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Neighborhood Conditions and k-Factors

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Abstract. Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k-1-4\sqrt{2(k-1)^2+2}$, kn is even, and the minimum degree is at least k. We prove that if $|N_G(u) \cup N_G(v)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices u, v of G, then G has a k-factor.

1. Introduction.

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G, we write $N_G(v)$ for the set of vertices of $V(G)$ adjacent to v, and $N_G[v]$ for $N_G(v) \cup \{v\}$. Further the degree of v, $\deg_G(v)$, is defined to be $|N_G(v)|$. In addition, we denote $|N_G(u) \cup N_G(v)|$ by $N(u, v)$. We define NC to be $\min N(u, v)$, where the minimum is taken over all pairs of nonadjacent vertices u, v. We use $\delta(G)$ for the minimum degree. Let A and B be disjoint subsets of $V(G)$. Then $e_G(A, B)$ denotes the number of edges that join a vertex in A and a vertex in B. We let $G-A$ denotes the subgraph of G obtained from G by deleting the vertices in A together with the edges incident with them. A spanning subgraph F of G is called a *k-factor* if $\deg_F(v)=k$ for all $v \in V(G)$. If G and H are disjoint graphs, the union and the join are denoted by $G \cup H$ and $G+H$, respectively. A vertex v is often identified with the set $\{v\}$. The definition of terms not defined here can be found in [1].

THEOREM 1.1. *Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k-1-4\sqrt{2(k-1)^2+2}$, kn is even, and the minimum degree is at least k. If G satisfies $NC \geq \frac{1}{2}(n+k-2)$, then G has a k-factor.*

The condition $NC \geq \frac{1}{2}(n+k-2)$ is best possible, as can be seen from the following examples.

First assume that k is even. Let $T=K_{k-1}$ with $V(T)=\{a_1, \cdots, a_{k-1}\}$, and $C_i=K_{k+2p}$ $(p>0, i=1, 2)$ with $V(C_i)=\{b_{i,1}, \cdots, b_{i,k-1}, \cdots, b_{i,k+2p}\}$. Now we define a graph

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G as follows:

$V(G)=V(T)\cup V(C_1)\cup V(C_2)$ and $E(G)=E(T)\cup E(C_1)\cup E(C_2)\cup\{a_jb_{ij}\mid i=1,2$ and $1\le j\le k-1\}$. Then G is connected and $\delta(G)\ge k$. Also G has no k-factor, because $\theta(\varnothing, T)=k\cdot 0+0\cdot|T|-h_G(\varnothing, T)=-2$, where θ and h_G are as will be defined in the statement of Lemma 1.3 (note that $k|C_i|+e_G(C_i, T)=k(k+2p)+k-1=k(k+2p+1)-1\equiv 1 \pmod 2$). However, for $u\in V(T)$ and $v\in C_1$ or C_2 with $uv\notin E(G)$, $N(u, v)=(k-2)+(k+2p-1)+1=(n+k-3)/2$, and for $u\in C_1$ and $v\in C_2$, $N(u, v)\ge 2(k+2p-1)>(n+k-3)/2$, and hence G satisfies $NC\ge(n+k-3)/2$.

Next assume that k is odd. Let $p>0$ be an integer. Let $S=K_1$, $T=K_k$, and $C_i=K_{k+2p}$ $(i=1, 2)$, and define a graph G by

$$G=S+(T\cup C_1\cup C_2)\,.$$

Then we have $\theta(S, T)=k+(k-1-k)|T|-h_G(S, T)=-2$ because $k|C_i|+e_G(C_i, T)=k(k+2p)+0\equiv 1 \pmod 2$. Also for $u, v\in T\cup C_1\cup C_2$ with $uv\notin E(G)$, we get $N(u, v)\ge |S|+(|T|-1)+(|C_2|-1)=2p+2k-1=\frac{1}{2}(n+k-3)$ (since $n=3k+4p+1$).

For the special case where $k=2$, we have the following theorem, in which all conditions are best (for example, $K_2+(3K_2)$ does not have a 2-factor).

