *Let $\chi \in \hat{G}$ be an even nontrivial character of conductor f_χ. Let $U(\chi)$ and $V(\chi)$ be the same as before. Then we have*

$$v_\wp\left(\frac{p}{\tau(\chi)}L_p(1,\chi)\right)=\begin{cases} v_\wp(\log_p U(\chi)) & \text{if } f_\chi \text{ is not a prime power},\\ v_\wp(\log_p V(\chi)) & \text{if } f_\chi \text{ is a prime power}.\end{cases}$$

PROOF. If p does not divide f_χ,

$$v_\wp\left(\frac{p}{\chi(p)-p}\frac{f_\chi}{\tau(\chi)}L_p(1,\chi)\right)=v_\wp\left(\frac{p}{\tau(\chi)}L_p(1,\chi)\right).$$

If p divides f_χ, as p is the exact power of f_χ and $\chi(p)=0$, we have

$$v_\wp\left(\frac{p}{\chi(p)-p}\frac{f_\chi}{\tau(\chi)}L_p(1,\chi)\right)=v_\wp\left(\frac{p}{-p}\frac{f_\chi}{\tau(\chi)}L_p(1,\chi)\right)=v_\wp\left(\frac{p}{\tau(\chi)}L_p(1,\chi)\right).$$

The proof is now completed by Lemmas 3 and 4.

PROPOSITION 6. *Let p, m, G and E be as before. Then we have*

$$e_\chi(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)\simeq \mathbf{Z}_p \qquad \textit{for all even nontrivial } \chi\in\hat{G}.$$

PROOF. From the definition of $U(\chi)$ and $V(\chi)$,

$$U(\chi)\in e_\chi(E\otimes_{\mathbf{Z}}\mathbf{Z}_p) \quad \text{or} \quad V(\chi)\in e_\chi(E\otimes_{\mathbf{Z}}\mathbf{Z}_p).$$

By the theorem of A. Brumer [1], we have $L_p(1,\chi)\neq 0$. So, from Proposition 5, $e_\chi(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)$ is not trivial and $\operatorname{rank}_{\mathbf{Z}_p} e_\chi(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)\geq 1$. From Lemma 2, $\operatorname{rank}_{\mathbf{Z}_p} e_\omega(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)=0$ and

$$\operatorname{rank}_{\mathbf{Z}_p}(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)\leq r=\#\{\chi\ ;\ \chi \text{ is even nontrivial character}\}.$$

Therefore, $\operatorname{rank}_{\mathbf{Z}_p} e_\chi(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)=1$ for all even nontrivial characters χ. Our proposition is proved.

LEMMA 7. *Let p, m, G and E be as above. Let $\chi\in\hat{G}$ be any even nontrivial character. Then we have*

$$v_{\wp}(\log_p \eta_\chi)\le v_{\wp}\left(\frac{p}{\tau(\chi)}L_p(1,\chi)\right),$$

for any generator η_χ of $e_\chi(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)$.

PROOF. From Proposition 6, $U(\chi)=\alpha\eta_\chi$ or $V(\chi)=\alpha\eta_\chi$ in $E\otimes_{\mathbf{Z}}\mathbf{Z}_p$, for some $\alpha\in\mathbf{Z}_p$. Then, it holds that

$$\log_p U(\chi)=\alpha\log_p\eta_\chi \quad \text{or} \quad \log_p V(\chi)=\alpha\log_p\eta_\chi\,.$$

Therefore

$$v_{\wp}(\log_p U(\chi))=v_{\wp}(\alpha)+v_{\wp}(\log_p\eta_\chi)\ge v_{\wp}(\log_p\eta_\chi)$$

or

$$v_{\wp}(\log_p V(\chi))=v_{\wp}(\alpha)+v_{\wp}(\log_p\eta_\chi)\ge v_{\wp}(\log_p\eta_\chi)\,.$$

By means of Proposition 5, the lemma is proved.

2. Proof of the theorem.

By our assumption, there exists a rational integer a such that $u\equiv a \bmod p^{M_\wp}$. First, we treat the case $u\equiv 1 \bmod \wp$. Then we have $a\equiv 1 \bmod p$. For each even nontrivial character $\chi\in\hat{G}$, from Proposition 6, there exists a generator η_χ of $e_\chi(E\otimes_{\mathbf{Z}}\mathbf{Z}_p)$. Then, from Lemma 2 and Proposition 6, we have

$$u\otimes 1=\sum_{1\ne\chi:\text{even}}\delta_\chi\eta_\chi+\delta_\omega(\zeta_p\otimes 1)$$

for some $\delta_\chi\in\mathbf{Z}_p$ and $\delta_\omega\in\mathbf{Z}$.

From our assumption, it holds that $u=a+p^{M_\wp}x$ for some $x\in\mathbf{Z}[\zeta_{mp}]$. So, for any $\sigma\in G$, we have

$$\begin{aligned}(\chi(\sigma)\sigma^{-1})(u\otimes 1)&=u^{\sigma^{-1}}\otimes\chi(\sigma)=(a+p^{M_\wp}x^{\sigma^{-1}})\otimes\chi(\sigma)\\&=a\left(1+\frac{x^{\sigma^{-1}}}{a}p^{M_\wp}\right)\otimes\chi(\sigma)=a\otimes\chi(\sigma)+\left(1+\frac{x^{\sigma^{-1}}}{a}p^{M_\wp}\right)\otimes\chi(\sigma)\,.\end{aligned}$$

