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Estimates for the Operators $V^{\alpha}(-\Delta+V)^{-\beta}$ and $V^{\alpha}\nabla(-\Delta+V)^{-\beta}$ with Certain Non-negative Potentials V

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1. Introduction and main results.

Let $V\in L^1_{loc}(\mathbf{R}^n)$, $n\geq 3$, be a non-negative potential and consider the Schrödinger operator $L=-\Delta+V$. If V belongs to the reverse Hölder class B_q, the L^p boundedness of the operators VL^{-1}, $V^{1/2}L^{-1/2}$, $V^{1/2}\nabla L^{-1}$, and $\nabla L^{-1/2}$ were proved by Shen ([Sh]). For operators of the type VL^{-1} and $V^{1/2}\nabla L^{-1}$, these results were generalized as follows ([KS]). We replace Δ by the second order uniformly elliptic operator L_0 and let $L=L_0+V$. Suppose V satisfy the same condition as above. Then, the operators VL^{-1} and $V^{1/2}\nabla L^{-1}$ are bounded on weighted L^p spaces and Morrey spaces.

The purpose of this paper is to extend Shen's results to another direction. More precisely, we shall investigate the L^p-L^q boundedness of the operators

$$T_1=V^{\alpha}(-\Delta+V)^{-\beta}\,,\qquad 0\leq\alpha\leq\beta\leq 1\,,$$
$$T_2=V^{\alpha}\nabla(-\Delta+V)^{-\beta}\,,\qquad 0\leq\alpha\leq 1/2\leq\beta\leq 1\,,\quad \beta-\alpha\geq 1/2$$

on $\mathbf{R}^n$, $n\geq 3$. We obtain weighted estimates for T_1 and T_2 and their boundedness on Morrey spaces.

Shen established the estimate of the fundamental solution of the Schrödinger operator by using the auxiliary function $m(x, V)$ which was introduced by himself. One of his idea is to combine the estimates of the fundamental solution with the technique of decomposing $\mathbf{R}^n$ into spherical shells $\{x\mid 2^{j-1}r<|x|\leq 2^j r\}$, $r=\{m(x, V)\}^{-1}$, for estimating various integral operator (see [Sh, Theorem 4.13, Theorem 5.10]). We shall prove our theorems by similar methods.

As we mentioned above, for the special values of α, β, the estimate for T_1 and T_2 were proved in [Sh] and [KS]. For the operator T_2, our theorem does not cover the case $(\alpha, \beta)=(0, 1/2)$. To prove this case Shen's advanced method is needed (see [Sh, Theorem 0.5] and its proof).

In their paper ([Sh], [KS]), the authors obtained pointwise estimates for the

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operators using the Hardy-Littlewood maximal operator M. In this paper we use the fractional maximal operator M_γ to obtain the estimates for T_1 and T_2. Note that $M_0 = M$.

We shall repeat the definitions of the potential class B_q (e.g. [Sh]) and the fractional maximal operator M_γ, weight class $A_{p,q}$ (e.g. [MW]), and the Morrey space (e.g. [CF]).

Throughout this paper we denote the ball centered at x with radius r by $B(x, r)$.

Definition 1. Let $V \geq 0$.

(1) For $1 < q < +\infty$ we say $V \in B_q$, if there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B V(x)^q dx\right)^{1/q} \leq \frac{C}{|B|}\int_B V(x)dx$$

for every ball $B \subset \mathbf{R}^n$.

(2) We say $V \in B_\infty$, if there exists a constant C such that

$$\|V\|_{L^\infty(B)} \leq \frac{C}{|B|}\int_B V(x)dx$$

for every ball $B \subset \mathbf{R}^n$.

Remark 1. (1) If V is a polynomial, then V belongs to B_∞ ([Fe]).

(2) For $1 < q < +\infty$, it is easy to see $\mathrm{B}_\infty \subset B_q$.

(3) If $V \in B_q$, then V is a Muckenhoupt A_∞ weight, and hence $V(x)dx$ is a doubling measure ([CoF]), that is, there is a constant C such that

$$\int_{B(x,2r)} V(y)dy \leq C\int_{B(x,r)} V(y)dy\,.$$

For other properties of the class B_q, see [KS, Remark 1].

Definition 2. Let $f \in L^1_{loc}(\mathbf{R}^n)$. For $0 \leq \gamma < n$ the fractional maximal operator is defined by

$$M_\gamma f(x) = \sup_{x \in B} \frac{1}{|B|^{1-\gamma/n}}\int_B |f(y)|dy\,, \qquad x \in \mathbf{R}^n\,,$$

where the supremum on the right side is taken over all balls $B \subset \mathbf{R}^n$.