THEOREM 1.2. *Let G be a connected graph of order $n\ge 9$ such that the minimum degree is at least 2. If $NC\ge n/2$, then G has a 2-factor.*

We conclude this introductory section by stating a criterion for the existence of a k-factor.

LEMMA 1.3 (Tutte). *A graph G has a k-factor if and only if*

$$\theta(S, T):=k|S|+\sum_{v\in T}\deg_{G-S}(v)-k|T|-h_G(S, T)\ge 0$$

for any disjoint subsets S, T of $V(G)$, where $h_G(S, T)$ denotes the number of connected components C of $G-(S\cup T)$ such that $k|C|+e_G(C, T)\equiv 1 \pmod 2$. Furthermore, whether G has a k-factor or not, we have $\theta(S, T)\equiv k|V(G)| \pmod 2$ for any disjoint subsets S and T of $V(G)$.

2. Proof of Theorem 1.1.

First we state some numerical results which are often applied in the proof of Theorem 1.1.

LEMMA 2.1. *Let n, s, t, m_1, m_2, and w_0 be nonnegative integers. Also, suppose that $m_i\ge 3$ $(i=1, 2)$ and $(m_1+m_2)w_0\le 2(n-s-t)$. Then the following hold.*

(i) *If $w_0\ge 4$, then $m_1+m_2+s+t-2\le\frac{1}{2}(n+s+t-3w_0+8)$.*

(ii) *If $w_0\ge 5$, then $m_1+m_2+s+t-2\le\frac{1}{5}(2n+3s+3t-6w_0+20)$.*

LEMMA 2.2.

$$9k-1-4\sqrt{2(k-1)^2+2}\begin{cases} >3k+5\,, & \text{for} \quad k\geq 4 \\ >3k+4\,, & \text{for} \quad k=3 \\ =3k+3\,, & \text{for} \quad k=2\,. \end{cases}$$

Let k, n, G be as in Theorem 1.1, and suppose that G has no k-factor. We aim at deducing a contradiction. By Lemma 1.3, we have $\theta(S, T)\leq -2$ for some disjoint subsets S and T of $V(G)$. We have $S\cup T\neq\varnothing$ because $\theta(\varnothing, \varnothing)=0$. We choose such subsets S and T so that $|S\cup T|$ is as large as possible. Then we have the following lemma.

LEMMA 2.3 ([7]). *We have* $\deg_{G-S}(u)\geq k+1$ *and* $e_G(u, T)\leq k-1$ *for all vertices* $u\in G-(S\cup T)$. *Further we have* $|C|\geq 3$ *for all components* C *of* $G-(S\cup T)$.

For convenience, we set $U:=G-(S\cup T)$ and let $C_1, \cdots, C_w$ be the components C of U, labelled so that $|C_1|\leq\cdots\leq|C_w|$, where w denotes the number of components of U. We also let $s=|S|$, $t=|T|$ and $m_i=|C_i|$. Since $w\geq h_G(S, T)$, it follows from the inequality $\theta(S, T)\leq -2$ that

$$w\geq ks+\sum_{v\in T}\deg_{G-S}(v)-kt+2\,. \tag{1}$$

Further, by Lemma 2.3, we also have

$$n-s-t\geq 3w\,. \tag{2}$$

Case 1. $T=\varnothing$. Since $t=0$, (1) becomes

$$w\geq ks+2\,. \tag{3}$$

From (2) and (3), we obtain $n-s\geq 3w\geq 3(ks+2)$. Therefore we have $s\leq(n-6)/(3k+1)$. By (3), $w\geq 2$. Also we have

$$m_1+m_2\leq\frac{2(n-s)}{w}\leq\frac{2(n-s)}{ks+2} \qquad \text{(by (3))}\,.$$

For $y_i\in V(C_i)$ $(i=1, 2)$, $N(y_1, y_2)\leq m_1+m_2-2+s$. When $s=1$,

$$N(y_1, y_2)\leq\frac{2(n-1)}{k+2}-2+1<\frac{1}{2}n\,.$$

When $s\geq 2$,

$$N(y_1, y_2)\leq\frac{2(n-1)}{2k+2}-2+\frac{n-6}{3k+1}<\frac{n}{3}-2+\frac{n}{7}<\frac{n}{2}\,.$$

Thus in either case, $N(y_1, y_2)<n/2$, which contradicts the assumption that G satisfies $NC\geq(n+k-2)/2$.

