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Time Periodic Solutions of the Navier-Stokes Equations under General Outflow Condition

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Abstract. It is known that there exists a time periodic solution of the Navier-Stokes equations with Dirichlet boundary conditions satisfying so-called stringent outflow condition (SOC). In this paper we will show the existence of periodic solutions of the Navier-Stokes equations with Dirichlet boundary conditions satisfying so-called general outflow condition (GOC).

1. Introduction

The purpose of this paper is to show that for a bounded domain in $\mathbf{R}^2$ the nonstationary Navier-Stokes equations with the Dirichlet boundary conditions has a time periodic solution. H. Morimoto [13] obtained the periodic solution with the time-independent Dirichlet boundary conditions. In this paper we treat such a problem with the time-dependent Dirichlet boundary conditions.

Let Ω be a bounded domain in $\mathbf{R}^2$. The domain Ω has a smooth boundary $\partial\Omega$. $\Gamma_0, \Gamma_1, \dots, \Gamma_J$ are connected components of $\partial\Omega$. The domain Ω is filled with an incompressible viscous fluid. We consider the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad (0,T)\times\Omega\,, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \quad (0,T)\times\Omega \tag{1.2}$$

with the Dirichlet boundary conditions

$$\boldsymbol{u} = \boldsymbol{\beta} \quad \text{on} \quad (0,T)\times\partial\Omega\,, \tag{1.3}$$

where $\boldsymbol{u} = (u_1(t,x), u_2(t,x))$ and $p = p(t,x)$ are the velocity and pressure of the fluid motion in Ω respectively, $\boldsymbol{f} = (f_1(t,x), f_2(t,x))$ is the prescribed external force and $\boldsymbol{\beta} = (\beta_1(t,x), \beta_2(t,x))$ is the prescribed function defined on $\partial\Omega$. The boundary condition $\boldsymbol{\beta}$ must satisfy

$$\int_{\partial\Omega} \boldsymbol{\beta}(t)\cdot\boldsymbol{n}\,d\sigma = 0 \quad (\forall t\in(0,T))\,, \tag{1.4}$$

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where $\boldsymbol{n}$ is the unit outer normal to $\partial\Omega$. We call the condition (1.4) "*General Outflow Condition*" (GOC). If $\boldsymbol{\beta}$ satisfies

$$\int_{\Gamma_j} \boldsymbol{\beta}(t)\cdot \boldsymbol{n}d\sigma = 0 \quad (\forall t\in(0,T),\ 0\le j\le J)\,, \tag{1.5}$$

the condition (1.5) is called "*Stringent Outflow Condition*" (SOC). We set the periodic condition

$$\boldsymbol{u}(0)=\boldsymbol{u}(T) \quad \text{in} \quad \Omega\,. \tag{1.6}$$

In this paper we suppose that the domain Ω satisfies the following assumption.

ASSUMPTION 1.1. A domain Ω is bounded, smooth and symmetric with respect to the x_2-axis and the boundary $\partial\Omega$ has connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_J$ and each $\Gamma_j\,(0\le j\le J)$ intersects with the x_2-axis.

In this paper we use following rules for function spaces. Let Y be a function space. Y^S is the set of Y functions symmetric with respect to the x_2-axis, that is to say, for a vector function $\boldsymbol{u}=(u_1,u_2)$ u_1 is an odd function and u_2 is an even function with respect to x_2-axis. Y_σ is the set of Y functions $\boldsymbol{\varphi}$ such that $\operatorname{div}\boldsymbol{\varphi}=0$. Y' is the dual space of Y.

$\mathbf{C}_0^\infty(\Omega)$ is the set of all real smooth vector functions with a compact support in Ω. $\mathcal{V}(\Omega)$ and $\mathcal{H}(\Omega)$ are the completion of $\mathbf{C}_{0,\sigma}^\infty(\Omega)$ with respect to the usual $\mathbf{H}^1(\Omega)$ and $\mathbf{L}^2(\Omega)$ norm respectively. $\mathbf{H}_0^1(\Omega)$ is the completion of $\mathbf{C}_0^\infty(\Omega)$ with respect to the $\mathbf{H}^1(\Omega)$ norm. $\|\cdot\|_2$ and $(\cdot,\cdot)$ denotes the $\mathbf{L}^2(\Omega)$ norm and inner product on Ω respectively. $\mathbf{H}_0^1(\Omega)$, $\mathbf{H}_0^{1,S}(\Omega)$, $\mathcal{V}(\Omega)$ and $\mathcal{V}^S(\Omega)$ are Hilbert spaces with respect to the inner product $((\boldsymbol{u},\boldsymbol{v}))=(\nabla\boldsymbol{u},\nabla\boldsymbol{u})$.

Let $\gamma\in\mathcal{L}(\mathbf{H}^1(\Omega),\mathbf{L}^2(\partial\Omega))$ be the trace operator. The space $\mathbf{H}^{\frac12}(\partial\Omega)$ denotes $\gamma(\mathbf{H}^1(\Omega))$. $\mathbf{H}^{\frac12}(\partial\Omega)$ is equipped with the norm $\|\boldsymbol{g}\|_{\mathbf{H}^{\frac12}(\partial\Omega)}=\inf_{\gamma\boldsymbol{u}=\boldsymbol{g},\boldsymbol{u}\in\mathbf{H}^1(\Omega)}\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)}$.

Let X be a Banach space. The space $C([0,T];X)$, $C^1([0,T];X)$, $L^2((0,T);X)$ and $L^\infty((0,T);X)$ are the usual Banach spaces. If $\boldsymbol{u}$ belongs to $C_\pi([0,T];X)$, $\boldsymbol{u}\in C([0,T];X)$ satisfies a periodic condition $\boldsymbol{u}(0)=\boldsymbol{u}(T)$ in X. $C_\pi^1([0,T];X)$ is similar to $C_\pi([0,T];X)$.

Our definition of a time periodic weak solution of the Navier-Stokes equations is as follows.

DEFINITION 1.1. Suppose that Ω satisfies Assumption 1.1, $\boldsymbol{\beta}\in C_\pi^1([0,T];\mathbf{H}^{\frac12,S}(\partial\Omega))$ satisfies (GOC) and $\boldsymbol{f}$ belongs to $L^2((0,T);(\mathcal{V}^S(\Omega))')$.

A measurable function $\boldsymbol{u}=\boldsymbol{u}(t,x)$ is called a weak solution of the Navier-Stokes equations (1.1), (1.2), (1.3), if $\boldsymbol{u}$ belongs to $L^2((0,T);\mathbf{H}_\sigma^{1,S}(\Omega))\cap L^\infty((0,T);\mathbf{L}^{2,S}(\Omega))$, satisfies

$$\begin{aligned}
&-\int_0^T(\boldsymbol{u},\boldsymbol{\varphi})\psi' dt+\int_0^T\{(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi})+((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{\varphi})\}\psi dt\\
&=\int_0^T {}_{(\mathcal{V}^S(\Omega))'}\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_{\mathcal{V}^S(\Omega)}\psi dt \quad (\boldsymbol{\varphi}\in\mathcal{V}^S(\Omega),\ \psi\in C_0^\infty(0,T))\,.
\end{aligned}\tag{1.7}$$

and

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{\beta} \quad \text{on} \quad (0, T) \times \partial\Omega \tag{1.8}$$

in the trace sense. The weak solution $\boldsymbol{u}$ is called a time periodic solution of the Navier-Stokes equations (1.1), (1.2), (1.3), (1.6), if $\boldsymbol{u}$ belongs to $C_\pi([0, T]; \mathbf{L}^{2,S}(\Omega))$. We call $\boldsymbol{u}$ "a time periodic weak solution of the Navier-Stokes equations (1.1), (1.2), (1.3), (1.6)."