Definition 3. Let $1 < p < +\infty$ and $1 < q < +\infty$. For a non-negative function $w(x)$, we say $w \in A_{p,q}$ if

$$\left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{1/q}\left(\frac{1}{|B|}\int_B w(x)^{-p/(p-1)}dx\right)^{(p-1)/p} \leq C$$

holds for all ball $B \subset \mathbf{R}^n$, where C is a constant independent of B.

Definition 4. For $0 \leq \lambda < n$ and $1 \leq p < +\infty$, the Morrey space is defined by

$$L^{p,\lambda}(\mathbf{R}^n)=\left\{f\in L^p_{loc}(\mathbf{R}^n)\,:\,\|f\|_{p,\lambda}=\sup_{\substack{r>0\\ x\in\mathbf{R}^n}}\left(\frac{1}{r^\lambda}\int_{B(x,r)}|f(y)|^p dy\right)^{1/p}<+\infty\right\}.$$

Note that $L^{p,0}(\mathbf{R}^n)=L^p(\mathbf{R}^n)$.

Next we state main results of this paper. If V belongs to B_∞, we obtain pointwise estimates for T_1 and T_2 and, if V belongs to B_q, we obtain pointwise estimates for the adjoint operators T_1^* and T_2^*.

THEOREM 1. *Suppose $V\in B_\infty$ and $0<\alpha\le\beta\le 1$. Then there exists a constant C such that*

$$|T_1 f(x)|\le CM_\gamma(|f|)(x)\,,\qquad f\in C_0^\infty(\mathbf{R}^n)\,,$$

where $\gamma=2(\beta-\alpha)$.

THEOREM 2. *Suppose $V\in B_\infty$, $0<\alpha\le 1/2<\beta\le 1$, and $\beta-\alpha\ge 1/2$. Then there exists a constant C such that*

$$|T_2 f(x)|\le CM_\gamma(|f|)(x)\,,\qquad f\in C_0^\infty(\mathbf{R}^n)\,,$$

where $\gamma=2(\beta-\alpha)-1$.

REMARK 2. In Theorem 1 the case $(\alpha,\beta)=(1,1)$ and in Theorem 2 the case $(\alpha,\beta)=(1/2,1)$ was shown in [KS, Theorem 1].

THEOREM 3. *Suppose $V\in B_{q_1}$ for some $q_1>n/2$, $0<\alpha\le\beta\le 1$, and let $1/q_2=1-\alpha/q_1$. Then there exists a constant C such that*

$$|T_1^* f(x)|\le C\{M_{\gamma q_2}(|f|^{q_2})(x)\}^{1/q_2}\,,\qquad f\in C_0^\infty(\mathbf{R}^n)\,,$$

where $\gamma=2(\beta-\alpha)$.

THEOREM 4. *Suppose $V\in B_{q_1}$ for some $q_1>n/2$, $0<\alpha\le 1/2<\beta\le 1$, and $\beta-\alpha\ge 1/2$. And let*

$$1/q_2=\begin{cases}1-\alpha/q_1\,, & \text{if}\ \ q_1\ge n\,,\\ 1-(\alpha+1)/q_1+1/n\,, & \text{if}\ \ n/2<q_1<n\,.\end{cases}$$

Then there exists a constant C such that

$$|T_2^* f(x)|\le C\{M_{\gamma q_2}(|f|^{q_2})(x)\}^{1/q_2}\,,\qquad f\in C_0^\infty(\mathbf{R}^n)\,,$$

where $\gamma=2(\beta-\alpha)-1$.

REMARK 3. In Theorem 3 the case $(\alpha,\beta)=(1,2)$ and the case $(\alpha,\beta)=(1/2,1/2)$ were shown in [KS, Theorem 2(1)], [Sh, Theorem 5.10] respectively. In Theorem 4 the case $(\alpha,\beta)=(1/2,1)$ was shown in [Sh, Theorem 4.13].

Weighted norm inequalities for fractional maximal operators and fractional integral operators and their boundedness in Morrey spaces are known (e.g. [MW], [Sa],

[CF], and [FR]). Therefore from these theorems we can obtain weighted estimates and boundedness on Morrey spaces for T_1 and T_2. We shall next state these estimates as corollaries.