We define h_1 to be equal to the minimum of the degree in $G-S$ of a vertex in T, and let $x_1\in\{v\in T\mid \deg_{G-S}(v)=h_1\}$.

Case 2. $T\neq\emptyset$ and $h_1\geq k+1$. We set $w_0:=ks+(h_1-k)t+2$. Then we clearly have $w\geq w_0$.

Subcase 2.1. $w_0\geq 4$. In this subcase $t\geq 2$ or $s\geq 1$, or $h_1\geq k+2$. Therefore for $y_i\in V(C_i)$ $(i=1,2)$, we have

$$\begin{aligned}N(y_1,y_2)&\leq m_1+m_2+s+t-2\\&\leq\tfrac{1}{2}(n+s+t-3w_0+8)\qquad\text{(by Lemma 2.1 (i))}\\&=\tfrac{1}{2}[n+s+t-3\{ks+(h_1-k)t+2\}+8]\\&=\tfrac{1}{2}\{n+(1-3k)s+(1-3h_1+3k)t+2\}<\tfrac{1}{2}n\,.\end{aligned}$$

This is a contradiction.

Subcase 2.2. $w_0=3$. Note that in this subcase $t=1$ and $s=0$ and $h_1=k+1$. Since $S\cup T=\{x_1\}$ and $\deg_{G-S}(x_1)=k+1$ by the assumptions of this subcase, it follows from Lemma 2.3 and the connectedness of G that $|V(C_i)|\geq k+2$ $(i=1,2,3)$. Hence there is a vertex y_i in C_i $(i=1,2,3)$ which is not adjacent to $x_1\in T$. For these vertices, we have $N(x_1,y_i)\leq m_i-1+e_G(T,C_{i+1}\cup C_{i+2})$ $(i=1,2,3)$ (we take $C_4=C_1$ and $C_5=C_2$). Therefore

$$\begin{aligned}\min N(x_1,y_i)&\leq\frac{1}{3}\sum_{i=1}^{3}N(x_1,y_i)\leq\frac{1}{3}\sum_{i=1}^{3}m_i-1+\frac{2}{3}e_G(T,U)\\&\leq\frac{1}{3}(n-1)-1+\frac{2}{3}(k+1)<\frac{1}{2}(n+k-2)\qquad(\text{since } n>k+2)\,.\end{aligned}$$

Case 3. $0\leq h_1\leq k$ and $T-N_T[x_1]=\emptyset$. Since $s\geq k-h_1$, we have $w\geq ks+(h_1-k)t+2\geq(k-h_1)(k-t)+2\geq 2$ (note that $t\leq h_1+1$). We claim that $C_i-N_G(x_1)\neq\emptyset$ for each $1\leq i\leq w$. Let $1\leq i\leq w$, and take $u\in C_i$. Then $k+1\leq\deg_{G-S}(u)\leq|C_i|-1+|T|$ by Lemma 2.3. Hence by the assumptions of Case 3, $|N_{G-S}(x_1)|=h_1<k+1\leq|C_i|-1+|T|=|C_i|+|N_T(x_1)|$, which implies $C_i-N_{G-S}(x_1)\neq\emptyset$, as desired.