Hereafter we represent ${}_{(\mathcal{V}^S(\Omega))'}\langle \cdot, \cdot \rangle_{\mathcal{V}^S(\Omega)}$ as $\langle \cdot, \cdot \rangle$. Our main result is a following.

THEOREM 1.1. *Suppose that Ω satisfies* Assumption 1.1, $\boldsymbol{\beta} \in C^1_\pi([0, T]; \mathbf{H}^{\frac{1}{2},S}(\partial\Omega))$ *satisfies* (GOC) *and* $\boldsymbol{f}$ *belongs to* $L^2((0, T); (\mathcal{V}^S(\Omega))')$.

Then there exist time periodic weak solutions $\boldsymbol{u}$ *of the Navier-Stokes equations* (1.1), (1.2), (1.3), (1.6).

REMARK 1.1. When the boundary condition $\boldsymbol{\beta}$ does not depend on time ($\boldsymbol{\beta} \in \mathbf{H}^{\frac{1}{2},S}(\partial\Omega)$) and satisfies (GOC), H. Morimoto [13] obtained time periodic weak solutions of the Navier-Stokes equations in the same domains. We use a following Theorem 2.2 for extensions of boundary conditions. She used Theorem 1 in H. Fujita [6] or Theorem 1 in H. Morimoto [14].

C. J. Amick [2] proved that there exist symmetric solutions of the stationary Navier-Stokes equations with the symmetric Dirichlet boundary conditions using a contradiction argument for a two dimensional bounded domain Ω satisfying Assumption 1.1. Under the same conditions as C. J. Amick [2], H. Fujita [6] proved the existence of symmetric solutions of the stationary Navier-Stokes equations with the symmetric Dirichlet boundary conditions using "*the Leray Inequality*". We know that "*the Leray Inequality*" does not hold true in general context. See A. Takeshita [15]. But we know that there exist the solutions of the nonstationary Navier-Stokes equations with the Dirichlet boundary conditions. See for example O. A. Ladyzhenskaya [10]. V. I. Yudovi č [19] proved that there exist periodic solutions of the Navier-Stokes equations under (SOC). S. Kaniel and M. Shinbrot [9] studied the uniqueness of the periodic solution of the Navier-Stokes equations with the $\mathbf{0}$ external force in three dimensional bounded domains. In infinite channels H. Beirão da Veiga [3] proved that there exist periodic solutions of the Navier-Stokes equations with a given time periodic flux. J. L. Lions [11] considered the time periodic problem for the Navier-Stokes equations with the homogeneous boundary conditions. A. Takeshita [16] studied the existence and uniqueness of periodic solutions of the Navier-Stokes equations in two dimensional bounded domains.

2. Preliminary

2.1. Some Lemmas. We use following Lemmas.

LEMMA 2.1 (the Poincaré inequality). *Let Ω be a bounded domain. Then there exists a constant $C(\Omega)$ depending only on Ω such that the inequality*

$$\|\boldsymbol{u}\|_2 \le C(\Omega)\|\nabla \boldsymbol{u}\|_2 \quad (\boldsymbol{u} \in \mathbf{H}_0^1(\Omega))$$

holds.

LEMMA 2.2 (R. Temam[17]). *For all $\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)$, the inequality*

$$\|\boldsymbol{u}\|^2_{\mathbf{L}^4(\Omega)} \le 2^{\frac{1}{2}}\|\boldsymbol{u}\|_2\|\nabla \boldsymbol{u}\|_2$$

holds.

For vector functions $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$, we define

$$((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \sum_{i,j=1}^{2} u_i \frac{\partial v_j}{\partial x_i} w_j dx\,.$$

Then the following Lemma holds.

LEMMA 2.3 (R. Temam [17]). *The inequality and the equalities hold.*

$$|((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{w})| \le C\|\nabla\boldsymbol{u}\|_2\|\nabla\boldsymbol{v}\|_2\|\nabla\boldsymbol{w}\|_2 \quad (\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w} \in \mathbf{H}_0^1(\Omega))\,,$$

$$((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{w}) = -((\boldsymbol{u}\cdot\nabla)\boldsymbol{w}, \boldsymbol{v}) \quad (\boldsymbol{u} \in \mathcal{V}(\Omega),\ \boldsymbol{v},\ \boldsymbol{w} \in \mathbf{H}^1(\Omega))\,,$$

$$((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{v}) = 0 \quad (\boldsymbol{u} \in \mathcal{V}(\Omega),\ \boldsymbol{v} \in \mathbf{H}^1(\Omega))\,.$$

LEMMA 2.4 (G. P. Galdi [7]). *For all $\varepsilon > 0$ there exist an $N \in \mathbf{N}$ and $\boldsymbol{\xi}_j \in \mathbf{L}^{2,S}(\Omega)(j = 1, \ldots, N)$ such that the following inequality holds true.*

$$\|\boldsymbol{\varphi}\|_2^2 \le \sum_{j=1}^{N} |(\boldsymbol{\varphi}, \boldsymbol{\xi}_j)|^2 + \varepsilon\|\nabla\boldsymbol{\varphi}\|_2^2 \quad (\boldsymbol{\varphi} \in \mathbf{H}_0^{1,S}(\Omega))\,. \tag{2.1}$$

This kind of inequality is called "*the Friedrichs inequality*" in general. The inequality (2.1) is a symmetric version of "*the Friedrichs inequality*".

LEMMA 2.5 (K. Masuda [12]). *For any $\varepsilon > 0$ and $\boldsymbol{w}_3 \in C([0,T]; \mathbf{L}^{2,S}(\Omega))$, there exist a constant M, an integer N and functions $\boldsymbol{\psi}_j \in \mathbf{L}^{2,S}(\Omega)$ $(j = 1, \ldots, N)$ such that the inequality holds true.*

$$\begin{aligned}\int_0^T |((\boldsymbol{w}_1\cdot\nabla)\boldsymbol{w}_2, \boldsymbol{w}_3)|dt \le{}& \varepsilon\int_0^T (\|\nabla\boldsymbol{w}_1\|_2^2 + \|\nabla\boldsymbol{w}_2\|_2^2 + \|\boldsymbol{w}_1\|_2\|\nabla\boldsymbol{w}_2\|_2)dt \\ &+ M\sum_{j=1}^{N}\int_0^T |(\boldsymbol{w}_1, \boldsymbol{\psi}_j)|^2 dt \quad (\boldsymbol{w}_1, \boldsymbol{w}_2 \in L^2((0,T); \mathcal{V}^S(\Omega)))\,.\end{aligned}$$

This kind of inequality appears in K. Masuda [12], p. 632, Lemma 2.5. The inequality is its two dimensional and symmetric version.

2.2. Extensions of boundary conditions. We use "*the Leray Inequality*" used for the proof of the existence of the periodic solutions of the Navier-Stokes equations.

