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Classification of Local Singularities on Torus Curves of Type $(2, 5)$

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Abstract. In this paper, we consider curves of degree 10 of torus type (2,5), $C := \{f_5(x, y)^2 + f_2(x, y)^5 = 0\}$. Assume that $f_2(0, 0) = f_5(0, 0) = 0$. Then $O = (0, 0)$ is a singular point of C which is called an inner singularity. In this paper, we give a topological classification of singularities of (C, O).

1. Introduction

A plane curve $C \subset \mathbf{P}^2$ is called a *curve of torus type* (p, q) if there is a defining polynomial F of C which can be written as $F = F_p^q + F_q^p$, where F_p, F_q are homogeneous polynomials of X, Y, Z of degree p and q respectively. In [6], D.T. Pho classified the local and global configurations of the singularities of sextics of torus type $(2, 3)$. In this paper, we will classify local inner singularities of torus type $(2, 5)$. We assume C is a reduced curve. Using affine coordinates $x = X/Z,\ y = Y/Z$, C is defined as

$$C := \{f_2(x, y)^5 + f_5(x, y)^2 = 0\}\,,$$

where $f_2(x, y) = F_2(x, y, 1)$, $f_5(x, y) = F_5(x, y, 1)$. Put $C_2 := \{f_2(x, y) = 0\}$ and $C_5 := \{f_5(x, y) = 0\}$. We assume that the origin $O = (0, 0)$ is an intersection point of C_2 and C_5 and O is an isolated singularity of C. We classify the topological types of the local singularity (C, O) following the method of [6]. If $f_2(x, y) = -\ell(x, y)^2$ for some linear form ℓ, the curve C is called a *linear torus curve* and it consists of two quintics, as $f_2^5 + f_5^2 = (f_5 + \ell^5)(f_5 - \ell^5)$.

First we recall the following notation.

$$A_n : x^{n+1} + y^2 = 0 \quad (n \geq 1)\,,$$
$$D_n : x^{n-1} + xy^2 = 0 \quad (n \geq 4)\,,$$
$$E_6 : x^3 + y^4 = 0\,, \quad E_7 : x^3 + xy^3 = 0\,, \quad E_8 : x^3 + y^5 = 0\,,$$
$$B_{n,m} : x^n + y^m = 0 \quad \text{(Brieskorn-Pham type)}\,.$$

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In the case $\gcd(m, n) > 1$, the equation of $B_{n,m}$ can contain other monomials on the Newton boundary. For example, $B_{2,4}$ has the form $C_t : x^2 + txy^2 + y^4 = 0$, for $t \neq \pm 2$. The germ (C_t, O) is topologically equivalent to $B_{2,4}$ (Oka [2]). In this notation, $A_n = B_{n+1,2}$ and $E_6 = B_{3,4}$.

We remark that every non-degenerate singularity in the sense of the Newton boundary is a union of Brieskorn-Pham type singularities. For example, $C_{p,q} : x^p + x^2y^2 + y^q,\ p, q \geq 4,\ p + q \geq 9$ is the union of two singularities: $x^{p-2} + y^2 = 0$ and $x^2 + y^{q-2} = 0$. So we introduce the notation: $B_{p-2,2} \circ B_{2,q-2}$ to express $C_{p,q}$.

For the classification, we use the local intersection multiplicity $\iota := I(C_2, C_5; O)$ effectively. The complete classifications is given by Theorem 1 in §5.

This paper consists of the following sections:

§1. Introduction.
§2. Preliminaries.
§3. Some lemmas for torus curves of type (p, q).
§4. Calculation of the local singularities.
§5. The classification.
§6. Linear torus curves of type (2, 5).
§7. Appendix.

2. Preliminaries

2.1. Toric modification. Throughout this paper, we follow the notation of Oka [4]. First we recall a toric modification. Let

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be a unimodular integral 2×2 matrix. We associate σ with a birational morphism $\pi_\sigma : \mathbf{C}^{*2} \to \mathbf{C}^{*2}$ by $\pi_\sigma(x, y) = (x^\alpha y^\beta, x^\gamma y^\delta)$. If $\alpha, \gamma \geq 0$ (respectively. $\beta, \delta \geq 0$), this map can be extended to $x = 0$ (resp. $y = 0$). Note that the morphisms $\{\pi_\sigma \mid \sigma : \text{unimodular}\}$ satisfy the equalities: $\pi_\sigma \circ \pi_\tau = \pi_{\sigma\tau}$ and $(\pi_\sigma)^{-1} = \pi_{\sigma^{-1}}$.

Let N be a free $\mathbf{Z}$-module of rank two with a fixed basis $\{E_1, E_2\}$. Through this basis, we identify $N_{\mathbf{R}} := N \otimes \mathbf{R}$ with $\mathbf{R}^2$. Thus N can be understood as the set of integral points in $\mathbf{R}^2$. We denote a vector in N by a column vector. Hereafter we fix two special vectors $E_1 = {}^t(1, 0)$ and $E_2 = {}^t(0, 1)$. Let N^+ be the space of positive vectors of N. Let $\{P_1, \ldots, P_m\}$ be given positive primitive integral vectors in N^+. Let $P_i = {}^t(a_i, b_i)$ and assume that $\det(P_i, P_{i+1}) > 0$ for each $i = 0, \ldots, m$. Here $P_0 = E_1$, $P_{m+1} = E_2$. We associate $\{P_0, P_1, \ldots, P_{m+1}\}$ with a simplicial cone subdivision Σ^* of $N_{\mathbf{R}}$ which has $m + 1$ cones of dimension two $\mathrm{Cone}(P_i, P_{i+1}),\ i = 0, \ldots, m$ where

$$\mathrm{Cone}(P_i, P_{i+1}) := \{tP_i + sP_{i+1} \mid t, s \geq 0\}.$$

We call $\{P_0, \dots, P_{m+1}\}$ *the vertices of* Σ^*. We say that Σ^* is a *regular simplicial cone subdivision* of N^+ if $\det(P_i, P_{i+1}) = 1$ for each $i = 0, \dots, m$.

Assume that Σ^* is a given regular simplicial cone subdivision with vertices $\{P_0, P_1, \dots, P_{m+1}\}$, $(P_0 = E_1, P_{m+1} = E_2)$ and put $P_i = {}^t(a_i, b_i)$. For each $\mathrm{Cone}(P_i, P_{i+1})$, we associate the unimodular matrix σ_i where

$$\sigma_i := \begin{pmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{pmatrix}.$$

We identify $\mathrm{Cone}(P_i, P_{i+1})$ with the unimodular matrix σ_i. Let (x, y) be a fixed system of coordinates of $\mathbf{C}^2$. Then we consider, for each σ_i, an affine space $\mathbf{C}^2_{\sigma_i}$ of dimension two with coordinates $(x_{\sigma_i}, y_{\sigma_i})$ and the birational map $\pi_{\sigma_i} : \mathbf{C}^2_{\sigma_i} \to \mathbf{C}^2$. First we consider the disjoint union of $\mathbf{C}^2_{\sigma_i}$ for $i = 0, \dots, m$ and we define the variety X as the quotient of this union by the following identification. Two points $(x_{\sigma_i}, y_{\sigma_i}) \in \mathbf{C}^2_{\sigma_i}$ and $(x_{\sigma_j}, y_{\sigma_j}) \in \mathbf{C}^2_{\sigma_j}$ are identified if and only if the birational map $\pi_{\sigma_j^{-1}\sigma_i}$ is well defined at the point $(x_{\sigma_i}, y_{\sigma_i})$ and $\pi_{\sigma_j^{-1}\sigma_i}(x_{\sigma_i}, y_{\sigma_i}) = (x_{\sigma_j}, y_{\sigma_j})$. It can be easily checked that X is non-singular and the maps $\{\pi_{\sigma_i} : \mathbf{C}^2_{\sigma_i} \to \mathbf{C}^2 \mid 0 \le i \le m\}$ glue into a proper analytic map $\pi : X \to \mathbf{C}^2$.