Subcase 3.1. $w\geq 3$. We have $n-s-t\geq 3w\geq 3\{ks+(h_1-k)t+2\}$. Hence

$$n+(3k-3h_1-1)t-6\geq(3k+1)s\,. \tag{4}$$

Let y_1 be a vertex of $C_1-N_G(x_1)$. From the hypotheses of the theorem and the assumption of this subcase, we obtain

$$\frac{n+k-2}{2}\leq N(x_1,y_1)\leq s+h_1+|C_1|-1\leq s+h_1+\frac{n-s-t}{3}-1\,.$$

Therefore we have

$$n+2t+3k-6h_1\leq 4s\,. \tag{5}$$

From (4) and (5), we have $(3k+1)(n+2t+3k-6h_1)\leq 4n+4(3k-3h_1-1)t-24$. Hence

$$\begin{aligned}(k-1)n &\le (6k+2-4t)h_1+(2k-2)t-k(3k+1)-8\\ &\le (6k+2-4t)k+(2k-2)t-k(3k+1)-8\\ &= (-2k-2)t+3k^2+k-8 \le 3k^2-k-10\,.\end{aligned}$$

Therefore, we obtain the following inequality which, in view of Lemma 2.2, contradicts the assumption $n \ge 9k-1-4\sqrt{2(k-1)^2+2}$:

$$n \le \frac{3k^2-k-10}{k-1} = 3k+2-\frac{8}{k-1} < 3k+2\,.$$

Subcase 3.2. $w=2$. We divide the proof of this subcase further into two subcases.

(i) $h_1=k-1$, $t=k$ and $s=1$. Take $y_1 \in C_1-N_G(x_1)$. Then

$$N(x_1, y_1) \le s+t-1+|C_1|-1 \le k+\frac{n-(k+1)}{2}-1=\frac{1}{2}(n+k-3)\,.$$

This is a contradiction.

(ii) $h_1=k$ and $s=0$. For $i=1, 2$, take $y_i \in C_i-N_G(x_1)$. Let $p_i=|N_G(x_1)\cap C_i|$ $(i=1, 2)$. Then $\deg_{G-S}(x_1)=p_1+p_2+t-1=k$. Therefore, we have $p_1+p_2=k+1-t$. Moreover, we have $|C_1|+|C_2|=n-t$. Hence we get the following inequalities:

$$\begin{aligned}\min(N(x_1, y_i) : i=1, 2) &\le \frac{1}{2}(N(x_1, y_1)+N(x_1, y_2))\\ &\le \frac{1}{2}[|C_1|-1+(t-1)+p_2+|C_2|-1+(t-1)+p_1]\\ &= \frac{n-t}{2}+t-2+\frac{1}{2}(k+1-t)=\frac{1}{2}(n+k-3)\,.\end{aligned}$$

This is a contradiction. This concludes the discussion for the case $T-N_T[x_1]=\varnothing$.

We henceforce assume $T-N_T[x_1]\neq\varnothing$. We define h_2 to be equal to the minimum of the degree in $G-S$ of a vertex in $T-N_T[x_1]$, and let $x_2 \in \{v \in T-N_T[x_1] \mid \deg_{G-S}(v)=h_2\}$. Since x_1 and x_2 are nonadjacent, $(n+k-2)/2 \le N(x_1, x_2) \le s+h_1+h_2$, and hence

$$2s \ge n+k-2(h_1+h_2+1)\,. \tag{6}$$

For convenience, we set $p=|N_T[x_1]|$.

Case 4. $0 \le h_1 \le h_2 \le k-1$. In this case, we have

$$(k-h_2)(n-s-t) \ge n-s-t \ge w \ge ks+(h_1-k)p+(h_2-k)(t-p)+2\,.$$

Since $p \le h_1+1$, this implies

$$(2k-h_2)s \le (k-h_2)n+(h_2-h_1)(h_1+1)-2\,. \tag{7}$$

Consequently,

$$\begin{aligned}0 &\le -h_2n+2(h_2-h_1)(h_1+1)-4-k(2k-h_2)+2(2k-h_2)(h_1+h_2+1) \quad \text{(by (6) and (7))}\\ &\le -4h_2^2+(9k-n-2)h_2-(2k^2-4k+4) \quad (\text{since } h_1\le h_2\le k-1)\\ &\le -4h_2^2+4\sqrt{2(k-1)^2+2}\,h_2-2(k-1)^2-2 \quad (\text{since } n>9k-2-4\sqrt{2(k-1)^2+2})\\ &= -4\left(h_2-\sqrt{\frac{(k-1)^2+1}{2}}\right)^2\le 0\,.\end{aligned}$$

Further we have strict inequality in the third inequality or in the last inequality according to whether $h_2\neq 0$ or $h_2=0$. This is a contradiction.