THEOREM 2.1. *Suppose that Ω is a bounded domain with a smooth boundary $\partial\Omega$ and $\boldsymbol{\beta} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ satisfies (SOC).*

Then for any $\varepsilon > 0$, there exist extensions $\boldsymbol{b}_\varepsilon \in \mathbf{H}^1_\sigma(\Omega)$ of $\boldsymbol{\beta}$ satisfying

$$|((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{b}_\varepsilon)| < \varepsilon\|\nabla\boldsymbol{v}\|_2^2 \quad (\boldsymbol{v} \in \mathcal{V}(\Omega))\,. \tag{2.2}$$

See H. Fujita [5] or O. A. Ladyzhenskaya [10] for the proof of Theorem 2.1. The estimate (2.2) is called "*the Leray Inequality*". But if a given function $\boldsymbol{\beta} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ satisfies (GOC) (not (SOC)), we cannot make an extension $\boldsymbol{b} \in \mathbf{H}^1_\sigma(\Omega)$ of $\boldsymbol{\beta}$ satisfying "*the Leray Inequality*". See A. Takeshita [15]. By using "*the Leray Inequality*" we can show the existence of a weak solution of the stationary Navier-Stokes equations with the Dirichlet boundary conditions satisfying (SOC). The following Corollary is the nonstationary, periodic and symmetric version of Theorem 2.1.

COROLLARY 2.1. *Suppose that Ω satisfies Assumption 1.1 and $\boldsymbol{\beta} \in C^1_\pi([0,T];\mathbf{H}^{\frac{1}{2}}(\partial\Omega))$ satisfies (SOC).*

Then there exists an extension $\boldsymbol{g} \in C^1_\pi([0,T];\mathbf{H}^1_\sigma(\partial\Omega))$ of $\boldsymbol{\beta}$ satisfying

$$|((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{g}(t))| \le \varepsilon\|\nabla\boldsymbol{v}\|_2^2 (\forall t \in [0,T], \boldsymbol{v} \in \mathcal{V}(\Omega))\,, \tag{2.3}$$

H. Fujita [6] proved that if a domain Ω satisfies Assumption 1.1, symmetric functions defined on the boundary with (GOC) have the extensions which satisfy symmetric version of "*the Leray Inequality*". The following Theorem 2.2 is the nonstationary and periodic version of Theorem 1 in H. Fujita [6] or Theorem 1 in H. Morimoto[14].

THEOREM 2.2. *Suppose that Ω satisfies Assumption 1.1 and $\boldsymbol{\beta} \in C^1_\pi([0,T];\mathbf{H}^{\frac{1}{2},S}(\partial\Omega))$ satisfies (GOC). Then for any $\varepsilon > 0$, there exist extensions $\boldsymbol{b}_\varepsilon \in C^1_\pi([0,T];\mathbf{H}^{1,S}_\sigma(\partial\Omega))$ of $\boldsymbol{\beta}$ satisfying*

$$|((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{b}_\varepsilon(t))| < \varepsilon\|\nabla\boldsymbol{v}\|_2^2 \quad (\forall\boldsymbol{v} \in \mathcal{V}^S(\Omega), t \in [0,T])\,. \tag{2.4}$$

The proof of Theorem 2.2 is similar to H. Fujita [6]. Theorem 2.2 is a special case where a given function on $\partial\Omega$ satisfying (GOC) is extended to Ω satisfying "*the Leray Inequality*".

Before stating the proof of Theorem 2.2, we prove the proof of Corollary 2.1.

PROOF OF COROLLARY 2.1. We know that for a fixed $t \in [0,T]$ and $\boldsymbol{\beta}(t) \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ there exist a $\boldsymbol{b}(t) \in \mathbf{H}^1(\Omega)$ such that

$$\boldsymbol{b}(t) = \boldsymbol{\beta}(t) \quad \text{on} \quad \partial\Omega\,,$$

$$\|\boldsymbol{b}(t)\|_{\mathbf{H}^1} \le C_0\|\boldsymbol{\beta}(t)\|_{\mathbf{H}^{\frac{1}{2}}}$$

holds true, where the constant C_0 is independent of $t \in [0, T]$. For example, we solve the Laplace equation

$$-\Delta \boldsymbol{u} = \mathbf{0} \quad \text{in} \quad \Omega\,,$$
$$\boldsymbol{u} = \boldsymbol{\beta}(t) \quad \text{on} \quad \partial\Omega\,.$$

For the proof, See D. Gilbarg and N. S. Trudinger [8], Theorem 8.6 and Corollary 8.7. Then we have

$$\int_\Omega \operatorname{div} \boldsymbol{b}(t) dx = \int_\Omega \boldsymbol{\beta}(t) \cdot \boldsymbol{n} ds = 0\,.$$

Then there exists a $\boldsymbol{b}_1(t) \in \mathbf{H}_0^1(\Omega)$ such that

$$\operatorname{div} \boldsymbol{b}_1(t) = \operatorname{div} \boldsymbol{b}(t) \quad \text{in} \quad \Omega\,,$$
$$\|\boldsymbol{b}_1(t)\|_{\mathbf{H}^1} \le c_0 \|\boldsymbol{\beta}(t)\|_{\mathbf{H}^{\frac{1}{2}}}$$

holds true, where the constant c_0 does not depend on $t \in [0, T]$. For the proof, See G. P. Galdi [7], section III.3 Theorem 3.1. We set

$$\boldsymbol{\psi}(t) = \boldsymbol{b}(t) - \boldsymbol{b}_1(t) \quad \text{in} \quad \Omega$$

Then we obtain that $\boldsymbol{\psi}(t) \in \mathbf{H}_\sigma^1(\Omega)$,

$$\boldsymbol{\psi}(t) = \boldsymbol{\beta}(t) \quad \text{on} \quad \partial\Omega\,,$$
$$\|\boldsymbol{\psi}(t)\|_{\mathbf{H}^1} \le c \|\boldsymbol{\beta}(t)\|_{\mathbf{H}^{\frac{1}{2}}}\,.$$

Furthermore we obtain that $\widetilde{\boldsymbol{\psi}}(t) \in \mathbf{H}_\sigma^1(\Omega)$,

$$\widetilde{\boldsymbol{\psi}}(t) = \boldsymbol{\beta}_t(t) \quad \text{on} \quad \partial\Omega\,,$$
$$\|\widetilde{\boldsymbol{\psi}}(t)\|_{\mathbf{H}^1} \le c \|\boldsymbol{\beta}_t(t)\|_{\mathbf{H}^{\frac{1}{2}}}$$

by using the same method as above. It is easy to obtain that

$$\|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(s)\|_{\mathbf{H}^1} \le c\, \|\boldsymbol{\beta}(t) - \boldsymbol{\beta}(s)\|_{\mathbf{H}^{\frac{1}{2}}}$$
$$\|\widetilde{\boldsymbol{\psi}}(t) - \widetilde{\boldsymbol{\psi}}(s)\|_{\mathbf{H}^1} \le c\, \|\boldsymbol{\beta}_t(t) - \boldsymbol{\beta}_t(s)\|_{\mathbf{H}^{\frac{1}{2}}}$$

holds true. Therefore $\boldsymbol{\psi}(t)$ and $\widetilde{\boldsymbol{\psi}}(t)$ is continuous with respect to t on $[0, T]$ in $\mathbf{H}^1(\Omega)$. For all $h \in \mathbf{R}$

$$\left\| \frac{\boldsymbol{\psi}(t+h) - \boldsymbol{\psi}(t)}{h} - \widetilde{\boldsymbol{\psi}}(t) \right\|_{\mathbf{H}^1} \le c \left\| \frac{\boldsymbol{\beta}(t+h) - \boldsymbol{\beta}(t)}{h} - \boldsymbol{\beta}_t(t) \right\|_{\mathbf{H}^{\frac{1}{2}}} \tag{2.5}$$

holds true. If h goes to 0, we obtain that the right hand side of (2.5) goes to 0. Consequently $\boldsymbol{\psi} = \boldsymbol{\psi}(t)$ has the derivative $\widetilde{\boldsymbol{\psi}}$ on $[0, T]$ in $\mathbf{H}^1(\Omega)$. Since it supposed that $\boldsymbol{\beta}(0) = \boldsymbol{\beta}(T)$ in $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$, it is easy to obtain that $\boldsymbol{\psi}(0) = \boldsymbol{\psi}(T)$ in $\mathbf{H}^1(\Omega)$. Therefore we have $\boldsymbol{\psi} \in C^1([0, T]; \mathbf{H}_\sigma^1(\Omega))$.