COROLLARY 1. *Assume $V \in B_\infty$ and $0 \le \alpha \le \beta \le 1$.*

(1) *Let $1 < p < n/\gamma$ and let $1/q = 1/p - \gamma/n$, where $\gamma = 2(\beta - \alpha)$. We assume $w \in A_{p,q}$. Then there exists a constant C such that*

$$\|(T_1 f)w\|_{L^q(\mathbf{R}^n)} \le C\|fw\|_{L^p(\mathbf{R}^n)}, \qquad f \in C_0^\infty(\mathbf{R}^n).$$

(2) *Let $1 < p < n/\gamma$ and let $0 \le \lambda < n - \gamma p$, where $\gamma = 2(\beta - \alpha)$. Then there exists a constant C such that*

$$\|T_1 f\|_{q,\lambda} \le C\|f\|_{p,\lambda}, \qquad f \in C_0^\infty(\mathbf{R}^n),$$

where $1/q = 1/p - \gamma/(n - \lambda)$.

COROLLARY 2. *Assume $V \in B_\infty$, $0 \le \alpha \le 1/2 \le \beta \le 1$, and $\beta - \alpha \ge 1/2$.*

(1) *Let $1 < p < n/\gamma$ and let $1/q = 1/p - \gamma/n$, where $\gamma = 2(\beta - \alpha) - 1$. We assume $w \in A_{p,q}$. Then there exists a constant C such that*

$$\|(T_2 f)w\|_{L^q(\mathbf{R}^n)} \le C\|fw\|_{L^p(\mathbf{R}^n)}, \qquad f \in C_0^\infty(\mathbf{R}^n).$$

(2) *Let $1 < p < n/\gamma$ and let $0 \le \lambda < n - \gamma p$, where $\gamma = 2(\beta - \alpha) - 1$. Then there exists a constant C such that*

$$\|T_2 f\|_{q,\lambda} \le C\|f\|_{p,\lambda}, \qquad f \in C_0^\infty(\mathbf{R}^n),$$

where $1/q = 1/p - \gamma/(n - \lambda)$.

REMARK 4. (1) If $\alpha = \beta$ in Corollary 1 and if $\beta - \alpha = 1/2$ in Corollary 2, the condition for w is $w \in A_{p,p}$, which is equivalent to that w^p belongs to the Muckenhoupt A_p class. Moreover Corollary 1 for $(\alpha, \beta) = (1, 1)$ and Corollary 2 for $(\alpha, \beta) = (1/2, 1)$ were shown in [KS, Corollary 1(1)].

(2) For $w(x) \equiv 1$, the cases $(\alpha, \beta) = (1, 1), (1/2, 1/2)$ in Corollary 1(1) and the case $(\alpha, \beta) = (1/2, 1)$ in Corollary 2(1) were shown in [Sh, Remark 2.9], [Sh, Theorem 5.10], [Sh, Theorem 4.13] respectively.

As a corollary of Theorem 3, we obtain

COROLLARY 3. *Suppose $V \in B_{q_1}$ for some $q_1 > n/2$. Let $0 \le \alpha \le \beta \le 1$, $\gamma = 2(\beta - \alpha)$. And let*

$$1 < p < \frac{1}{\alpha/q_1 + \gamma/n}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}, \quad \frac{1}{q_2} = 1 - \frac{\alpha}{q_1}.$$

In addition assume $w^{-q_2} \in A_{q'/q_2, p'/q_2}$, where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$. Then there exists a constant C such that

$$\|(T_1 f)w\|_{L^q(\mathbf{R}^n)} \le C\|fw\|_{L^p(\mathbf{R}^n)}, \qquad f \in C_0^\infty(\mathbf{R}^n).$$

REMARK 5. (1) In Corollary 3 the case $(\alpha, \beta)=(1, 1)$ was shown in [KS, Theorem 2 (1)]. For $w(x)\equiv 1$, the cases $(\alpha, \beta)=(1, 1)$ and $(\alpha, \beta)=(1/2, 1/2)$ were shown in [Sh, Theorem 3.1], [Sh, Theorem 5.10] respectively.

(2) It is well known that if $V\in B_q$ for some $q>1$ then there exists $\epsilon>0$ such that $V\in B_{q+\epsilon}$ ([Ge]). Using this fact, under the assumption $V\in B_{n/2}$, we can also obtain weighted estimates by Theorem 3.

(3) If we take the limit $q_1\to+\infty$, then the condition $w^{-q_2}\in A_{q'/q_2,p'/q_2}$ becomes $w^{-1}\in A_{q',p'}$, which is equivalent to $w\in A_{p,q}$. Therefore, Corollary 3 is an extension of Corollary 1 (1).