Case 5. $0\le h_1\le k$ and $k\le h_2\le k+1$.

Subcase 5.1. $k\ge 3$. We have

$$n-s-2\ge n-s-t\ge 3w\ge 3\{ks+(h_1-k)p+(h_2-k)(t-p)+2\}\,.$$

Therefore, we get

$$\begin{aligned}(3k+1)s &\le n-3(h_1-k)p-3(h_2-k)-8 \quad (\text{since } t\ge p+1)\\ &\le n+3(k-h_1)(h_1+1)-3(h_2-k)-8 \end{aligned}\tag{8}$$

(the second inequality follows from the assumption that $k\ge h_1$ and the fact that $p\le h_1+1$). From (6) and (8), we obtain

$$(3k+1)\{n+k-2(h_1+h_2+1)\}\le 2n+6(k-h_1)(h_1+1)-6(h_2-k)-16\,.$$

Hence we have

$$\begin{aligned}(3k-1)n &\le 2(3k+1)(h_1+h_2+1)-k(3k+1)+6(k-h_1)(h_1+1)-6(h_2-k)-16\\ &\le 2(3k+1)(h_1+k+2)-k(3k+1)+6(k-h_1)(h_1+1)-22\\ &\le 2(3k+1)(2k+2)-3k^2-k-22=9k^2+15k-18\,.\end{aligned}$$

Therefore, we obtain $n\le 3k+5-12/(3k-1)$. This implies that

$$n\le\begin{cases}3k+3 & (3\le k\le 4)\\ 3k+4 & (k\ge 5)\,.\end{cases}$$

In view of Lemma 2.2, this contradicts the assumption $n\ge 9k-1-4\sqrt{2(k-1)^2+2}$.

Subcase 5.2. $k=2$. By (6), we have

$$n\le 2(s+h_1+h_2)\,. \tag{9}$$

Further, since $t\ge p+1$, we obtain the following inequalities:

$$\begin{aligned}n-s-(p+1) &\ge 3w\ge 3\{ks+(h_1-k)p+(h_2-k)(t-p)+2\}\\ &\ge 3\{ks+(h_1-k)p+(h_2-k)+2\}\,.\end{aligned}$$

Since $k=2$, this implies

$$n \geq 7s+(3h_1-5)p+3h_2+1 . \tag{10}$$

From (9) and (10), we have $2(s+h_1+h_2) \geq 7s+(3h_1-5)p+3h_2+1$. Therefore we have $5s \leq h_1-(3h_1-5)p-1$. When $0 \leq h_1 \leq k-1=1$, from $p \leq h_1+1$, we have $5s \leq h_1+(5-3h_1)(h_1+1)-1=4$. Since s is a nonnegative integer, $s=0$. When $h_1=k=2$, we have $5s \leq 2-p-1 \leq 0$ (because $p \geq 1$). Therefore, we again have $s=0$. This implies that $h_1=k=2$ and $h_2=3$ because otherwise, by (9), we have $n \leq 8$, which is against our assumption $n \geq 9$. But then from (9), we get $n \leq 10$, and from (10), we get $n \geq p+3h_2+1 \geq 3h_2+2 \geq 11$. This is a contradiction.

Case 6. $0 \leq h_1 \leq k$ and $h_2 \geq k+2$. In this case, we have

$$\begin{aligned} w &\geq ks+(h_1-k)p+(h_2-k)(t-p)+2 \\ &\geq (k-h_1)(k-p)+2(t-p)+2 \geq 4 . \end{aligned}$$