For a fixed $(t, x_0) \in [0, T] \times \Omega$, we define functions ϕ and $\widetilde{\phi}$ as

$$\phi(t, y) = \int_{x_0}^{y} \psi_1(t, x)dx_2 - \psi_2(t, x)dx_1, \; y \in \Omega ,$$

$$\widetilde{\phi}(t, y) = \int_{x_0}^{y} \psi_{t,1}(t, x)dx_2 - \psi_{t,2}(t, x)dx_1, \; y \in \Omega .$$

Since $\operatorname{div} \boldsymbol{\psi}(t) = \operatorname{div} \boldsymbol{\psi}_t(t) = 0$ in Ω, ϕ and $\widetilde{\phi}$ are independent of path of integration. We set

$$\varphi(t, x) = \phi(t, x) - \frac{1}{|\Omega|} \int_{\Omega} \phi(t, y)dy, \; x \in \Omega ,$$

$$\widetilde{\varphi}(t, x) = \widetilde{\phi}(t, x) - \frac{1}{|\Omega|} \int_{\Omega} \widetilde{\phi}(t, y)dy, \; x \in \Omega .$$

Then an easy calculation yields

$$\mathrm{rot}\varphi(t) = (\frac{\partial}{\partial x_2}\varphi(t), -\frac{\partial}{\partial x_1}\varphi(t)) = \boldsymbol{\psi}(t) \quad \text{in} \quad \Omega ,$$

$$\mathrm{rot}\widetilde{\varphi}(t) = \boldsymbol{\psi}_t(t) \quad \text{in} \quad \Omega .$$

Since

$$\int_{\Omega} \varphi(t, x)dx = 0, \; \int_{\Omega} \widetilde{\varphi}(t, x)dx = 0$$

holds true, the Poincaré inequality holds true for $\varphi(t)$ and $\widetilde{\varphi}(t)$. Of course $\|\nabla\varphi(t)\|_2 = \|\boldsymbol{\psi}(t)\|_2$ and $\|\nabla^2\varphi(t)\|_2 = \|\nabla\boldsymbol{\psi}(t)\|_2$ holds true and φ is periodic in $H^2(\Omega)$. Consequently the inequalities

$$\|\varphi(t) - \varphi(s)\|_{H^2} \leq C\|\boldsymbol{\psi}(t) - \boldsymbol{\psi}(s)\|_{\mathbf{H}^1} ,$$

$$\|\widetilde{\varphi}(t) - \widetilde{\varphi}(s)\|_{H^2} \leq C\|\boldsymbol{\psi}_t(t) - \boldsymbol{\psi}_t(s)\|_{\mathbf{H}^1} ,$$

$$\left\| \frac{\varphi(t + h) - \varphi(t)}{h} - \widetilde{\varphi}(t) \right\|_{H^2} \leq C \left\| \frac{\boldsymbol{\psi}(t + h) - \boldsymbol{\psi}(t)}{h} - \boldsymbol{\psi}_t(t) \right\|_{\mathbf{H}^1}$$

holds true, where the constant C is independent of $t \in [0, T]$. Therefore we obtain $\varphi \in C^1_\pi([0, T]; H^2(\Omega))$ and the equality and inequalities

$$\mathrm{rot}\varphi = \boldsymbol{\beta} \quad \text{on} \quad [0, T] \times \partial\Omega$$

$$\|\varphi(t)\|_{H^2} \leq C\|\boldsymbol{\beta}(t)\|_{\mathbf{H}^{\frac{1}{2}}} ,$$

$$\|\varphi_t(t)\|_{H^2} \leq C\|\boldsymbol{\beta}_t(t)\|_{\mathbf{H}^{\frac{1}{2}}} .$$

Let $\gamma > 0$ and $0 < \kappa < \frac{1}{4}$. We suppose that $j \in C_0^\infty(0, \infty)$ satisfies

$$j(s) = \begin{cases} \dfrac{1}{s} & (2\kappa\gamma < s < (1 - 2\kappa)\gamma) \\ 0 & (0 \leq s \leq \kappa\gamma, (1 - \kappa)\gamma \leq s) \end{cases} ,$$

$$0 \le j(s) \le \frac{1}{s} \quad (s \in (0,\infty)) .$$

We set

$$h(\tau) = 1 - \frac{\int_0^\tau j(s)ds}{\int_0^\infty j(s)ds} \quad (\tau \ge 0) .$$

Then for any $\varepsilon > 0$ there exists $\kappa > 0$ such that

$$\tau |h'(\tau)| < \varepsilon \quad (\forall \tau \in (0,\infty))$$

holds true. We set

$$\begin{aligned} \rho(x) &= \operatorname{dist}(x, \partial\Omega) && x \in \Omega , \\ \boldsymbol{g}(t,x) &= \operatorname{rot}(h(\rho(x))\varphi(t,x)) && (t,x) \in [0,T] \times \Omega . \end{aligned} \tag{2.6}$$

Then (2.3) holds true. q.e.d.

Using Corollary 2.1, we prove Theorem 2.2.

PROOF OF THEOREM 2.2. The proof of *First step* and *Second step* are same as H. Fujita[6]. But for the convenience of the readers, we follow his argument.

First step

We suppose that $0 < \kappa < \frac{1}{2}$, ξ_κ belongs to $C_0^\infty(\mathbf{R})$ such that

$$\begin{aligned} \xi_\kappa(s) &= \xi_\kappa(-s) && (s \in \mathbf{R}) , \\ 0 \le \xi_\kappa(s) &\le \frac{1}{|s|} && (\forall s \in \mathbf{R}\backslash\{0\}) , \\ \xi_\kappa(s) &= \frac{1}{s} && (\kappa \le |s| \le \frac{1}{2}) , \\ \xi_\kappa(s) &= 0 && (1 \le |s|) . \end{aligned}$$

We set

$$\gamma_\kappa = \int_{\mathbf{R}} \xi_\kappa(s) ds .$$

The constant γ_κ is finite because the support of ξ_κ is contained in $(-1,1)$. If κ goes to 0, then γ_κ goes to infinity. We define

$$\eta(s) = \frac{1}{\gamma_\kappa} \frac{1}{\delta} \xi_\kappa \left(\frac{s}{\delta} \right) \quad (s \in \mathbf{R})$$

for any $\delta > 0$. Then η belongs to $C_0^\infty(\mathbf{R})$, the support of η is included in $[-\delta, \delta]$, η is positive on $\mathbf{R}$, η is symmetric with respect to the origin and

$$\int_{-\delta}^{\delta} \eta(s) ds = 1 .$$

Furthermore, for any $s \in \mathbf{R}$

$$|s|\eta(s) \leq |s|\frac{1}{\gamma_\kappa}\frac{\delta}{|s|} = \frac{1}{\gamma_\kappa} \to 0 \quad (\kappa \to 0)$$

holds true, that is to say, $|s|\eta(s)$ goes to 0 uniformly on $\mathbf{R}$ if κ goes to 0.