As a corollary of Theorem 4, we obtain

COROLLARY 4. *Suppose $V\in B_{q_1}$ for some $q_1>n/2$. Let*

$$\begin{cases} 0\le\alpha\le 1/2\le\beta\le 1\,, & \text{if } q_1\ge n\,,\\ 0\le\alpha\le 1/2<\beta\le 1\,, & \text{if } n/2<q_1<n\,. \end{cases}$$

And let $\beta-\alpha\ge 1/2$, $\gamma=2(\beta-\alpha)-1$,

$$1<p<\frac{1}{1/p_1+\gamma/n}\,,\quad \frac{1}{q}=\frac{1}{p}-\frac{\gamma}{n}\,,\quad \frac{1}{q_2}=1-\frac{1}{p_1}\,,$$

where

$$1/p_1=\begin{cases} \alpha/q_1\,, & \text{if } q_1\ge n\,,\\ (\alpha+1)/q_1-1/n\,, & \text{if } n/2<q_1<n\,. \end{cases}$$

In addition assume $w^{-q_2}\in A_{q'/q_2,p'/q_2}$, where $1/p+1/p'=1$, $1/q+1/q'=1$. Then there exists a constant C such that

$$\|(T_2f)w\|_{L^q(\mathbf{R}^n)}\le C\|fw\|_{L^p(\mathbf{R}^n)}\,,\qquad f\in C_0^\infty(\mathbf{R}^n)\,.$$

REMARK 6. In Corollary 4 the case $w(x)\equiv 1$ and $(\alpha, \beta)=(1/2, 1)$ was shown in [Sh, Theorem 4.13].

We first note that the case $(\alpha, \beta)=(0, 1/2)$ for T_2 can be derived from the well known results. In fact, singular integral operators are bounded on $L^p(\mathbf{R}^n; w(x)dx)$ and Morrey spaces ([St, pages 205 and 221] and [CF]) and hence Corollary 2 and Corollary 4 for this case immediately follow from the fact that T_2 is a Calderón-Zygmund operator under the assumption $V\in B_{q_1}$ for some $q_1\ge n$ ([Sh, Theorem 0.8]).

For the case $V\in B_{q_1}$ for some $n/2<q_1<n$ Corollary 4 does not cover the case $(\alpha, \beta)=(0, 1/2)$ since it is not known whether T_2 is a Calderón-Zygmund operator or not. Under this assumption L^p boundedness of T_2 was directly proved by Shen but only for $w(x)\equiv 1$ ([Sh, Theorem 0.5]).

The plan of this paper is as follows. In section 2, we recall the estimates of the fundamental solution of the operator $-\Delta+(V(x)+i\tau)$, $\tau\in\mathbf{R}$, which was established by Shen. With these estimates, we prove the theorems in section 3, which contain the proof of corollaries.

2. Preliminaries.

First we recall the definition of the auxiliary function $m(x, V)$ and its properties which was obtained by Shen.

DEFINITION 5 ([Sh]). For $x \in \mathbf{R}^n$, the function $m(x, V)$ is defined by

$$\frac{1}{m(x, V)} = \sup\left\{r > 0 : \frac{r^2}{|B(x, r)|} \int_{B(x,r)} V(y)dy \le 1\right\}.$$

LEMMA 1 ([Sh]). *Assume that $V \in B_q$ for some $q > n/2$. Then there exists a constant C such that, for $0 < r < R < +\infty$,*

$$\frac{r^2}{|B(x, r)|} \int_{B(x,r)} V(y)dy \le C\left(\frac{R}{r}\right)^{n/q-2} \frac{R^2}{|B(x, R)|} \int_{B(x,R)} V(y)dy.$$

By Lemma 1 we see that

$$0 < m(x, V) < +\infty \qquad \text{for} \quad V \in B_q,\ q > n/2,$$

$$\frac{r^2}{|B(x, r)|} \int_{B(x,r)} V(y)dy = 1 \qquad \text{for} \quad r = \frac{1}{m(x, V)}.$$

Next we recall the estimate of the fundamental solution of the Schrödinger operator ([Sh]). Let $\Gamma(x, y, \tau)$ denote the fundamental solution for the operator $-\Delta + (V(x) + i\tau)$, $\tau \in \mathbf{R}$. Then we have

LEMMA 2 ([Sh]). *Let $k > 0$ be an integer.*

(1) *Under the assumption $V \in B_{n/2}$, there exists a constant C_k such that*

$$|\Gamma(x, y, \tau)| \le \frac{C_k}{\{1 + |\tau|^{1/2}|x-y|\}^k \{1 + m(x, V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-2}}.$$

(2) *Under the assumption $V \in B_n$, there exists a constant C_k such that*

$$|\nabla_x \Gamma(x, y, \tau)| \le \frac{C_k}{\{1 + |\tau|^{1/2}|x-y|\}^k \{1 + m(x, V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-1}}.$$

REMARK 7. These estimates (Lemma 2) also hold when $m(x, V)$ is replaced by $m(y, V)$. We see this by $\Gamma(x, y, \tau) = \Gamma(y, x, -\tau)$ or by the following estimate: There exist positive constants C, k_0 such that

$$C\{1 + |x-y| m(y, V)\}^{1/(k_0+1)} \le 1 + |x-y| m(x, V)$$

(see [Sh, Corollary 1.5]).