DEFINITION 1. The map $\pi : X \to \mathbf{C}^2$ is called *the toric modification associated with* $\{\Sigma^*, (x, y), O\}$ where Σ^* is a regular simplicial cone subdivision of N^+ and (x, y) is a coordinate system of $\mathbf{C}^2$ centered at the origin O.

Recall that this modification has the following properties.

(1) $\{\mathbf{C}^2_{\sigma_i}, (x_{\sigma_i}, y_{\sigma_i})\}$, $(0 \le i \le m)$ give coordinate charts of X and we call them *the toric coordinate charts* of X.

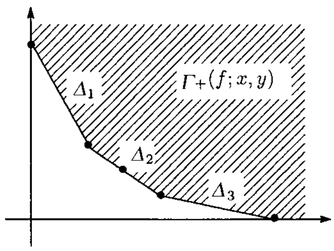
(2) Two affine divisors $\{y_{\sigma_{i-1}} = 0\} \subset \mathbf{C}^2_{\sigma_{i-1}}$ and $\{x_{\sigma_i} = 0\} \subset \mathbf{C}^2_{\sigma_i}$ are glued together to make a compact divisor isomorphic to $\mathbf{P}^1$ for $1 \le i \le m$. We denote this divisor by $\hat{E}(P_i)$.

(3) $\pi^{-1}(O) = \bigcup_{i=1}^m \hat{E}(P_i)$ and $\pi : X - \pi^{-1}(O) \to \mathbf{C}^2 - \{O\}$ is an isomorphism. The non-compact divisor $x_{\sigma_0} = 0$ (respectively $y_{\sigma_m} = 0$) is isomorphically mapped onto the divisor $x = 0$ (resp. $y = 0$).

(4) $\hat{E}(P_i) \cap \hat{E}(P_j) \neq \emptyset$ if and only if $i - j = \pm 1$. If $i - j = \pm 1$, they intersect transversely at a point.

2.2. Toric modification with respect to an analytic function. Let $\mathcal{O}$ be the ring of germs of analytic functions at the origin. Recall that $\mathcal{O}$ is isomorphic to the ring of convergent power series in x, y. Let $f \in \mathcal{O}$ be a germ of a complex analytic function and suppose that $f(O) = 0$. Let $f(x, y) = \sum c_{i,j} x^i y^j$ be the Taylor expansion of f at the origin. We assume that $f(x, y)$ is reduced as a germ. *The Newton polygon* $\Gamma_+(f; x, y)$ of f, with respect to the coordinate system (x, y), is the convex hull of the union $\bigcup_{i,j}\{(i, j) + \mathbf{R}^2_+\}$ where the union is taken for (i, j) such that $c_{i,j} \neq 0$ and *the Newton boundary* $\Gamma(f; x, y)$ is the union of compact faces of the Newton polygon $\Gamma_+(f; x, y)$. For each compact face Δ of $\Gamma(f; x, y)$, *the face function* $f_\Delta(x, y)$ is defined by $f_\Delta(x, y) := \sum_{(i,j)\in\Delta} c_{i,j} x^i y^j$.

In the space M where the Newton polygon $\Gamma_+(f; x, y)$ is contained, we use (ν_1, ν_2) as the coordinates. For any positive weight vector $P = {}^t(a, b) \in N$, we consider P as a linear function on M by $P(\nu_1, \nu_2) = a\nu_1 + b\nu_2$. We define $d(P; f)$ to be the smallest value of the restriction of P to the Newton polygon $\Gamma_+(f; x, y)$ and let $\Delta(P; f)$ be the face where P takes the smallest value. For simplicity we shall write f_P instead of $f_{\Delta(P;f)}$. By the definition, f_P is a weighted homogeneous polynomial of degree $d(P; f)$ with the weight $P = {}^t(a, b)$. For each face $\Delta \in \Gamma(f; x, y)$ there is a unique primitive integral vector $P = {}^t(a, b)$ such that $\Delta = \Delta(P; f)$. The Newton boundary $\Gamma(f; x, y)$ has a finite number of faces.



Let $\Delta_1, \dots, \Delta_m$ denote these faces and let $P_i = {}^t(a_i, b_i)$ be the corresponding positive primitive integral vector, i.e., $\Delta_i = \Delta(P_i; f)$. We call P_i the weight vector of the face Δ_i. Then we can factor $f_{P_i}(x, y)$ as

$$f_{P_i}(x, y) = c x^{r_i} y^{s_i} \prod_{j=1}^{k_i} (y^{a_i} + \gamma_{i,j} x^{b_i})^{\nu_{i,j}}, \quad c \neq 0$$

with distinct non-zero complex numbers $\gamma_{i,1}, \dots, \gamma_{i,k_i}$. We define

$$\tilde{f}_{P_i}(x, y) = f_{P_i}(x, y)/x^{r_i} y^{s_i} = c \prod_{j=1}^{k_i} (y^{a_i} + \gamma_{i,j} x^{b_i})^{\nu_{i,j}}.$$

The polynomial $\sum_{(i,j)\in\Gamma(f;x,y)} c_{i,j} x^i y^j$ is called *the Newton principal part* of $f(x, y)$ and we denote it by $\mathcal{N}(f; x, y)$. We say that $f(x, y)$ is *convenient* if the intersection $\Gamma(f; x, y)$ with each axis is non-empty. Note that $\tilde{f}_{\Delta_i}(x, y)$ is always convenient. We say that f is *non-degenerate on a face* Δ_i if the function $f_{\Delta_i} : \mathbf{C}^{*2} \to \mathbf{C}$ has no critical points. This is equivalent to $\nu_{i,j} = 1$ for all $j = 1, \dots, k_i$. We say that f is *non-degenerate* if f is non-degenerate on any face Δ_i for $i = 1, \dots, m$.

We introduce an equivalence relation $\sim$ in N^+ which is defined by $P \sim Q$ if and only if $\Delta(P; f) = \Delta(Q; f)$. The equivalence classes define a conical subdivision of N^+. This gives a simplicial cone subdivision of N^+ with $m + 2$ vertices $\{P_0, \dots P_{m+1}\}$ with $P_0 = E_1$, $P_{m+1} = E_2$. We denote this subdivision by $\Gamma^*(f; x, y)$ and we call it *the dual Newton* *diagram* of f with respect to the system of coordinates (x, y). The dual Newton diagram $\Gamma^*(f; x, y)$ has $m+1$ cones of dimension 2, $\mathrm{Cone}(P_i, P_{i+1})$, $i = 0, \ldots, m$. Note that these cones are not regular in general.

DEFINITION 2. A regular simplicial cone subdivision Σ^* is *admissible for* $f(x, y)$ if Σ^* is a subdivision of $\Gamma^*(f; x, y)$. The corresponding toric modification $\pi : X \to \mathbf{C}^2$ is called *an admissible toric modification for* $f(x, y)$ with respect to the system of coordinates (x, y).

There exists a unique canonical regular simplicial cone subdivision (Lemma 3.3 of [3]). We call the corresponding toric modification *the canonical toric modification* with respect to $f(x, y)$. Let C be a germ of a reduced curve defined by $f(x, y) = 0$ and let $\pi : X \to \mathbf{C}^2$ be a good resolution. Recall that the dual graph of the resolution is defined as follows. Let $E_1, \ldots, E_r$ be the exceptional divisors and put $\pi^* f^{-1}(0) = \sum_{i=1}^r m_i E_i + \sum_j^s \widetilde{C}_j$. To each E_i, we associate a vertex v_i of $\mathcal{G}(\pi)$ denoted by a black circle. We give an edge joining v_i and v_j if $E_i \cap E_j \neq \emptyset$. For the extended dual graph $\widetilde{\mathcal{G}}(\pi)$, we add vertices w_i to each irreducible components C_i, $i = 1, \ldots, s$ and we join w_j and v_i if $E_i \cap C_j \neq \emptyset$ by a dotted arrow line. It is also important to remember the multiplicities of $\pi^* f$ along E_i, which we denote by m_i. So we put weights m_i to each vertex and call $\mathcal{G}(\pi)$ with weight *the weighted dual graph of the resolution* $\pi : X \to \mathbf{C}^2$.