Second step

Let $(0, y_j)$ and $(0, y_j^*)$ be the cross points of x_2-axis and $\Gamma_j(1 \leq j \leq J)$. Without loss of generality, we suppose $y_j^* > y_j(1 \leq j \leq J)$ and $y_{j-1} > y_j$, $y_{j-1}^* > y_j^*(2 \leq j \leq J)$. Let $(0, y_0)$ and $(0, y_0^*)$ be the cross points of x_2-axis and Γ_0. Without loss of generality, we suppose $y_0^* < y_j^*$ and $y_j < y_0$ for $1 \leq j \leq J$. We choose a small $\delta_1 > 0$ such that the points $(0, y_j + \delta_1)(1 \leq j \leq J)$ is the inside of Γ_j. Then we define

$$\begin{aligned} Q &= [-\delta, \delta] \times \mathbf{R}\,, \\ Q_j &= [-\delta, \delta] \times [y_0 - \delta_1, y_j + \delta_1] \quad (j = 1, \dots, J)\,, \\ K_j &= Q_j \cap \overline{\Omega}\,. \end{aligned}$$

Third Step

We set

$$\mu_j(t) = \int_{\Gamma_j} \boldsymbol{\beta}(t) \cdot \boldsymbol{n} d\sigma \quad (j = 1, \dots, J, t \in [0, T])\,.$$

Then μ_j is a C^1 class function on $[0, T]$ and periodic and satisfies

$$\int_{\Gamma_0} \boldsymbol{\beta}(t) \cdot \boldsymbol{n} d\sigma = -\sum_{j=1}^{J} \mu_j(t) \quad (t \in [0, T])\,.$$

We set

$$\tilde{\boldsymbol{b}}_j(t, x) = \begin{cases} (0, -\mu_j(t)\eta(x_1)) & \text{in} \quad [0, T] \times K_j \\ (0, 0) & \text{in} \quad [0, T] \times (\overline{\Omega} \backslash K_j) \end{cases}\,.$$

We obtain that

$$\operatorname{div} \tilde{\boldsymbol{b}}_j = 0 \quad \text{in} \quad [0, T] \times \Omega$$

and

$$\int_{\Gamma_k} \tilde{\boldsymbol{b}}_j(t) \cdot \boldsymbol{n} d\sigma = \mu_j(t)\delta_{jk} \quad (j, k = 1, \dots, J, t \in [0, T])$$

holds true. By the same calculations as H. Fujita[6] we have

$$\begin{aligned} |((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \tilde{\boldsymbol{b}}_j(t))| &\leq \max_{t \in [0,T]} |\mu_j(t)| \sup_{x_1}(|x_1|\eta(x_1)) 2\sqrt{2} \|\nabla \boldsymbol{v}\|_2^2 \\ &\qquad (\boldsymbol{v} \in \mathcal{V}^S(\Omega), j = 1, \dots, J, t \in [0, T])\,. \end{aligned}$$

We choose a parameter $\kappa > 0$ such that

$$\max_{1\le j\le J}\max_{t\in[0,T]}|\mu_j(t)|2\sqrt{2}\sup_{x_1}(|x_1|\eta(x_1)) < \frac{2\varepsilon}{J}$$

holds true. Therefore

$$|((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\tilde{\boldsymbol{b}}_j(t))| < \frac{\varepsilon}{2J}\|\nabla\boldsymbol{v}\|_2^2 \quad (\boldsymbol{v}\in\mathcal{V}^S(\Omega),\ j=1,\dots,J,\ t\in[0,T])$$

holds true. We set

$$\tilde{\boldsymbol{b}} = \sum_{j=1}^{J}\tilde{\boldsymbol{b}}_j \quad \text{in} \quad [0,T]\times\Omega$$

and

$$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} - \tilde{\boldsymbol{b}}|_{\partial\Omega} \quad \text{on} \quad [0,T]\times\partial\Omega\,.$$

Then $\hat{\boldsymbol{\beta}} \in C^1_\pi([0,T];\mathbf{H}^{\frac{1}{2},S}(\partial\Omega))$ and satisfies (SOC). Using Corollary 2.1, there exists an extension $\hat{\boldsymbol{b}} \in C^1_\pi([0,T];\mathbf{H}^{1,S}_\sigma(\Omega))$ of $\hat{\boldsymbol{\beta}}$ satisfying

$$|((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\hat{\boldsymbol{b}}(t))| < \frac{\varepsilon}{2}\|\nabla\boldsymbol{v}\|_2^2 \quad (\boldsymbol{v}\in\mathcal{V}^S(\Omega),\ t\in[0,T])\,.$$

We set

$$\boldsymbol{b}_\varepsilon = \hat{\boldsymbol{b}} + \tilde{\boldsymbol{b}} \quad \text{in} \quad [0,T]\times\Omega\,.$$

Then we obtain

$$\begin{aligned} \boldsymbol{b}_\varepsilon &\in C^1_\pi([0,T];\mathbf{H}^{1,S}_\sigma(\Omega))\,,\\ \boldsymbol{b}_\varepsilon &= \boldsymbol{\beta} \quad \text{on} \quad [0,T]\times\partial\Omega\,,\\ |((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{b}_\varepsilon(t))| &< \varepsilon\|\nabla\boldsymbol{v}\|_2^2 \quad (\boldsymbol{v}\in\mathcal{V}^S(\Omega),\ t\in[0,T])\,. \end{aligned}$$

q.e.d.

3. Proof of Theorem 1.1

In this section $\boldsymbol{b}_\varepsilon \in C^1_\pi([0,T];\mathbf{H}^{1,S}_\sigma(\Omega))$ is the extension of $\boldsymbol{\beta} \in C^1_\pi([0,T];\mathbf{H}^{\frac{1}{2},S}(\partial\Omega))$ established by Theorem 2.2 for ε satisfying $1-2\varepsilon>0$.

We suppose that $\{\boldsymbol{\varphi}_k\}_{k\in\mathbf{N}}$ is the basis in $\mathcal{V}^S(\Omega)$, satisfying $(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_k)=\delta_{jk}$. Let $\boldsymbol{v}_0$ be $\mathcal{H}^S(\Omega)$, then there exists $\{a_k\}\subset\mathbf{R}$ such that

$$\boldsymbol{v}_0 = \sum_{k=1}^{\infty} a_k\boldsymbol{\varphi}_k\,.$$

Set

$$\boldsymbol{v}_{0m} = \sum_{k=1}^{m} a_k \boldsymbol{\varphi}_k \,.$$

We look for a solution

$$\boldsymbol{v}_m(t,x) = \sum_{k=1}^{m} c_k(t)\boldsymbol{\varphi}_k(x)$$

to the ordinary differential equations

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{v}_m, \boldsymbol{\varphi}_j) &+ ((\boldsymbol{v}_m, \boldsymbol{\varphi}_j)) + ((\boldsymbol{v}_m \cdot \nabla)\boldsymbol{v}_m, \boldsymbol{\varphi}_j) + ((\boldsymbol{v}_m \cdot \nabla)\boldsymbol{b}_\varepsilon, \boldsymbol{\varphi}_j) + ((\boldsymbol{b}_\varepsilon \cdot \nabla)\boldsymbol{v}_m, \boldsymbol{\varphi}_j) \\ &= \langle \boldsymbol{f}, \boldsymbol{\varphi}_j \rangle - (\boldsymbol{b}_{\varepsilon,t}, \boldsymbol{\varphi}_j) - (\nabla \boldsymbol{b}_\varepsilon, \nabla \boldsymbol{\varphi}_j) - ((\boldsymbol{b}_\varepsilon \cdot \nabla)\boldsymbol{b}_\varepsilon, \boldsymbol{\varphi}_j) \quad (j = 1, \dots, m) \qquad (3.1)\end{aligned}$$

with the initial condition

$$\boldsymbol{v}_m(0) = \boldsymbol{v}_{0m} \,. \qquad (3.2)$$

We know that the initial value problem (3.1) with (3.2) has one and only one solution $\boldsymbol{c}^m(t) = (c_1^m(t), \dots, c_m^m(t))$ on $[0, T]$.