3. Proofs.

PROOF OF THEOREM 1. By the functional calculus, we may write, for all $0<\beta<1$,

$$(-\Delta+V)^{-\beta}=\frac{1}{2\pi}\int_{\mathbf{R}}(-i\tau)^{-\beta}(-\Delta+V+i\tau)^{-1}d\tau\,. \tag{1}$$

Let $f\in C_0^\infty$. From $(-\Delta+V+i\tau)^{-1}f(x)=\int_{\mathbf{R}^n}\Gamma(x,y,\tau)f(y)dy$, it follows that

$$T_1f(x)=\int_{\mathbf{R}^n}K_1(x,y)V(x)^\alpha f(y)dy\,, \tag{2}$$

where

$$K_1(x,y)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_{\mathbf{R}}(-i\tau)^{-\beta}\Gamma(x,y,\tau)d\tau\,, & \text{for } 0<\beta<1\,,\\ \Gamma(x,y,0)\,, & \text{for } \beta=1\,.\end{cases} \tag{3}$$

By Lemma 2 (1), for all $0<\beta\le 1$ and all integer $k\ge 2$, there exists a constant C_k such that

$$|K_1(x,y)|\le\frac{C_k}{\{1+m(x,V)|x-y|\}^k}\cdot\frac{1}{|x-y|^{n-2\beta}}\,. \tag{4}$$

Let $r=1/m(x,V)$. Since $V\in B_\infty$, $V(x)\le Cm(x,V)^2$ a.e. Therefore we obtain

$$\begin{aligned}|T_1f(x)|&\le\int_{\mathbf{R}^n}\frac{C_k}{\{1+m(x,V)|x-y|\}^k}\cdot\frac{1}{|x-y|^{n-2\beta}}V(x)^\alpha|f(y)|dy\\&\le CC_k\sum_{j=-\infty}^{+\infty}\int_{2^{j-1}r<|x-y|\le 2^jr}\frac{1}{\{1+r^{-1}|x-y|\}^k}\cdot\frac{1}{|x-y|^{n-2\beta}}\cdot\frac{1}{r^{2\alpha}}|f(y)|dy\\&\le CC_k\sum_{j=-\infty}^{+\infty}\frac{2^{2\alpha j}}{(1+2^{j-1})^k}\cdot\frac{1}{(2^jr)^{n-2(\beta-\alpha)}}\int_{B(x,2^jr)}|f(y)|dy\le CM_\gamma(|f|)(x)\,,\end{aligned}$$

where $\gamma=2(\beta-\alpha)$ and we choose $k\ge 3$. $\square$

PROOF OF THEOREM 2. Let $f\in C_0^\infty$. As in the proof of Theorem 1, we have

$$T_2f(x)=\int_{\mathbf{R}^2}K_2(x,y)V(x)^\alpha f(y)dy\,, \tag{5}$$

where

$$K_2(x,y)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_{\mathbf{R}}(-i\tau)^{-\beta}\nabla_x\Gamma(x,y,\tau)d\tau\,, & \text{for } 1/2<\beta<1\,,\\ \nabla_x\Gamma(x,y,0)\,, & \text{for } \beta=1\,.\end{cases} \tag{6}$$

By Lemma 2 (2), for all $1/2<\beta\le 1$ and all integer $k\ge 2$, there exists a constant C_k such that

$$|K_2(x, y)| \leq \frac{C_k}{\{1+m(x, V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-2\beta+1}} . \tag{7}$$

Then as in the proof of Theorem 1, we obtain

$$|T_2 f(x)| \leq CM_\gamma(|f|)(x) ,$$

where $\gamma = 2(\beta-\alpha)-1$. □

PROOF OF THEOREM 3. Let $f \in C_0^\infty$. Using (2) and (3), the adjoint of T_1 is given by

$$T_1^* f(x) = \int_{\mathbf{R}^n} \overline{K_1(y, x)} V(y)^\alpha f(y) dy .$$

And by Lemma 2 (1), for all integer $k \geq 2$, there exists a constant C_k such that

$$|\overline{K_1(y, x)}| \leq \frac{C_k}{\{1+m(x, V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-2\beta}} .$$