EXAMPLE 1. Consider the curve $C := \{y^2 - x^3 = 0\}$. The Newton boundary consists of one face Δ. Then $P = {}^t(2, 3)$ is the weight vector corresponding to the face Δ. The dual Newton diagram $\Gamma^*(f; x, y)$ has three vertices $\{E_1, P, E_2\}$. We take a regular simplicial cone subdivision Σ^* of $\Gamma^*(f; x, y)$ and we consider an admissible toric modification $\pi : X \to \mathbf{C}^2$ associated to $\{\Sigma^*, (x, y), O\}$. Vertices of Σ^* are $\{E_1, T_1, P, T_2, E_2\}$ where $T_1 = {}^t(1, 1)$ and $T_2 = {}^t(1, 2)$ are new vertices which are added to $\Gamma^*(f; x, y)$ to make Σ^* regular.

The proper transform $\widetilde{C}$ intersects transversely with the exceptional divisor $\hat{E}(P)$ at 1-point. Take a system of toric coordinates (u, v) which corresponds to $\mathrm{Cone}(P, T_2)$ so that $u = 0$ defines $\hat{E}(P)$ and let ξ be the intersection point. Then $\widetilde{C}$ is defined by $v - 1 = 0$ and $\widetilde{C}$ is smooth at ξ. We have $(\pi^* f) = 2\hat{E}(T_1) + 6\hat{E}(P) + 3\hat{E}(T_2) + \widetilde{C}$.

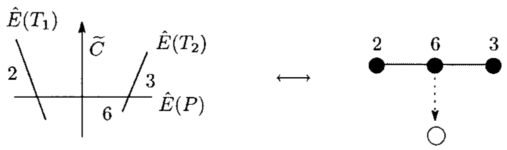


FIGURE 1

3. Some lemmas for torus curves of type *(p,q)*

3.1. Notation. Throughout this paper, we use the same notation as in [2], unless otherwise stated. We also use the fact that the topological equivalence class of a non-degenerate germ depends only on its Newton boundary (Theorem 2.1 of [2]). In the process of the classification of the topological type of inner singularities of a torus curve of type (2.5), we have the following possibilities:

(1) (C, O) is non-degenerate and the Newton boundary has one face,
(2) (C, O) is non-degenerate (in some local coordinate system) but the Newton boundary has several faces, or
(3) (C, O) has some degenerate faces for any choice of coordinates (x, y).

To make the expression of these singularity classes simpler, we introduce some notation. The class of singularities (1) can be expressed by the class of $B_{n,m}$:

$$B_{n,m}: \quad x^n + y^m = 0\,.$$

The class of singularities (2) can be understood as a union of singularities of type (1). For example, $x^n + x^2y^2 + y^m = 0$ is topologically equivalent to $(x^{n-2} + y^2)(x^2 + y^{m-2}) = 0$ and thus we denote this class by $B_{n-2,2} \circ B_{2,m-2}$, as is already introduced in §1. The last class (3) is most complicated. For example, we consider the singularity germ (C, O) which is defined by $f(x, y) = (\lambda x^3 + y^2)^2 + x^3y^3$. Then $\mathcal{N}(f; x, y) = (\lambda x^3 + y^2)^2$ and $\Gamma(f; x, y)$ consists of one face Δ with the weight vector $P = {}^t(2, 3)$ and f is degenerate on Δ. We take a regular simplicial subdivision Σ^* as in Example 1 and we take a toric modification $\pi_1 : X_1 \to \mathbf{C}^2$. Then we can see that $\widetilde{C_1}$ intersects transversely with $\hat{E}(P)$ at $\xi_1 = (0, -\lambda)$ in the system of the toric coordinates (u, v) corresponding to $\mathrm{Cone}(P, T_2)$ and $\hat{E}(P) = \{u = 0\}$ with multiplicity 12. (Figure 2).

To express the strict transform $\widetilde{C}$ at ξ_1, we choose the coordinate (u, v_1) where $v_1 = v + \lambda$. We call these coordinates (u, v_1) *the translated toric coordinates* for $(\widetilde{C}, \xi_1)$. Now we find that $(\widetilde{C}, \xi_1)$ is defined by $B_{3,2} : v_1^2 - \lambda u^3 + \text{(higher terms)} = 0$ where $u = 0$ defines the exceptional divisor which contains ξ_1. Observe that $\hat{E}(P)$ is defined by $u = 0$ and the tangent cone of $\widetilde{C}$ is $v_1^2 = 0$. Thus the tangent cone is transverse to $\hat{E}(P)$ at ξ_1. Again we take a toric blow-up $\pi_2 : X_2 \to (X_1, \xi_1)$. This is essentially the same as the one for the cusp singularity $v_1^2 - \lambda u^3 = 0$. As $\pi_1^* f(0)$ is non-degenerate in (u, v_1), π_2 gives a good resolution of $(\widetilde{C}, \xi_1)$ and therefore the composition $\pi_1 \circ \pi_2 : X_2 \to \mathbf{C}^2$ gives a good resolution of (C, O). The

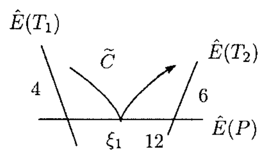


FIGURE 2

resolution graph is simply obtained by adding a bamboo for this blowing up (See (1) in Figure 3). We denote this class of singularity by $(B^2_{3,2})^{B_{3,2}}$.

Sometimes, we need to take a coordinate change of the type (u, v_2) where $v_2 = v_1 + h(u)$ for some polynomial $h(u)$. The important point here is that we do not change the first coordinate u, as it defines the exceptional divisor $\hat{E}(P)$. We call such a coordinate system (u, v_2) *admissible translated toric coordinates at* ξ_1.

EXAMPLE 2. Consider $f(x, y) = (y^5 - xy^2 + x^2y + x^5)^2 + (x - 2y)^5(x - 3y)^5$. Then $\mathcal{N}(f; x, y) = -31y^{10} - 2xy^7 + x^2y^2(x - y)^2 + 2x^7y + 2x^{10}$ and $\Gamma(f; x, y)$ consists of three faces Δ_i $(i = 1, 2, 3)$ with weight vectors $P_1 = {}^t(3, 1)$, $P_2 = {}^t(1, 1)$ and $P_3 = {}^t(1, 3)$. Note that $f(x, y)$ is degenerate on $\Delta(P_2; f)$. By adding vertices $T_1 = {}^t(2, 1)$ and $T_2 = {}^t(1, 2)$, we get the canonical regular subdivision. We take the associated toric modification and we can easily see that the strict transform $\widetilde{C}$ splits into three germs $\widetilde{C_1}$, $\widetilde{C_2}$ and $\widetilde{C_3}$ so that for $i = 1, 3$, $\widetilde{C_i}$ intersects transversely with the exceptional divisor $\hat{E}(P_i)$ at two points and $\widetilde{C_i}$ is smooth at these points. (Thus $\widetilde{C_1}$ and $\widetilde{C_3}$ are the union of two irreducible components of C.) The germ $\widetilde{C_2}$ which intersects with $\hat{E}(P_2)$ is still singular and intersects with $\hat{E}(P_2)$ transversely at $\xi_1 = (0, 1)$ in the toric coordinate (u, v) corresponding to Cone(P_2, T_2). Thus taking the translated toric coordinate (u, v_1), $v_1 = v - 1$ at ξ_1, we can write the defining equation of $\widetilde{C_2}$ as $v_1^2 - 4u^2v_1 - 239u^4 +$ (higher terms) $= 0$, while the exceptional divisor $\hat{E}(P_2)$ is defined by $u = 0$. Note that $(\widetilde{C_2}, \xi_1) = B_{4,2}$. Thus $\pi_1^* f$ is non-degenerate and we need one more toric modification centered at ξ_1. Then the resolution graph is given by (2) of Figure 3. We denote this class of singularity as $B_{6,2} \circ (B^2_{1,1})^{B_{4,2}} \circ B_{2,6}$.