Multiplying $c_j(t)$ to (3.1) and adding these equations with respect to $j = 1, \dots, m$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_m\|_2^2 + \|\nabla \boldsymbol{v}_m\|_2^2 + ((\boldsymbol{v}_m \cdot \nabla)\boldsymbol{b}_\varepsilon, \boldsymbol{v}_m) = \langle \boldsymbol{F}, \boldsymbol{v}_m \rangle \,, \qquad (3.3)$$

where $\langle \boldsymbol{F}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle - (\boldsymbol{b}_{\varepsilon,t}, \boldsymbol{\varphi}) - (\nabla \boldsymbol{b}_\varepsilon, \nabla \boldsymbol{\varphi}) - ((\boldsymbol{b}_\varepsilon \cdot \nabla)\boldsymbol{b}_\varepsilon, \boldsymbol{\varphi})$ $(\boldsymbol{\varphi} \in \mathcal{V}^S(\Omega))$. Using the Leray Inequality (2.4), then we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_m\|_2^2 + (1-\varepsilon)\|\nabla \boldsymbol{v}_m\|_2^2 \le \langle \boldsymbol{F}, \boldsymbol{v}_m \rangle \,, \qquad (3.4)$$

Integrating (3.4) on $[0, t]$, the solution $\boldsymbol{v}_m$ satisfies the estimates

$$\|\boldsymbol{v}_m(t)\|_2^2 \le \|\boldsymbol{v}_0\|_2^2 + \frac{4}{\varepsilon}\int_0^T L^2(t)dt \,, \qquad (3.5)$$

where

$$L(t) = \|\boldsymbol{f}(t)\|_{(\mathcal{V}^S)'} + \|\boldsymbol{\beta}_t(t)\|_{\mathbf{H}^{\frac{1}{2}}} + \|\boldsymbol{\beta}(t)\|_{\mathbf{H}^{\frac{1}{2}}} + \|\boldsymbol{\beta}(t)\|_{\mathbf{H}^{\frac{1}{2}}}^2 \,.$$

Integrating (3.4) on $[0, T]$, the solution $\boldsymbol{v}_m$ satisfies the estimates

$$(1-2\varepsilon)\int_0^T \|\nabla \boldsymbol{v}_m\|_2^2 dt \le \|\boldsymbol{v}_0\|_2^2 + \frac{4}{\varepsilon}\int_0^T L^2(t)dt \,. \qquad (3.6)$$

Let us show that $(\boldsymbol{v}_m(t), \boldsymbol{\varphi}_j)$ is uniformly bounded and equicontinuous on $[0, T]$ with respect to m. A simple calculation yields

$$|(\boldsymbol{v}_m(t), \boldsymbol{\varphi}_j) - (\boldsymbol{v}_m(s), \boldsymbol{\varphi}_j)|$$

$$
\begin{aligned}
&= \left| \int_s^t \frac{d}{d\tau}(\boldsymbol{v}_m(\tau), \boldsymbol{\varphi}_j) d\tau \right| \\
&\leq \int_s^t |((\boldsymbol{v}_m, \boldsymbol{\varphi}_j))| + |((\boldsymbol{v}_m \cdot \nabla)\boldsymbol{v}_m, \boldsymbol{\varphi}_j)| + |((\boldsymbol{v}_m \cdot \nabla)\boldsymbol{b}_\varepsilon, \boldsymbol{\varphi}_j)| + |((\boldsymbol{b}_\varepsilon \cdot \nabla)\boldsymbol{v}_m, \boldsymbol{\varphi}_j)| \\
&\qquad + |\langle \boldsymbol{f}, \boldsymbol{\varphi}_j \rangle| + |(\boldsymbol{b}_{\varepsilon,t}, \boldsymbol{\varphi}_j)| + |(\nabla \boldsymbol{b}_\varepsilon, \nabla \boldsymbol{\varphi}_j)| + |((\boldsymbol{b}_\varepsilon \cdot \nabla)\boldsymbol{b}_\varepsilon, \boldsymbol{\varphi}_j)| d\tau \\
&\leq \int_s^t (\|\nabla \boldsymbol{v}_m\|_2 + 2^{\frac{1}{2}} \|\boldsymbol{v}_m\|_2 \|\nabla \boldsymbol{v}_m\|_2 + C_1 \|\nabla \boldsymbol{v}_m\|_2 \|\boldsymbol{b}_\varepsilon\|_{\mathbf{H}^1} + L(t)) \|\nabla \boldsymbol{\varphi}_j\|_2 d\tau \\
&\leq |t-s|^{\frac{1}{2}} \|\nabla \boldsymbol{\varphi}_j\| \left(\frac{1}{1-2\varepsilon} M_1^2 M_2 + \int_s^t L^2(\tau) d\tau \right)^{\frac{1}{2}}, \qquad (3.7)
\end{aligned}
$$

where

$$
\begin{aligned}
M_1^2 &:= (\|\boldsymbol{v}_0\|_2^2 + \frac{4}{\varepsilon} \int_0^T L^2(t) dt), \\
M_2 &:= 1 + 2^{\frac{1}{2}} M_1^2 + C_1 C_\varepsilon \sup_{t \in [0,T]} \|\boldsymbol{\beta}(t)\|_{\mathbf{H}^{\frac{1}{2}}}.
\end{aligned}
$$

We obtain that $\{\boldsymbol{v}_m\}$ is a bounded sequence in $L^\infty((0,T); \mathcal{H}^S(\Omega)) \cap L^2((0,T); \mathcal{V}^S(\Omega))$ from (3.5) and (3.6). Therefore there exist a subsequence $\{\boldsymbol{v}_{mk}\}_k$ of $\{\boldsymbol{v}_m\}_m$ and some $\boldsymbol{v} \in L^\infty((0,T); \mathcal{H}^S(\Omega)) \cap L^2((0,T); \mathcal{V}^S(\Omega))$ such that

$$
\boldsymbol{v}_{mk} \to \boldsymbol{v} \quad \text{in} \quad \begin{cases} L^\infty((0,T); \mathcal{H}^S(\Omega)) & \textit{weak star} \\ L^2((0,T); \mathcal{V}^S(\Omega)) & \textit{weakly} \end{cases} \quad (k \to \infty).
$$

On the other hand, *the Ascoli-Arzelà* Theorem and the diagonal method assure that $\boldsymbol{v}_{mk}$ converges to $\boldsymbol{v}$ in the weak topology of $\mathbf{L}^2(\Omega)$. We can establish the convergence

$$
\boldsymbol{v}_{mk} \to \boldsymbol{v} \quad \text{in} \quad L^2((0,T); \mathbf{L}^4(\Omega)) \quad \textit{strongly}
$$

by Lemma 2.4 (the Friedrichs inequality) and Lemma 2.2.