Let $r = 1/m(x, V)$. By Hölder's inequality, it follows that

$$\begin{aligned} |T_1^* f(x)| &\leq \int_{\mathbf{R}^n} \frac{C_k}{\{1+m(x, V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-2\beta}} V(y)^\alpha |f(y)| dy \\ &\leq \sum_{j=-\infty}^{+\infty} \int_{2^{j-1}r < |x-y| \leq 2^j r} \frac{C_k}{(1+2^{j-1})^k (2^{j-1}r)^{n-2\beta}} V(y)^\alpha |f(y)| dy \\ &\leq CC_k \sum_{j=-\infty}^{+\infty} \frac{(2^j r)^{2\beta}}{(1+2^{j-1})^k} \left\{ \frac{1}{(2^j r)^n} \int_{B(x, 2^j r)} V(y)^{q_1} dy \right\}^{\alpha/q_1} \\ &\quad \cdot \left\{ \frac{1}{(2^j r)^n} \int_{B(x, 2^j r)} |f(y)|^{q_2} dy \right\}^{1/q_2} . \end{aligned}$$

From the assumptions we have $0 \leq 2(\beta-\alpha) q_2 < n$. Thus, letting $\gamma = 2(\beta-\alpha)$ we see that

$$|T_1^* f(x)| \leq CC_k \{M_{\gamma q_2}(|f|^{q_2})(x)\}^{1/q_2} \sum_{j=-\infty}^{+\infty} \frac{(2^j r)^{2\alpha}}{(1+2^{j-1})^k} \left\{ \frac{1}{|B(x, 2^j r)|} \int_{B(x, 2^j r)} V(y) dy \right\}^\alpha .$$

Since $V(x)dx$ is a doubling measure, for the case $j \geq 1$ there exists a constant C_0 such that

$$\begin{aligned} \frac{(2^j r)^2}{|B(x, 2^j r)|} \int_{B(x, 2^j r)} V(y) dy &\leq 2^{2j} C_0^j C \cdot 2^{-jn} \frac{r^2}{|B(x, r)|} \int_{B(x, r)} V(y) dy \\ &= C(2^j)^{k_0} , \end{aligned} \tag{8}$$

where $k_0 = 2-n+\log_2 C_0$. For the case $j \leq 0$, by Lemma 1 we obtain

$$\begin{aligned} \frac{(2^j r)^2}{|B(x, 2^j r)|} \int_{B(x, 2^j r)} V(y) dy &\leq C \left(\frac{r}{2^j r} \right)^{n/q_1 - 2} \frac{r^2}{|B(x, r)|} \int_{B(x, r)} V(y) dy \\ &= C(2^j)^{2-n/q_1} . \end{aligned} \tag{9}$$

Hence if we take k sufficiently large, we can conclude

$$|T_1^* f(x)| \leq C\{M_{\gamma q_2}(|f|^{q_2})(x)\}^{1/q_2}. \quad \square$$

PROOF OF THEOREM 4. Let $f \in C_0^\infty$. Using (5) and (6), the adjoint of T_2 is given by

$$T_2^* f(x) = \int_{\mathbf{R}^n} \overline{K_2(y, x)} V(y)^\alpha f(y) dy .$$

Case $q_1 \geq n$: By Lemma 2 (2), for all integer $k \geq 2$, there exists a constant C_k such that

$$|\overline{K_2(y, x)}| \leq \frac{C_k}{\{1 + m(x, V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-2\beta+1}} .$$

Let $r = 1/m(x, V)$. Then as in the proof of Theorem 3, it follows that

$$|T_2^* f(x)| \leq CC_k \sum_{j=-\infty}^{+\infty} \frac{(2^j r)^{2\beta-1}}{(1+2^{j-1})^k} \left\{\frac{1}{(2^j r)^n} \int_{B(x, 2^j r)} V(y)^{q_1} dy\right\}^{\alpha/q_1}$$
$$\cdot \left\{\frac{1}{(2^j r)^n} \int_{B(x, 2^j r)} |f(y)|^{q_2} dy\right\}^{1/q_2} .$$

From the assumptions we have $0 \leq \{2(\beta-\alpha)-1\} q_2 < n$. Thus, letting $\gamma = 2(\beta-\alpha)-1$ and using (8) and (9) we can conclude $|T_2^* f(x)| \leq C\{M_{\gamma q_2}(|f|^{q_2})(x)\}^{1/q_2}$.