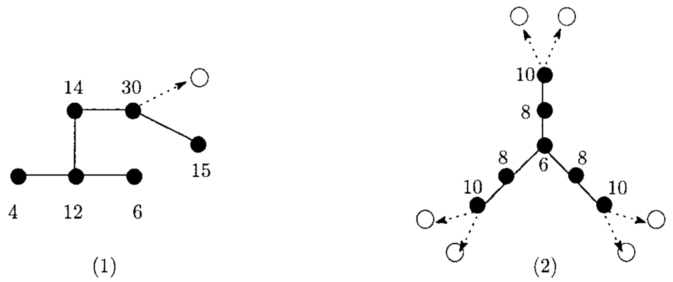


FIGURE 3

3.2. Some lemmas for general torus curves. First we prepare some lemmas for the general torus curves of type (p, q). Let $C = \{f = 0\}$ be a curve of torus type (p, q) which can be written as $f = f_q^p + f_p^q$ where f_p and f_q are polynomial of degree p and q respectively. We assume that $p \geq q \geq 2$. We put $C_q := \{f_q = 0\}$ and $C_p := \{f_p = 0\}$. Suppose that $O \in C_q \cap C_p$ and let ι be the local intersection multiplicity $I(C_p, C_q; O)$. We recall the following key lemmas. Hereafter we denote the tangent cone of C_p at O by $T_O C_p$.

LEMMA 1 (Lemma 1 of [1]). *Suppose that C_p is non-singular at O. Then the singularity (C, O) is topologically equivalent to the Brieskorn-Pham singularity $B_{p\iota,q}$.*

LEMMA 2 (Lemma 4.3 of [5]). *Suppose that (C_p, O) is singular with multiplicity m and (C_q, O) is smooth. If C_q intersects transversely with C_p and $p < qm$, the singularity (C, O) is topologically equivalent to $B_{mq,p}$.*

LEMMA 3. *Suppose that (C_p, O) is singular with multiplicity m and $T_O C_p$ consist of m distinct lines. Then C_p consists of m smooth components at O. Consider the local factorization $f_p = \prod_{i=1}^m g_i$.*

(1) *Suppose that C_q is smooth at O and $p < qm$.*

  (a) *If $\iota < \dfrac{p}{p-q}(m-1)$, then $(C, O) \sim B_{q\iota,p}$.*

  (b) *If $\iota > \dfrac{p}{p-q}(m-1)$, then $(C, O) \sim B_{\beta,\, q} \circ B_{q(m-1),\, p-q}$ where $\beta = p(\iota - m + 1) - q(m-1)$.*

  (c) *If $\iota = \dfrac{p}{p-q}(m-1)$ and the coefficients are generic, then $(C, O) \sim B_{q\iota,p}$.*

(2) *Suppose that $m = 2$, C_q is smooth and $p > 2q$. Then $(C, O) \sim B_{p(\iota-1)-q,q} \circ B_{q,p-q}$.*

(3) *Suppose that (C_q, O) is singular with multiplicity 2 and $T_O C_q$ consists of two distinct lines and let $f_q = h_1 h_2$ be the local factorization. Put $\ell_i = \{h_i = 0\}$, $i = 1, 2$. We assume that $g_1(x_2, y_2) = x_2$, $g_2(x_2, y_2) = y_2$ for a local coordinate system (x_2, y_2) and*

$$I(\ell_1, g_i; O) = 1\,, \quad (i \neq 1) \quad I(\ell_2, g_j; O) = 1, \quad (j \neq 2)\,.$$

Put $\nu_1 = I(\ell_1, g_1; O)$ and $\nu_2 = I(\ell_2, g_2; O)$. Assume that $2p > qm$. Then

$$(C, O) \sim B_{\beta_1,2} \circ (B^q_{m-2,\, m-2})^{(m-2)B_{2p-qm,q}} \circ B_{2,\beta_2}\,, \quad \beta_i = p(\nu_i + 1) - q(m-1)\,.$$

PROOF. By the assumption, $T_O C_p$ consists of m distinct lines. This implies (C_p, O) has m smooth components which are transverse with each other.

First we consider the case (1). First we take a local coordinate system (x_1, y_1) so that C_q is defined by $y_1 = 0$. Let $f_p(x_1, y_1) = \prod_{i=1}^m g_i(x_1, y_1)$ be the factorization in $\mathcal{O}$ such that $g_1(x_1, y_1) = y_1 + \alpha_1 x_1^\nu +$ (higher terms) and $g_i(x_1, y_1) = y_1 - \alpha_i x_1 +$ (higher terms) with $\alpha_i \neq 0$ $(2 \leq i \leq m)$. In this expression, we have $\iota = \nu + m - 1$. In the case of (a): $\iota < \frac{p}{p-q}(m-a)$, the Newton boundary of $f_p(x_1, y_1)^q$, $f_q(x_1, y_1)^p$ are on the left hand side of Figure 4. Thus it is easy to see that $f(x_1, y_1)$ is non-degenerate in this system of coordinates and we have $(C, O) \sim B_{q\iota,p}$.

In the case of (b) : $\iota > \frac{p}{p-q}(m-a)$, $f(x_1, y_1)$ are degenerate in this coordinate. We take another coordinate: (x_2, y_2) with $x_2 = x_1$, $y_2 = g_1(x_1, y_1)$ so that $y_2 | f_p$. Then in these coordinates (x_2, y_2), the Newton boundaries of $f_p(x_2, y_2)$ and $f_q(x_2, y_2)$ are given on the right hand side of Figure 4 so that $f(x_2, y_2)$ are now non-degenerate and the assertion follows.

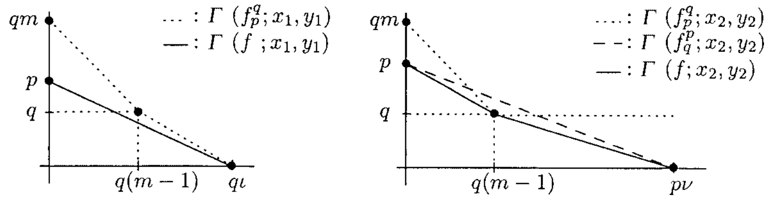


FIGURE 4

For the proof of (2), we take a local coordinate system so that $f_p(x, y) = c\, x_1 y_1$ where $c \neq 0$. Then we may assume that $f_q(x_1, y_1) = y_1 + c' x^{\iota - 1} +$ (higher terms). Then the assertion is immediate from the Newton boundary argument.

Next we consider the case (3). We have chosen a local coordinate system (x_2, y_2) so that $g_1(x_2, y_2) = c\, x_2$ $(c \neq 0)$ and $g_2(x_2, y_2) = y_2$. Put $g_i(x_2, y_2) = y_2 - \alpha_i x_2 +$ (higher terms) $(3 \leq i \leq m)$. By the assumption, we can write

$$h_1(x_2, y_2) = x_2 + d_1\, y_2^{\nu_1} + \text{(higher terms)}\,, \quad h_2(x_2, y_2) = y_2 + d_2\, x_2^{\nu_2} + \text{(higher terms)}$$

with $d_1,\ d_2 \neq 0$. Then the Newton principal part of f can be written as:

$$\mathcal{N}(f, x_2, y_2) = d_1^p\, y_2^{p(\nu_1+1)} + c^q\, x_2^q y_2^q \prod_{i=3}^{m} (y_2 - \alpha_i x_2)^q + d_2^p\, x_2^{p(\nu_2+1)}\,.$$