It is easy to prove that $\boldsymbol{v}$ satisfies

$$
\begin{aligned}
&-\int_0^T (\boldsymbol{v}, \boldsymbol{\varphi}) \phi' dt + \int_0^T \{((\boldsymbol{v}, \boldsymbol{\varphi})) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi}) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{b}_\varepsilon, \boldsymbol{\varphi}) + ((\boldsymbol{b}_\varepsilon \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi})\} \phi dt \\
&= (\boldsymbol{v}_0, \boldsymbol{\varphi}) \phi(0) + \int_0^T \langle \boldsymbol{F}, \boldsymbol{\varphi} \rangle \phi dt \quad (\boldsymbol{\varphi} \in \mathcal{V}^S(\Omega), \phi \in C_0^\infty([0,T)))
\end{aligned}
$$

and

$$
\frac{d}{dt}(\boldsymbol{v}, \boldsymbol{\varphi}) + ((\boldsymbol{v}, \boldsymbol{\varphi})) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi}) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{b}_\varepsilon, \boldsymbol{\varphi}) + ((\boldsymbol{b}_\varepsilon \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi}) = \langle \boldsymbol{F}, \boldsymbol{\varphi} \rangle
$$

$$
(\boldsymbol{\varphi} \in \mathcal{V}^S(\Omega)) \quad (3.8)
$$

in the distribution sense on $(0,T)$, namely $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{b}_\varepsilon$ is the weak solution of the Navier-Stokes equations. Since the terms except the first term of (3.8) belong to

$L^2((0,T);(\mathcal{V}^S(\Omega))')$, $\boldsymbol{v}$ has a weak derivative $\boldsymbol{v}' \in L^2((0,T);(\mathcal{V}^S(\Omega))')$ and $\boldsymbol{v}$ belongs to $C([0,T];\mathcal{H}^S(\Omega))$. Moreover we can show that $\boldsymbol{v}$ is unique.

Let us show that we can choose the initial value $\boldsymbol{v}_0 \in \mathcal{H}^S(\Omega)$ such that the weak solution $\boldsymbol{v}$ of the initial value problem of the Navier-Stokes equations (3.8) is periodic. Let $\mathcal{F}$ be a map from the initial value $\boldsymbol{v}_0 \in \mathcal{H}^S(\Omega)$ to the last value $\boldsymbol{v}(T) \in \mathcal{H}^S(\Omega)$, that is to say,

$$\mathcal{F};\ \boldsymbol{v}_0 \in \mathcal{H}^S(\Omega) \to \boldsymbol{v}(T) \in \mathcal{H}^S(\Omega) \tag{3.9}$$

We see that the fixed point of the map $\mathcal{F}$ is a periodic solution of the Navier-Stokes equations. We use the Leray-Schauder Theorem in order to prove the existence of the fixed point. For the theorem See D. Gilbarg and N. S. Trudinger[8], p.280, Theorem 11.3.

Firstly, we prove that the map $\mathcal{F}$ is compact. Suppose that $\{\boldsymbol{w}_{0m}\} \subset \mathcal{H}^S(\Omega)$ converges weakly to $\boldsymbol{w}_0 \in \mathcal{H}^S(\Omega)$. We must show that there exists a subsequence $\{\boldsymbol{w}_{0m_k}\}$ such that $\mathcal{F}\boldsymbol{w}_{0m_k}$ converge to $\mathcal{F}\boldsymbol{w}_0$ in $\mathcal{H}^S(\Omega)$. Let $\boldsymbol{w}$ and $\boldsymbol{w}_m$ be weak solutions of the Navier-Stokes equations (3.8) with the initial value $\boldsymbol{w}_0$ and $\boldsymbol{w}_{0m}$ respectively, that is to say, $\boldsymbol{w}$ and $\boldsymbol{w}_m$ satisfy

$$\frac{d}{dt}(\boldsymbol{w},\boldsymbol{\varphi}) + ((\boldsymbol{w},\boldsymbol{\varphi})) + ((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi}) + ((\boldsymbol{w}\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi}) + ((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi}) = \langle \boldsymbol{F},\boldsymbol{\varphi}\rangle$$
$$(\boldsymbol{\varphi} \in \mathcal{V}^S(\Omega)), \tag{3.10}$$

$$\frac{d}{dt}(\boldsymbol{w}_m,\boldsymbol{\varphi}) + ((\boldsymbol{w}_m,\boldsymbol{\varphi})) + ((\boldsymbol{w}_m\cdot\nabla)\boldsymbol{w}_m,\boldsymbol{\varphi}) + ((\boldsymbol{w}_m\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi}) + ((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{w}_m,\boldsymbol{\varphi}) = \langle \boldsymbol{F},\boldsymbol{\varphi}\rangle$$
$$(\boldsymbol{\varphi} \in \mathcal{V}^S(\Omega)) \tag{3.11}$$

respectively. Subtracting the equation (3.11) from (3.10), we have

$$\begin{aligned}&\frac{d}{dt}(\boldsymbol{w}-\boldsymbol{w}_m,\boldsymbol{\varphi}) + ((\boldsymbol{w}-\boldsymbol{w}_m,\boldsymbol{\varphi})) + ((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi}) - ((\boldsymbol{w}_m\cdot\nabla)\boldsymbol{w}_m,\boldsymbol{\varphi})\\ &+ (((\boldsymbol{w}-\boldsymbol{w}_m)\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi}) + ((\boldsymbol{b}_\varepsilon\cdot\nabla)(\boldsymbol{w}-\boldsymbol{w}_m),\boldsymbol{\varphi}) = 0 \quad (\boldsymbol{\varphi} \in \mathcal{V}^S(\Omega)).\end{aligned} \tag{3.12}$$

Substituting $\boldsymbol{w}-\boldsymbol{w}_m$ for $\boldsymbol{\varphi}$ and using the Leray inequality (2.4), then we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}-\boldsymbol{w}_m\|_2^2 \le -(((\boldsymbol{w}-\boldsymbol{w}_m)\cdot\nabla)\boldsymbol{w},\boldsymbol{w}-\boldsymbol{w}_m). \tag{3.13}$$

Multiplying (3.13) by $\phi \in C_0^\infty((0,T])$ such that $0 \le \phi \le 1$ and $\phi(T) = 1$ and integrating on $(0,T)$, then we obtain

$$\|\boldsymbol{w}(T)-\boldsymbol{w}_m(T)\|_2^2 \le \int_0^T \|\boldsymbol{w}-\boldsymbol{w}_m\|_2^2|\phi'| + 2|(((\boldsymbol{w}-\boldsymbol{w}_m)\cdot\nabla)\boldsymbol{w},\boldsymbol{w}-\boldsymbol{w}_m)|dt.$$