Case $n/2 < q_1 < n$: We fix x_0, $y_0 \in \mathbf{R}^n$. Let $R = |x_0 - y_0|/4$. Then following estimate was obtained by Shen: For all positive integer k, there exists a constant C_k such that

$$|\nabla_y \Gamma(y_0, x_0, \tau)|$$
$$\leq \frac{C_k}{(1+|\tau|^{1/2} R)^k \{1 + m(x_0, V) R\}^k} \left\{\frac{1}{R^{n-2}} \int_{B(y_0, R)} \frac{V(y)}{|y-y_0|^{n-1}} dy + \frac{1}{R^{n-1}}\right\}$$

(see [Sh, page 538]). Thus, by (6), for all integer $k \geq 2$, there exists a constant C_k such that

$$|\overline{K_2(y_0, x_0)}| \leq \frac{C_k}{\{1 + m(x_0, V) R\}^k} \left\{\frac{1}{R^{n-2\beta}} \int_{B(y_0, R)} \frac{V(y)}{|y-y_0|^{n-1}} dy + \frac{1}{R^{n-2\beta+1}}\right\} .$$

Let $r = 1/m(x, V)$. We choose p_1 such that $1/p_1 = 1/q_1 - 1/n$. Then from the assumptions we have $1/p_1 + \alpha/q_1 + 1/q_2 = 1$. By Hölder's inequality, it follows that

$$|T_2^* f(x)| \leq \sum_{j=-\infty}^{+\infty} \int_{2^{j-1} r < |x-y| \leq 2^j r} |\overline{K_2(y, x)}| \, V(y)^\alpha |f(y)| dy$$
$$\leq \sum_{j=-\infty}^{+\infty} (2^j r)^n \left\{\frac{1}{(2^j r)^n} \int_{2^{j-1} r < |x-y| \leq 2^j r} |\overline{K_2(y, x)}|^{p_1} dy\right\}^{1/p_1}$$
$$\cdot \left\{\frac{1}{(2^j r)^n} \int_{B(x, 2^j r)} V(y)^{q_1} dy\right\}^{\alpha/q_1} \left\{\frac{1}{(2^j r)^n} \int_{B(x, 2^j r)} |f(y)|^{q_2} dy\right\}^{1/q_2} .$$

Using Minkowski's inequality and Hardy-Littlewood-Sobolev's inequality, we obtain

$$(2^j r)^n \left\{ \frac{1}{(2^j r)^n} \int_{2^{j-1}r < |x-y| \le 2^j r} \overline{|K_2(y, x)|^{p_1}} dy \right\}^{1/p_1}$$

$$\le \frac{CC_k(2^j r)^n}{(1+2^{j-3})^k} \left[\frac{1}{(2^j r)^{n-2\beta-1}} \left\{ \frac{1}{(2^j r)^n} \int_{|x-y| \le 2^{j+1}r} V(y)^{q_1} dy \right\}^{1/q_1} + \frac{1}{(2^j r)^{n-2\beta+1}} \right]$$

$$\le CC_k \frac{(2^j r)^{2\beta-1}}{(1+2^{j-3})^k} \left\{ \frac{(2^j r)^2}{|B(x, 2^j r)|} \int_{B(x,2^j r)} V(y) dy + 1 \right\}.$$

For the case $j \ge 1$, using (8) we obtain

$$(2^j r)^n \left\{ \frac{1}{(2^j r)^n} \int_{2^{j-1}r < |x-y| \le 2^j r} \overline{|K_2(y, x)|^{p_1}} dy \right\}^{1/p_1} \le CC_k \frac{(2^j)^{k_0}(2^j r)^{2\beta-1}}{(1+2^{j-3})^k}.$$

For the case $j \le 0$, using (9) we obtain

$$(2^j r)^n \left\{ \frac{1}{(2^j r)^n} \int_{2^{j-1}r < |x-y| \le 2^j r} \overline{|K_2(y, x)|^{p_1}} dy \right\}^{1/p_1} \le CC_k \frac{(2^j r)^{2\beta-1}}{(1+2^{j-3})^k}.$$

Then it follows that

$$|T_2^* f(x)| \le CC_k \{M_{\gamma q_2}(|f|^{q_2})(x)\}^{1/q_2} \left\{ \sum_{j=1}^{+\infty} \frac{(2^j)^{k_0}(2^j r)^{2\alpha}}{(1+2^{j-3})^k} + \sum_{j=-\infty}^{0} \frac{(2^j r)^{2\alpha}}{(1+2^{j-3})^k} \right\}$$
$$\cdot \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V(y) dy \right\}^{\alpha},$$

where $\gamma = 2(\beta - \alpha) - 1$. Using (8) and (9) once more, and taking k sufficiently large, we finally obtain $|T_2^* f(x)| \le C\{M_{\gamma q_2}(|f|^{q_2})(x)\}^{1/q_2}$. □

PROOF OF COROLLARY 1. (1). Case $\alpha > 0$: Under the assumption $w \in A_{p,q}$, weighted L^p-L^q estimate holds for the fractional maximal operator ([MW]). This estimate and Theorem 1 yield the desired estimate.