The Newton boundary of f consists of three faces Δ_1, Δ_2 and Δ_3 and f is non-degenerate on Δ_1, Δ_3 but degenerate on Δ_2. We take an admissible toric blowing-up $\pi : X \to \mathbf{C}^2$ with respect to some regular simplicial cone subdivision $\Sigma^* = \{P_0, \dots, P_k\}$. Assume that $P_\gamma = {}^t(1, 1)$, the weight vector of the homogeneous face Δ_2 and put $P_{\gamma+1} = {}^t(n, n{+}1)$. Then the pull-back of f_p and f_q to the coordinate chart $(\mathbf{C}^2_\sigma, (u_\gamma, v_\gamma))$ with $\sigma = \mathrm{Cone}(P_\gamma, P_{\gamma+1})$ are given by

$$\pi_\sigma^* f_p(u_\gamma, v_\gamma) = u_\gamma^m\, v_\gamma \prod_{i=3}^{m} \{g_i(u_\gamma v_\gamma^n, u_\gamma v_\gamma^{n+1})/u_\gamma\}$$
$$\pi_\sigma^* f_q(u_\gamma, v_\gamma) = u_\gamma^2 (h_1(u_\gamma v_\gamma^n, u_\gamma v_\gamma^{n+1})/u_\gamma) \times (h_2(u_\gamma v_\gamma^n, u_\gamma v_\gamma^{n+1})/u_\gamma)$$

where

$$g_i(u_\gamma v_\gamma^n, u_\gamma v_\gamma^{n+1})/u_\gamma \equiv v_\gamma - \alpha_i \mod u_\gamma$$
$$h_1(u_\gamma v_\gamma^n, u_\gamma v_\gamma^{n+1})/u_\gamma \equiv 1 \mod u_\gamma$$
$$h_2(u_\gamma v_\gamma^n, u_\gamma v_\gamma^{n+1})/u_\gamma \equiv v_\gamma \mod u_\gamma.$$

Thus we get

$$\begin{aligned}\pi_\sigma^*(f(u_\gamma, v_\gamma)) &= u_\gamma^{qm}(\bar{f}_p^q(u_\gamma, v_\gamma) + u_\gamma^{2p-qm}\bar{f}_q^p(u_\gamma, v_\gamma)) \\ &= u_\gamma^{qm}\left(\prod_{i=3}^{m}(v_\gamma - \alpha_i)^q + u^{2p-qm}\bar{f}_q(u_\gamma, v_\gamma) + (\text{higher terms})\right)\end{aligned}$$

and $\bar{f}_q(\alpha_i, 0) \neq 0$. We put $\xi_i = (0, \alpha_i)$ and we take local coordinates $(u_\gamma, v_{\gamma,i})$ at ξ_i with $v_{\gamma,i} = v_\gamma - \alpha_i$. The above expression implies $(\widetilde{C}, \xi_i) \sim B_{2p-qm,q}$. □

The assertion (1) of Lemma 3 can be generalized for the case where $T_O C_p$ may have some factors with multiplicity as follows.

LEMMA 4. *Suppose that C_p is singular at O and C_q is smooth at O. Let m be the multiplicity of (C_p, O). We take a local coordinate system (x_1, y_1) so that C_q is defined by $y_1 = 0$. We assume that $p < qm$ and*

- *either $y_1 \nmid (f_p)_m$ (this implies that C_q intersects transversely with $T_O C_p$), or*
- *$y_1 = 0$ is a simple tangent line of $T_O C_p$ (this is equivalent to $y_1 | (f_p)_m,\ y_1^2 \nmid (f_p)_m$) where $(f_p)_m$ is the homogeneous part of degree m of f_p.*

Then we have:

(1) *If $\iota < \dfrac{p}{p-q}(m-1)$, then $(C, O) \sim B_{q\iota,p}$.*

(2) *If $\iota > \dfrac{p}{p-q}(m-1)$, then $(C, O) \sim B_{\beta,\, q} \circ B_{q(m-1),\, p-q}$ where $\beta = p(\iota - m + 1) - q(m-1)$.*

(3) *If $\iota = \dfrac{p}{p-q}(m-1)$ and the coefficients are generic, then $(C, O) \sim B_{q\iota,p}$.*

PROOF. The proof is completely parallel to that of Lemma 3. □

4. Calculation of the local singularities

We come back to our original situation of a torus curve of type (2, 5):

$$C = \{f(x, y) = f_5(x, y)^2 + f_2(x, y)^5 = 0\},$$
$$C_5 = \{f_5(x, y) = 0\}, \quad C_2 = \{f_2(x, y) = 0\}.$$

First we consider the case that C_2 is reduced in Section 4 and 5. Next we consider the case of C being a linear torus curve in Section 6. For the classification, we start from the following generic equations:

$$f_2(x, y) = \sum_{i+j \le 2} a_{ij} x^i y^j, \quad f_5(x, y) = \sum_{i+j \le 5} b_{ij} x^i y^j.$$

Hereafter x, y are the affine coordinates $x = X/Z,\ y = Y/Z$ on $\mathbf{C}^2 := \mathbf{P}^2 \setminus \{Z = 0\}$. As we assume that $C_2,\ C_5$ pass through the origin, we have $a_{00} = b_{00} = 0$. We study the inner singularity $O \in C_2 \cap C_5$. We denote hereafter the multiplicities of C_2 and C_5 at the origin O by m_2 and m_5 respectively and the intersection multiplicity $I(C_2, C_5; O)$ of C_2 and C_5 at O by ι. By the Bézout theorem, we have the inequalities:

$$1 \leq \iota \leq 10\,, \quad m_2 m_5 \leq \iota\,.$$

The tangent cone of C_p at the origin is denoted by $T_O C_p$ for $p = 2, 5$.

For the classification of possible topological types of the singularity (C, O), we divide the situations into the 5 cases, corresponding to the values of m_5. Then for a fixed m_5, we consider the subcases, corresponding to ι, $m_5 \leq \iota \leq 10$ taking the geometry of the intersection of C_2 and C_5 at O into account. And each case has several subcases by type of the singularity of (C_5, O).

(1) Case I. $m_5 = 1$. The quintic C_5 is smooth.
(2) Case II. $m_5 = 2$. We divide this case into two subcases (a) and (b) by the type of the tangent cone $T_O C_5$.
   (a) The tangent cone $T_O C_5$ consists of two distinct lines i.e., $(C_5, O) \sim A_1$.
   (b) The tangent cone $T_O C_5$ consists of a single line with multiplicity 2.
(3) Case III. $m_5 = 3$. We divide this case into three subcases by the type of the tangent cone $T_O C_5$.
   (a) The tangent cone $T_O C_5$ consists of three distinct lines.
   (b) The tangent cone $T_O C_5$ consists of a line with multiplicity 2 and another line.
   (c) The tangent cone $T_O C_5$ consists of a single line with multiplicity 3.
(4) Case IV. $m_5 = 4$. We divide this case into five subcases by the type of the tangent cone $T_O C_5$.
   (a) The tangent cone $T_O C_5$ consists of four distinct lines.
   (b) The tangent cone $T_O C_5$ consists of a line with multiplicity 2 and two distinct lines.
   (c) The tangent cone $T_O C_5$ consists of a line with multiplicity 3 and another line.
   (d) The tangent cone $T_O C_5$ consists of a single line with multiplicity 4.
   (e) The tangent cone $T_O C_5$ consists of two lines with multiplicity 2.
(5) Case V. $m_5 = 5$. The quintic C_5 consists of five lines.

4.1. Case I: $m_5 = 1$**.** The quintic C_5 is smooth. This case is determined by Lemma 1 as follows.

PROPOSITION 1. *Suppose that* C_5 *is smooth at* O *and let* $\iota = I(C_5, C_2; O)$ *be the local intersection multiplicity. Then* $(C, O) \sim B_{5\iota,2}$ *for* $\iota = 1, \ldots, 10$.

4.2. Case II: $m_5 = 2$**.** We divide Case II into two subcases (a) and (b) by the type of $T_O C_5$.

(a) $T_O C_5$ consists of two distinct lines i.e., $(C_5, O) \sim A_1$.
(b) $T_O C_5$ consists of a line with multiplicity 2.

Case II-(a): We consider the subcase (a). In this case, we assume that $T_O C_5$ is given $xy = 0$ so that $f_5(x, y) = xy +$ (higher terms).