By Lemma 2.4, for any $\delta > 0$ there exist an integer N_1 and functions $\boldsymbol{\xi}_j \in \mathbf{L}^{2,S}(\Omega)$ $(j = 1,\dots,N_1)$ such that the inequality

$$\int_0^T \|\boldsymbol{w}-\boldsymbol{w}_m\|_2^2|\phi'|dt \le \sup_{[0,T]}|\phi'|\int_0^T \sum_{j=1}^{N_1}|(\boldsymbol{w}-\boldsymbol{w}_m,\boldsymbol{\xi}_j)|^2 + \delta\|\nabla\boldsymbol{w}-\nabla\boldsymbol{w}_m\|_2^2 dt$$

holds. By using Lemma 2.5, there exist a constant M, an integer N_2 and functions $\boldsymbol{\psi}_j \in \mathbf{L}^{2,S}(\Omega)$ $(j = 1, \dots, N_2)$ such that the inequality

$$\begin{aligned}
&\int_0^T |(((\boldsymbol{w} - \boldsymbol{w}_m) \cdot \nabla)\boldsymbol{w}, \boldsymbol{w} - \boldsymbol{w}_m)|dt \\
&\qquad \le \delta \int_0^T (\|\nabla \boldsymbol{w} - \nabla \boldsymbol{w}_m\|_2^2 + \|\nabla \boldsymbol{w}\|_2^2 + \|\boldsymbol{w} - \boldsymbol{w}_m\|_2 \|\nabla \boldsymbol{w}\|_2)dt \\
&\qquad\quad + M \sum_{j=1}^{N_2} \int_0^T |(\boldsymbol{w} - \boldsymbol{w}_m, \boldsymbol{\psi}_j)|^2 dt
\end{aligned} \tag{3.14}$$

holds. The solution $\boldsymbol{w}_m$ is a bounded sequence of $L^2((0,T); \mathcal{V}^S(\Omega)) \cap L^\infty((0,T); \mathcal{H}^S(\Omega))$ with respect to m, because the estimates (3.5) and (3.6) hold true with respect to $\boldsymbol{w}_m$ and $\boldsymbol{w}_{0m}$ and $\|\boldsymbol{w}_{0m}\|_2$ is less than a certain constant which does not depend on m. There exist constants M_3 and M_4 which do not depend on m such that the inequality

$$\|\mathcal{F}\boldsymbol{w}_0 - \mathcal{F}\boldsymbol{w}_{0m}\|_2^2 \le M_3 \int_0^T \sum_{j=1}^{N_1} |(\boldsymbol{w} - \boldsymbol{w}_m, \boldsymbol{\xi}_j)|^2 + \sum_{j=1}^{N_2} |(\boldsymbol{w} - \boldsymbol{w}_m, \boldsymbol{\psi}_j)|^2 dt + M_4 \delta$$

holds true because $\mathcal{F}\boldsymbol{w}_0 = \boldsymbol{w}(T)$. Consequently if $\boldsymbol{w}_m$ converges to $\boldsymbol{w}$ uniformly on $[0, T]$ in the weak topology of $\mathbf{L}^{2,S}(\Omega)$, then we obtain the map $\mathcal{F}$ is compact. Now it is easy to prove that $(\boldsymbol{w}_m, \boldsymbol{\varphi}_j)$ satisfies the estimate similar to (3.7). Therefore $(\boldsymbol{w}_m, \boldsymbol{\varphi}_j)$ is equicontinuous on $[0, T]$ with respect to m. Hence for any $\eta > 0$ there exist a $\delta_1 > 0$ and K_0 such that for all $|t| \le \delta_1$ and $m \ge K_0$

$$\begin{aligned}
|(\boldsymbol{w}_m(t), \boldsymbol{\varphi}_j) - (\boldsymbol{w}_{0m}, \boldsymbol{\varphi}_j)| &< \frac{\eta}{3}, \\
|(\boldsymbol{w}_{0m}, \boldsymbol{\varphi}_j) - (\boldsymbol{w}_0, \boldsymbol{\varphi}_j)| &< \frac{\eta}{3}, \\
|(\boldsymbol{w}_0, \boldsymbol{\varphi}_j) - (\boldsymbol{w}(t), \boldsymbol{\varphi}_j)| &< \frac{\eta}{3}
\end{aligned}$$

hold and we obtain

$$|(\boldsymbol{w}_m(t) - \boldsymbol{w}(t), \boldsymbol{\varphi}_j)| < \eta \quad (|t| \le \delta_1,\ m \ge K_0).$$

Similarly we obtain that there exists a K_1 such that for any $t \in [\delta_1, 2\delta_1]$ and $m \ge K_1$

$$\begin{aligned}
|(\boldsymbol{w}_m(t), \boldsymbol{\varphi}_j) - (\boldsymbol{w}_m(\delta_1), \boldsymbol{\varphi}_j)| &< \frac{\eta}{3}, \\
|(\boldsymbol{w}_m(\delta_1), \boldsymbol{\varphi}_j) - (\boldsymbol{w}(\delta_1), \boldsymbol{\varphi}_j)| &< \frac{\eta}{3}, \\
|(\boldsymbol{w}(\delta_1), \boldsymbol{\varphi}_j) - (\boldsymbol{w}(t), \boldsymbol{\varphi}_j)| &< \frac{\eta}{3}
\end{aligned}$$

hold true. So we obtain

$$|(\boldsymbol{w}_m(t) - \boldsymbol{w}(t), \boldsymbol{\varphi}_j)| < \eta \quad (t \in [\delta_1, 2\delta_1],\ m \ge K_1).$$

If we repeat this process, we see that $(\boldsymbol{w}_m, \boldsymbol{\varphi}_j)$ converges to $(\boldsymbol{w}, \boldsymbol{\varphi}_j)$ uniformly on $[0, T]$. Therefore it is obvious that $\boldsymbol{w}_m$ converges $\boldsymbol{w}$ uniformly on $[0, T]$ in the weak topology of $\mathbf{L}^{2,S}(\Omega)$.

Secondly, we prove that there exists a constant ρ such that $\|\boldsymbol{w}_0\|_2 \leq \rho$ for all $\boldsymbol{w}_0 \in \mathcal{H}(\Omega)$ and $\sigma \in [0, 1]$ satisfying $\boldsymbol{w}_0 = \sigma \mathcal{F} \boldsymbol{w}_0$. It is easy to obtain that

$$\|\boldsymbol{w}_0\|_2 \leq \|\mathcal{F} \boldsymbol{w}_0\|_2 . \tag{3.15}$$

Let $\boldsymbol{\varphi} = \boldsymbol{w}$ in (3.10). Then we obtain

$$\frac{1}{2} \frac{d}{dt} \|\boldsymbol{w}\|_2^2 + \|\nabla \boldsymbol{w}\|_2^2 + ((\boldsymbol{w} \cdot \nabla) \boldsymbol{b}_\varepsilon, \boldsymbol{w}) = \langle \boldsymbol{F}, \boldsymbol{w} \rangle .$$

Therefore the inequality

$$\frac{d}{dt} \|\boldsymbol{w}(t)\|_2^2 + 2 \frac{1 - 2\varepsilon}{C(\Omega)^2} \|\boldsymbol{w}(t)\|_2^2 \leq \frac{C}{2\varepsilon} L^2(t) \quad (\forall t \in [0, T]) \tag{3.16}$$

holds true by using *the Leray Inequality* and *the Poincaré* inequality. Setting

$$\alpha = 2 \frac{1 - 2\varepsilon}{C(\Omega)^2} ,$$

$$H = \int_0^T \frac{C}{2\varepsilon} L^2(t) e^{\alpha t} dt .$$

Multiplying the inequality (3.16) by $e^{\alpha t}$ and integrating on $[0, T]$, we obtain the inequality

$$\|\boldsymbol{w}(T)\|_2^2 \leq \|\boldsymbol{w}_0\|_2^2 e^{-\alpha T} + e^{-\alpha T} H .$$

By using (3.15), the estimate

$$\|\boldsymbol{w}_0\|_2^2 \leq \frac{e^{-\alpha T} H}{1 - e^{-\alpha T}}$$

holds true. Therefore we put

$$\rho^2 := \frac{e^{-\alpha T} H}{1 - e^{-\alpha T}} .$$

Then $\|\boldsymbol{w}_0\|_2$ is less than ρ, where $\boldsymbol{w}_0 \in \mathcal{H}^S(\Omega)$ satisfies $\boldsymbol{w}_0 = \sigma \mathcal{F} \boldsymbol{w}_0$. Consequently the map $\mathcal{F}$ has at least one fixed point by the Leray-Schauder Theorem.

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