Case $\alpha = 0$: Let $K_1(x, y)$ be the kernel of the operator T_1. Note that if $V \in L_{loc}^{n/2}(\mathbf{R}^n)$, $V \ge 0$, there is a constant C such that

$$|K_1(x, y)| \le \frac{C}{|x-y|^{n-2\beta}}, \tag{10}$$

where $0 < \beta \le 1$. Since under the assumption $w \in A_{p,q}$ weighted L^p-L^q estimate holds for the fractional integral operator ([MW]), the desired estimate follows.

(2). Case $\alpha > 0$: For $0 \le \gamma < n$ the operator M_γ is bounded on Morrey spaces ([CF] and [FR]). By Theorem 1 T_1 is also bounded there.

Case $\alpha = 0$: The fractional integral operator is bounded on Morrey space ([CF]). Hence by (10) T_1 is also bounded there. □

At the end of Section 1 we have mentioned that Corollary 2 and Corollary 4 are true for the case $(\alpha, \beta)=(0, 1/2)$. Hence we prove these corollaries for other cases in this section.

PROOF OF COROLLARY 2. (1). Case $\alpha>0$: Using Theorem 2 and the same idea of the proof of Corollary 1(1), we arrive at the desired estimate.

Case $\alpha=0$ and $1/2<\beta\leq 1$: Let $K_2(x, y)$ be the kernel of the operator T_2. Note that if $V\in L^{n/2}_{loc}(\mathbf{R}^n)$, $V\geq 0$, there is a constant C such that

$$|K_2(x, y)|\leq \frac{C}{|x-y|^{n-2\beta+1}}, \tag{11}$$

where $1/2<\beta\leq 1$. Since under the assumption $w\in A_{p,q}$ weighted L^p-L^q estimate holds for the fractional integral operator ([MW]), the desired estimate follows.

(2). Using Theorem 2 and the same idea of the proof of Corollary 1(2), we arrive at the desired estimate. □

PROOF OF COROLLARY 3. Case $\alpha>0$: Let $\gamma=2(\beta-\alpha)$ and let $1/q_2=1-\alpha/q_1$. And for q' such that $q_2<q'<n/\gamma$ let $1/p'=1/q'-\gamma/n$. Then from the assumptions we have

$$0\leq \gamma q_2<n\,, \quad 1<\frac{q'}{q_2}<\frac{n}{\gamma q_2}\,, \quad \frac{1}{p'/q_2}=\frac{1}{q'/q_2}-\frac{\gamma q_2}{n}\,.$$

By Theorem 3 and the boundedness of the fractional maximal operator, there exists a constant C such that

$$\|(T_1^* f)w^{-1}\|_{L^{p'}(\mathbf{R}^n)}\leq C\|fw^{-1}\|_{L^{q'}(\mathbf{R}^n)}\,, \qquad f\in C_0^\infty(\mathbf{R}^n)\,.$$

Now the desired estimate follows since $1/p+1/p'=1$ and $1/q+1/q'=1$.

Case $\alpha=0$: Since $q_2=1$ the condition for w is $w^{-1}\in A_{q',p'}$, which is equivalent to $w\in A_{p,q}$. Then as in the proof of Corollary 1(1), we arrive at the desired estimate. □

PROOF OF COROLLARY 4. Case $\alpha>0$: Let $\gamma=2(\beta-\alpha)-1$ and let $1/q_2=1-1/p_1$, where

$$1/p_1=\begin{cases}\alpha/q_1\,, & \text{if } q_1\geq n\,,\\ (\alpha+1)/q_1-1/n\,, & \text{if } n/2<q_1<n\,.\end{cases}$$

And for q' such that $q_2<q'<n/\gamma$ let $1/p'=1/q'-\gamma/n$. Then from the assumptions we have

$$0\leq \gamma q_2<n\,, \quad 1<\frac{q'}{q_2}<\frac{n}{\gamma q_2}\,, \quad \frac{1}{p'/q_2}=\frac{1}{q'/q_2}-\frac{\gamma q_2}{n}\,.$$

Then the desired estimate follows from Theorem 4 as in the proof of Corollary 3.

Case $\alpha=0$ and $1/2<\beta\leq 1$: Using (11) and the same idea of the proof of Corollary 3, we arrive at the desired estimate. □

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