PROPOSITION 2. *Under the situation in* (a)*, we have the following possibilities.*

(1) *Assume that the conic* C_2 *is smooth and* $T_O C_2$ *is defined by* $y = 0$ $(a_{10} = 0,\ a_{01} \neq 0)$ *for simplicity. Then* (C, O) *is equivalent to one of* $B_{5\iota-7,2} \circ B_{2,3}$ *for* $2 \leq \iota \leq 10$.

(2) *If* C_2 *consists of two lines* ℓ_1, ℓ_2 *(i.e.,* $a_{01} = a_{10} = 0$*), the generic singularity is* $B_{8,2} \circ B_{2,8}$. *Further degeneration occurs when these lines are tangent to one or both tangent cones of* C_5. *Put* $\iota_i = I(\ell_i, C_5; O)$ *for* $i = 1, 2$. *Then* $4 \leq \iota_1 + \iota_2 \leq 10$ *and the corresponding singularity is* $B_{5\iota_2-2,2} \circ B_{2,5\iota_1-2}$.

PROOF. Both assertions are immediate from Lemma 2 and Lemma 3 □

Case II-(b): We assume that $m_5 = 2$ and $T_O C_5$ is given by $L : y^2 = 0$ (with multiplicity 2). We divide this subcase (II-b) into two subcases: (b-1) $m_2 = 1$ and (b-2) $m_2 = 2$.

(b-1) Assume that $m_5 = 2$, $m_2 = 1$ and $T_O C_5$ is defined by $y^2 = 0$.

PROPOSITION 3. *Suppose that the tangent cone* $T_O C_5$ *is a line with multiplicity* 2 *and* C_2 *is smooth. Then we have the following possibilities.*

(1) *If* C_2 *and* $T_O C_5$ *are transverse at* O $(\iota = 2)$, *then* $(C, O) \sim B_{5,4}$.

(2) *If* $(C_5, O) \sim B_{3,2}$ *and* $\iota = 3$, *then we have* $(C, O) \sim (B_{3,2}^2)^{B_{3,2}}$.

(3) *If* $(C_5, O) \sim B_{4,2}$, *then we have* $(C, O) \sim (B_{4,2}^2)^{(B_{5\iota-18,2}+B_{2,2})}$ *for* $\iota = 4, \dots, 10$. *(Here the upper* $(B_{5\iota-18,2}+B_{2,2})$ *implies we have two non-degenerate singularities* $B_{5\iota-18,2}$, $B_{2,2}$ *sitting on two different points on the exceptional divisor* $\widehat{E}(P)$, $P = {}^t(1, 2)$, *after one toric modification.)*

(4) *If* $(C_5, O) \sim B_{5,2}$, *then we have*
 (a) $(C, O) \sim B_{10,4}$ *generically and* $B_{k,2} \circ B_{5,2}$, $6 \leq k \leq 15$ *for* $\iota = 4$,
 (b) $(B_{5,2}^2)^{B_{5,2}}$ *for* $\iota = 5$.

(5) *If* $(C_5, O) \sim B_{6,2}$, *then we have*
 (a) $(C, O) \sim B_{10,4}$ *for* $\iota = 4$ *and*
 (b) $(C, O) \sim (B_{6,2}^2)^{(B_{5\iota-27,2}+B_{3,2})}$ *for* $\iota = 6, \dots, 10$.

(6) *If* $(C_5, O) \sim B_{7,2}$, *then we have*
 (a) $(C, O) \sim B_{10,4}$ *for* $\iota = 4$ *and*
 (b) $(C, O) \sim (B_{7,2}^2)^{B_{5\iota-28,2}}$ *for* $\iota = 6, 7$.

(7) *If* $(C_5, O) \sim B_{8,2}$, *then we have*
 (a) $(C, O) \sim B_{10,4}$ *for* $\iota = 4$,
 (b) $(C, O) \sim B_{15,4}$ *for* $\iota = 6$,
 (c) $(B_{8,2}^2)^{(B_{5\iota-36,2}+B_{4,2})}$ *for* $\iota = 8, 9, 10$.

(8) *If* $(C_5, O) \sim B_{9,2}$, *then we have*
 (a) $(C, O) \sim B_{10,4}$ *for* $\iota = 4$,
 (b) $(C, O) \sim B_{15,4}$ *for* $\iota = 6$,
 (c) $(C, O) \sim (B_{9,2}^2)^{B_{5\iota-35,2}}$ *for* $\iota = 8, 9$.

(9) *If* $(C_5, O) \sim B_{10,2}$, *then we have*
 (a) $(C, O) \sim B_{10,4}$ *for* $\iota = 4$,
 (b) $(C, O) \sim B_{15,4}$ *for* $\iota = 6$,

(c) $(C, O) \sim B_{20,4}$ *and* $B_{k,2} \circ B_{10,2}$ $(k = 11, 12)$ *for* $\iota = 8$,

(d) $(B_{10,2}^2)^{2B_{5,2}}$ *for* $\iota = 10$.

(10) *If* $(C_5, O) \sim B_{11,2}$, *then we have*

(a) $(C, O) \sim B_{10,4}$ *for* $\iota = 4$,

(b) $(C, O) \sim B_{15,4}$ *for* $\iota = 6$,

(c) $(C, O) \sim B_{20,4}$ *for* $\iota = 8$,

(d) $(C, O) \sim (B_{11,2}^2)^{B_{6,2}}$ *for* $\iota = 10$.

(11) *If* $(C_5, O) \sim B_{12,2}$, *then we have*

(a) $(C, O) \sim B_{10,4}$ *for* $\iota = 4$,

(b) $(C, O) \sim B_{15,4}$ *for* $\iota = 6$,

(c) $(C, O) \sim B_{20,4}$ *for* $\iota = 8$,

(d) $(C, O) \sim (B_{12,2}^2)^{2B_{1,2}}$ *for* $\iota = 10$.

(12) *If* $(C_5, O) \sim B_{13,2}$, *then we have*

(a) $(C, O) \sim B_{10,4}$ *for* $\iota = 4$,

(b) $(C, O) \sim B_{15,4}$ *for* $\iota = 6$,

(c) $(C, O) \sim B_{20,4}$ *for* $\iota = 8$,

(d) $(C, O) \sim B_{25,4}$ *for* $\iota = 10$.

PROOF. We can proceed with the classification mainly using the local intersection multiplicity $\iota = I(C_2, C_5; O)$ and the geometry of C_2 and C_5. By the assumption $m_5 = 2$, we have $\iota \geq 2$. When $\iota = 2$, C_2 intersects transversely with $T_O C_5$ at the origin, then $(C, O) \sim B_{5,4}$. Thus hereafter we assume that $\iota \geq 3$.

Assume that $(C_5, O) \sim A_{\ell-1}$. Then by taking local coordinates (x, y_1) where $y_1 = y + c_2x^2 + \cdots + c_{k-1}x^{k-1}$, we can write f_5 as

$$f_5(x, y_1) = \alpha y_1^2 + \beta x^\ell + (\text{higher terms}), \quad \ell \geq 2(k-1), \quad \alpha, \beta \neq 0.$$

A simple computation shows that $\ell \leq 13$. Now we can write f_2 in this coordinates as

$$f_2(x, y_1) = \gamma y_1 + \delta x^\nu + (\text{higher terms}), \quad \nu \geq 2, \quad \gamma, \delta \neq 0.$$

(Here $\delta = 0$ if $\nu = \infty$, i.e., $y_1 | f_2$.) First we notice that

$(\star)$: $\iota \geq \min(2\nu, \ell)$ and the equality holds except for the case $\ell = 2\nu$ and $\alpha\delta^2 + \beta\gamma^2 = 0$.