Then, we have

$$\begin{aligned}\left(\sum_{1\ne\chi:\text{even}}\chi(\sigma)\sigma^{-1}\right)(u\otimes 1)&=\sum_{1\ne\chi:\text{even}}(a\otimes\chi(\sigma))+\sum_{1\ne\chi:\text{even}}\left(1+\frac{x^{\sigma^{-1}}}{a}p^{M_\wp}\right)\otimes\chi(\sigma)\\&=\sum_{1\ne\chi:\text{even}}\left(1+\frac{x^{\sigma^{-1}}}{a}p^{M_\wp}\right)\otimes\chi(\sigma)\,.\end{aligned}$$

We define $\chi(\sigma)_{M_\wp}$ to be a natural number such that $\chi(\sigma)_{M_\wp} \equiv \chi(\sigma) \bmod p^{M_\wp+1}$ and $\chi(\sigma)'_{M_\wp}$ the element of $p^{M_\wp+1}\mathbf{Z}_p$ such that $\chi(\sigma) = \chi(\sigma)_{M_\wp} + \chi(\sigma)'_{M_\wp}$. Then, from above, it holds that

$$\left(\sum_{1 \neq \chi:\text{even}} \chi(\sigma)\sigma^{-1}\right)(u \otimes 1) = \sum_{1 \neq \chi:\text{even}} \left(1 + \frac{x^{\sigma^{-1}}}{a} p^{M_\wp}\right) \otimes (\chi(\sigma)_{M_\wp} + \chi(\sigma)'_{M_\wp})$$
$$= \sum_{1 \neq \chi:\text{even}} \left(1 + \frac{x^{\sigma^{-1}}}{a} p^{M_\wp}\right) \otimes \chi(\sigma)_{M_\wp} + \sum_{1 \neq \chi:\text{even}} \left(1 + \frac{x^{\sigma^{-1}}}{a} p^{M_\wp}\right) \otimes \chi(\sigma)'_{M_\wp}.$$

Consequently,

$$\log_p\left(\left(\sum_{1 \neq \chi:\text{even}} \chi(\sigma)\sigma^{-1}\right)(u \otimes 1)\right) = \sum_{1 \neq \chi:\text{even}} \chi(\sigma)_{M_\wp} \log_p\left(1 + \frac{x^{\sigma^{-1}}}{a} p^{M_\wp}\right)$$
$$+ \sum_{1 \neq \chi:\text{even}} \chi(\sigma)'_{M_\wp} \log_p\left(1 + \frac{x^{\sigma^{-1}}}{a} p^{M_\wp}\right)$$
$$\equiv \sum_{1 \neq \chi:\text{even}} \chi(\sigma)_{M_\wp} \log_p\left(1 + \frac{x^{\sigma^{-1}}}{a} p^{M_\wp}\right) \quad \bmod p^{M_\wp+1}.$$

Because

$$v_\wp\left(\log_p\left(1 + \frac{x^{\sigma^{-1}}}{a} p^{M_\wp}\right)\right) = v_\wp\left(\frac{x^{\sigma^{-1}}}{a} p^{M_\wp}\right) \geq M_\wp(p-1),$$

we see that $v_\wp(\log_p(e_\chi(u \otimes 1))) \geq M_\wp(p-1)$.

On the other hand, since $e_\chi(u \otimes 1) = \delta_\chi \eta_\chi$, we have, from Lemma 7,

$$v_\wp(\log_p(e_\chi(u \otimes 1))) = v_\wp(\delta_\chi) + v_\wp(\log_p \eta_\chi) \leq v_\wp(\delta_\chi) + v_\wp\left(\frac{p}{\tau(\chi)} L_p(1, \chi)\right)$$
$$< v_\wp(\delta_\chi) + M_\wp(p-1).$$

So, we have

$$M_\wp(p-1) < v_\wp(\delta_\chi) + M_\wp(p-1).$$

Therefore, $v_\wp(\delta_\chi) > 0$, so that $\delta_\chi \equiv 0 \bmod p$. Now we have proved that

$$E \otimes_{\mathbf{Z}} p\mathbf{Z}_p \ni u \otimes 1 - \delta_\omega(\zeta_p \otimes 1) = u\zeta_p^{-\delta_\omega} \otimes 1 = \psi(u\zeta_p^{-\delta_\omega}).$$

Lemma 1 means that $u\zeta_p^{-\delta_\omega} \in E^p$. Let v be a unit such that $u\zeta_p^{-\delta_\omega} = v^p$ and

$$v = z_0 + z_1(\zeta_p - 1) + \cdots + z_{p-2}(\zeta_p - 1)^{p-2} \quad \text{and} \quad z_0 \not\equiv 0 \bmod \wp,$$

where $z_i \in \mathbf{Z}[\zeta_m]$ $(0 \leq i \leq p-2)$. Then, $a \equiv u \equiv \zeta_p^{\delta_\omega} v^p \equiv \zeta_p^{\delta_\omega} z_0^p \bmod p$. Since $\zeta_p^{\delta_\omega} \equiv a/z_0^p \bmod p$ is Δ-invariant, we have $\delta_\omega \equiv 0 \bmod p$. As a result, $u \in E^p$ is proved.

Next, we assume $u \not\equiv 1 \bmod \wp$. Then, clearly, $u^{p^f-1} \equiv 1 \bmod \wp$ and further $u^{p^f-1} \equiv a^{p^f-1} \bmod p^{M_\wp}$. From the same discussion as above, u^{p^f-1} is a pth power, and so is u. The proof of the theorem is completed.

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