Now we set $s=k-h_1+\varepsilon_1$, $h_2=k+2+\varepsilon_2$, and $t=p+1+\varepsilon_3$. Then the ε_i $(i=1, 2, 3)$ are nonnegative integers. First assume at least one of the ε_i is a positive integer. Then we have $w \geq 5$. For $y_i \in C_i$ $(i=1, 2)$, we have

$$\begin{aligned} \frac{n+k-2}{2} &\leq N(y_1, y_2) \leq m_1+m_2-2+s+t \\ &\leq \frac{1}{5}(2n+3s+3t-6w_1+20) \qquad \text{(by Lemma 2.1 (ii))} , \end{aligned}$$

where w_1 stands for $ks+(h_1-k)p+(h_2-k)(t-p)+2$. From the above inequalities, we obtain

$$\begin{aligned} n &\leq 6s+6t+50-5k-12\{ks+(h_2-k)t+(h_1-h_2)p+2\} \\ &= -6(2k-1)s+\{6-12(h_2-k)\}t+26-5k+12(h_2-h_1)p \\ &= -6(2k-1)(k-h_1+\varepsilon_1)+6(-3-2\varepsilon_2)(p+1+\varepsilon_3)-5k+26+12(k+2+\varepsilon_2-h_1)p \\ &\leq -6(2k-1)(k-h_1+\varepsilon_1)-6(3+2\varepsilon_2)(\varepsilon_3+1)-5k+26+\{12(k-h_1)+6\}(h_1+1) \\ &\leq -6(2k-1)\varepsilon_1-6(3+2\varepsilon_2)(\varepsilon_3+1)-5k+26+6(k+1) \\ &= k+14-6(2k-1)\varepsilon_1-12\varepsilon_2\varepsilon_3-12\varepsilon_2-18\varepsilon_3 \leq k+2 . \end{aligned}$$

This is a contradiction. Finally, assume $\varepsilon_1=\varepsilon_2=\varepsilon_3=0$. Then we have $w \geq (k-h_1)(k-p)+4 \geq 4$. For $y_i \in C_i$ $(i=1, 2)$, we have

$$\frac{n+k-2}{2} \leq N(y_1, y_2) \leq m_1+m_2-2+s+t \leq \frac{2}{4}(n-s-t)-2+s+t .$$

Hence, we have $k-2 \leq s+t-4=k-h_1+p+1-4$, that is to say, $p \geq h_1+1$. Since $p \leq h_1+1$, this implies $p=h_1+1$. So any vertex in C_1 is independent of x_1. Hence if we let $y_1 \in C_1$, then we have

$$\frac{n+k-2}{2} \le N(x_1, y_1) \le s+t+|C_1|-1$$
$$\le \frac{1}{4}(n+3s+3t-4) \qquad (\text{since } |C_1| \le \tfrac{1}{4}(n-s-t)).$$

Consequently, we obtain $2n+2k-4 \le n+3s+3t-4$. Therefore,

$$n \le 3s+3t-2k = 3s+3(p+1)-2k = 3(k-h_1)+3(h_1-2)-2k$$
$$= k+6 < 3k+3 \qquad (\text{since } k \ge 2).$$

This is a contradiction, and this completes the proof of Theorem 1.1.

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References

[1] G. CHARTRAND and L. LESNIAK, *Graphs & Digraphs*. Wadsworth (1986).
[2] Y. EGAWA and H. ENOMOTO, Sufficient conditions for the existence of k-factors, *Recent Studies in Graph Theory* (ed. V. Kulli), Vishwa (1989), 96–105.
[3] T. IIDA, A neighborhood condition for the existence of k-factors, *Proc. JSPS Workshop on Graph Theory and Combinatorics* (1993), 21–35.
[4] T. IIDA and T. NISHIMURA, An Ore-type condition for the existence of k-factors in graphs, Graphs Combin. 7 (1991), 353–361.
[5] M. KANO and N. TOKUSHIGE, Binding numbers and f-factors of graphs, J. Combin. Theory Ser. B **54** (1992), 213–221.
[6] P. KATERINIS, Minimum degree of a graphs and the existence of k-factors, Proc. Indian Acad. Sci. **94** (1985), 123–127.
[7] P. KATERINIS and D. R. WOODALL, Binding numbers of graphs and the existence of k-factors, Quart. J. Math. **38** (1987), 221–228.
[8] W. T. TUTTE, The factors of graphs, Canad. J. Math. **4** (1952), 314–328.

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