We observe that

1. If $5\nu < 2\ell$, then f is non-degenerate in these coordinates and $(C, O) \sim B_{5\nu,4}$.
2. If $5\nu = 2\ell$ (in this situation, the possible pairs of (ν, ℓ) which satisfy this condition are $(\nu, \ell) = (2, 5),\ (4, 10)$), then we have $\mathcal{N}(f, x, y) = \alpha^2 y_1^4 + 2\alpha\beta x^\ell y_1^2 + (\beta^2 + \delta^5)x^{2\ell}$ and if $(\beta^2 + \delta^5) \neq 0$, then we have $(C, O) \sim B_{2\ell,4}$. If $(\beta^2 + \delta^5) = 0$, then the Newton boundary has two faces with $R = (\ell, 2)$ as the common vertex and f can be non-degenerate on these faces, after taking a suitable triangular change of coordinates (x, y_2).

3. If $5\nu > 2\ell$, then it is easy to see that the Newton principal part of $f(x, y_1)$ is given by $(\alpha y_1^2 + \beta x^\ell)^2$ which implies that $f(x, y_1)$ is degenerate in this coordinate. So we first need to take a toric modification $\pi : X \to \mathbf{C}^2$ with respect to the canonical regular simplicial cone subdivision $\{P_0, \dots, P_m\}$ and we have to study the equation of the total transform $\pi^* f$ in X.

The weight vector of $A_{\ell-1}$ is given as $P = {}^t(2, \ell),\ {}^t(1, \ell/2)$ for ℓ is odd or even respectively. Note also the germ $A_{\ell-1}$ has two smooth components if ℓ is even. Thus the description for the toric modification has to be divided in two cases.

Case A. ℓ is odd. The weight vector of $\Gamma(f_5; x, y_1)$ is given by $P = {}^t(2, \ell)$. We may assume that $P = P_s$ and we consider the cone $\sigma := \mathrm{Cone}(P_s, P_{s+1})$ and corresponding unimodular matrix

$$\sigma := \begin{pmatrix} 2 & a \\ \ell & b \end{pmatrix}, \quad 2b - a\ell = 1\,.$$

Let (u, v) be the toric coordinates of this chart. Then in these coordinates, we have $x = u^2 v^a,\ y_1 = u^\ell v^b$. We can write

$$\pi^* f_5(u, v) = u^{2\ell} v^{a\ell} \tilde{f}_5(u, v)\,, \quad \tilde{f}_5(u, v) = \alpha v + \beta + h_5(u, v)\,,$$
$$\pi^* f_2(u, v) = u^\mu v^{\mu'} \tilde{f}_2(u, v)\,, \quad \mu = \min(\ell, 2\nu)\,,\ \mu' = \min(b, a\nu)\,.$$

Putting $\xi = (0, -\beta/\alpha),\ \eta = 5\mu - 4\ell$, we can write $\pi^* f$ as

$$\pi^* f(u, v) = u^{4\ell} v^{2a\ell} \tilde{f}(u, v)\,,$$
$$\tilde{f}(u, v) = \tilde{f}_5(u, v)^2 + u^\eta v^{5\mu' - 2a\ell} \tilde{f}_2(u, v)^5\,.$$

Thus using admissible translated toric coordinates (u, v_2), $v_2 = v_1 + h(u),\ v_1 = v + \beta/\alpha$ for some polynomial h, the strict transform is defined as

$$\alpha^2 v_2^2 + \varepsilon u^{\eta'} + (\text{higher terms}) = 0\,,\ \varepsilon \neq 0$$

which implies $(\tilde{C}, \xi) \sim B_{\eta', 2}$ and the tangent cone is transverse to the exceptional divisor $u = 0$ where $\eta' \geq \eta$.

Case B. ℓ is even. The weight vector of $\Gamma(f_5; x, y_1)$ is given by $P = {}^t(1, k)$ where $\ell = 2k$. We may assume that there is a cone corresponding to a unimodular matrix

$$\sigma := \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}.$$

Let (u, v) be the toric coordinates of this chart. Then in these coordinates, we have $x = u,\ y_1 = u^k v$. Then we can write

$$\pi^* f_5(u, v) = u^\ell \tilde{f}_5(u, v)\,, \quad \tilde{f}_5(u, v) = \alpha(v + \alpha_1)(v + \alpha_2) + uvh_5(u, v)\,,$$
$$\pi^* f_2(u, v) = u^\mu \tilde{f}_2(u, v),\ \mu = \min(k, \nu)\,.$$

Putting $\xi_i = (0, \alpha_i),\ (i = 1, 2),\ \eta = 5\mu - 4\ell$, we can write $\pi^* f$ as

$$\pi^* f(u, v) = u^{2\ell} \left(\tilde{f}_5(u, v)^2 + u^{\eta} \tilde{f}_2(u, v)^5 \right) .$$

Then the strict transform $\widetilde{C}$ has two components. Thus using admissible translated toric coordinates $(u, v_i'),\ v_i' = v_i + h(u),\ v_i = v + \alpha_i\ (i = 1, 2)$ in a neighborhood of $(u, v) = (0, -\alpha_i)$ for some polynomial h, the total transform $\pi^* f$ is described as

$$\pi^* f(u, v_i') = u^{2\ell}\Big(\alpha^2(\alpha_1 - \alpha_2)^2 v_i'^2 + \varepsilon u^{\eta'} + \text{(higher terms)}\Big),\ \varepsilon \neq 0$$

which implies $(\tilde{C}, \xi_i) \sim B_{\eta',2}$ where $\eta' \geq \eta$. Putting the strategy above into consideration, we will explain several cases in more detail.

First we consider the case (2) in Proposition 3: $(C_5, O) \sim B_{3,2},\ \iota = 3\ (\ell = 3,\ \nu \geq 2)$.

We have $f_5(x, y) = \alpha\, y^2 + \beta x^3 +$ (higher terms). We have to consider the toric modification in the toric coordinate chart:

$$\sigma = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \quad \pi(u, v) = (u^2 v, u^3 v^2)$$

in the observation above. Then taking the translated coordinates $(u, v_1),\ v_1 = v + \beta/\alpha$, we have

$$\begin{aligned}
\pi^* f_5(u, v_1) &= u^6 (v_1 - \beta/\alpha)^3 (\alpha v_1 + cu + h_5(u, v_1)) ,\\
\pi^* f_2(u, v_1) &= u^3 (v_1 - \beta/\alpha)^2 (\delta u + \gamma + h_2(u, v_1)) ,\\
\pi^* f(u, v_1) &= u^{12} (v_1 - \beta/\alpha)^6 \tilde{f}(u, v_1) ,\\
\tilde{f}(u, v_1) &= (\alpha v_1 + cu + h_5(u, v_1))^2 + u^3 (v_1 - \beta/\alpha)^4 (\delta u + \gamma + h_2(u, v_1))^5 .
\end{aligned}$$

Now we can see that

$$\tilde{f}(u, v_2) = v_2^2 + c' u^3 + \text{(higher terms)}, \quad c' \neq 0, \quad v_2 = \alpha v_1 + cu$$

which implies that the corresponding singularity is $(B_{3,2}^2)^{B_{3,2}}$.

Next we consider the case (3) in Proposition 3. Thus we assume $(C_5, O) \sim B_{4,2}$ and $\iota \geq 4$. Then

$$\begin{aligned}
f_5(x, y_1) &= \alpha\, y_1^2 + \beta\, x^4 + \text{(higher terms)} ,\\
f_2(x, y_1) &= \gamma y_1 + \delta x^{\nu} + \text{(higher terms)} ,\ \nu \geq 2 .
\end{aligned}$$

Note that $\nu \geq 2$ and the case $\iota > 4$ only if $\nu = 2$ and $\alpha\, \delta^2 + \beta\, \gamma^2 = 0$. Thus for simplicity, we assume that $\nu = 2$. For the simplicity of the calculation, we put:

$$\begin{aligned}
f_5(x, y_1) &= \alpha y_1^2 + \beta x^4 + \text{(higher terms)} = \alpha(y_1 + \alpha_1 x)(y_1 + \alpha_2 x) + \text{(higher terms)} ,\\
f_2(x, y_1) &= \gamma y_1 + \delta x^2 + \text{(higher terms)}) .
\end{aligned}$$

The corresponding toric chart is associated with:

$$\sigma = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \pi(u, v) = (u, u^2 v)$$

by the above consideration. Note that $\iota = 4$ if and only if $\alpha\,\delta^2 + \beta\,\gamma^2 \neq 0$. Then taking the translated coordinates $(u, v_1),\ v_1 = v + \alpha_1$ (respectively $(u, v_2),\ v_2 = v + \alpha_1$), we have

$$\pi^* f_5(u, v_1) = u^4(\alpha(\alpha_1 - \alpha_2)v_1 + c_1 u + h_5(u, v_1)),$$

$$\pi^* f_2(u, v_1) = u^2((\gamma v_1 - \gamma\alpha_1 + \delta) + u h_2(u, v_1)),$$

$$\pi^* f(u, v_1) = u^8 \tilde{f}(u, v_1), \quad (\text{resp. } \pi^* f(u, v_2) = u^8 \tilde{f}(u, v_2)),$$

$$\tilde{f}(u, v_1) = (\alpha(\alpha_1 - \alpha_2)v_1 + c_1 u)^2 + (\delta - \gamma\alpha_1)^5\, u^2 + (\text{higher terms}),$$

$$(\text{resp. } \tilde{f}(u, v_2) = (\alpha(\alpha_2 - \alpha_1)v_1 + c_2 u)^2 + (\delta - \gamma\alpha_2)^5\, u^2 + (\text{higher terms}))$$

where c_i is constant for $i = 1, 2$. Then if $\iota = 4$, we have $\alpha\,\delta^2 + \beta\,\gamma^2 \neq 0$ and we see that $(\widetilde{C}, \xi_i) = A_1$ for $i = 1, 2$. Thus $(C, O) \sim (B_{4,2}^2)^{2B_{2,2}}$ and the resolution graph is given by Figure 5.

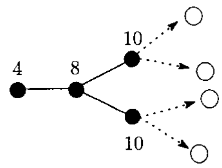


FIGURE 5

The case $\nu > 2$ gives the same conclusion as above.

If $\nu = 2$, $\iota > 4$ and if $\alpha\,\delta^2 + \beta\,\gamma^2 \neq 0$, then we have $(\widetilde{C}, \xi_2) = A_1$ but $(\widetilde{C}, \xi_1)$ is bigger than A_1. Thus we have to take a triangular change of coordinates (u, v_1') so that $\widetilde{C}$ is defined at ξ_1 as $(v_1')^2 + c'\, u^k + (\text{higher terms}) = 0$. The explicit computation shows that the possibilities of k are $5\iota - 18$ for $4 \leq \iota \leq 10$.

Next we consider the assertion (4) in Proposition 3. Assume $(C_5, O) \sim B_{5,2}$ and $\iota \geq 4$. We put as above

$$f_5(x, y_1) = \alpha y_1^2 + \beta x^5 + (\text{higher terms}), \qquad f_2(x, y_1) = \gamma y_1 + \delta x^\nu + (\text{higher terms}).$$

Note that $\iota = 4$ if and only if $\nu = 2$. If $\nu > 2$, $\iota = 5$ and $\mathcal{N}(f, x, y_1) = (\alpha y_1^2 + \beta x^5)^2$ and we have to take a toric modification. If $\nu = 2$, then $\iota = 4$ and as $\mathcal{N}(f, x, y_1) = y_1^4 + 2\beta x^5 y_1^2 + (\beta^2 + \delta^5)x^{10}$, we see that $(C, O) \sim B_{10,4}$ if $\iota = 4$ and $\beta^2 + \delta^5 \neq 0$. If $\beta^2 + \delta^5 = 0$, the Newton boundary has two faces.

First we consider the case $\iota = 4$ (so $\delta \neq 0$) and $\beta^2 + \delta^5 = 0$. Then $\mathcal{N}(f, x, y_1) = y_1^4 + 2\beta x^5 y_1^2 + \gamma_8 x^8 y_1 + \gamma_{11} x^{11}$ and $\Gamma(f; x, y_1)$ consists of two faces Δ_1 and Δ_2. Clearly

f is non-degenerate on Δ_1. If f is degenerate on Δ_2, we take a suitable triangular change of coordinates (x, y_2) so that $\mathcal{N}(f; x, y_2) = \alpha^2 y_2^4 + 2\alpha\beta x y_2^2 + \gamma' x^{11+k}$, $k = 0, \dots, 9$. This implies $(C, O) \sim B_{k+6,2} \circ B_{5,2}$.

Secondly, we consider the case $\iota \geq 5$ (i.e., $\nu > 2$). In this case, due to the previous consideration, we see that $\iota = 5$. Then $\mathcal{N}(f, x, y_1) = (\alpha y_1^2 + \beta x^5)^2$ and $\Gamma(f; x, y_1)$ consists of one face Δ with the weight vector $P = {}^t(2, 5)$ and f is degenerate on Δ. We consider the toric modification with respect to the canonical regular subdivision Σ^* of $\Gamma^*(f; x, y_1)$. The toric coordinate chart which intersects the strict transform $\widetilde{C}$ is described by a unimodular matrix

$$\sigma = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, \quad \pi(u, v) = (u^2 v, u^5 v^3) .$$

Then taking admissible translated toric coordinates (u, v_2), $v_2 = v_1 + h(u)$, $v_1 = \alpha v + \beta$ for a suitable polynomial h, we have

$$\pi^* f(u, v_2) = u^{20}(v_2 - \beta/\alpha + h(u))\Big(\alpha^2 v_2^2 + \beta'' u^5 + \text{(higher terms)}\Big), \quad \beta'' \neq 0 .$$

Thus we get $(C, O) \sim (B_{5,2}^2)^{B_{5,2}}$. Hence we have the assertion (4) of Proposition 3.

The assertions (5), ..., (12) of Proposition 3 can be shown in a similar manner. □

REMARK 1. Note that the singularity $B_{25,4}$ in case (12) has the Milnor number 72 and 72 is the maximum Milnor number of an irreducible curve of degree 10. Thus in this case, C is a rational curve.

The classification (Proposition 3) can be rewritten as follows from the viewpoint of ι.

(1) If $\iota = 2$, then we have $(C, O) \sim B_{5,4}$.

(2) If $\iota = 3$, then we have $(C, O) \sim (B_{3,2}^2)^{B_{3,2}}$.

(3) If $\iota = 4$, then we have $(C, O) \sim (B_{4,2}^2)^{2B_{2,2}}$, $B_{10,4}$ and $B_{k,2} \circ B_{5,2}$ $(6 \leq k \leq 15)$.

(4) If $\iota = 5$, then we have $(C, O) \sim (B_{4,2}^2)^{(B_{7,2}+B_{2,2})}$ and $(B_{5,2}^2)^{B_{5,2}}$.

(5) If $\iota = 6$, then we have $(C, O) \sim (B_{4,2}^2)^{(B_{12,2}+B_{2,2})}$, $(B_{6,2}^2)^{2B_{3,2}}$, $(B_{7,2}^2)^{B_{2,2}}$ and $B_{15,4}$.

(6) If $\iota = 7$, then we have $(C, O) \sim (B_{4,2}^2)^{(B_{17,2}+B_{2,2})}$, $(B_{6,2}^2)^{(B_{8,2}+B_{3,2})}$ and $(B_{7,2}^2)^{B_{7,2}}$.

(7) If $\iota = 8$, then we have $(C, O) \sim (B_{4,2}^2)^{(B_{22,2}+B_{2,2})}$, $(B_{6,2}^2)^{(B_{13,2}+B_{3,2})}$, $(B_{8,2}^2)^{2B_{4,2}}$, $(B_{9,2}^2)^{B_{5,2}}$, $B_{20,4}$, $B_{11,2} \circ B_{10,2}$ and $B_{12,2} \circ B_{10,2}$.

(8) If $\iota = 9$, then we have $(C, O) \sim (B_{4,2}^2)^{(B_{27,2}+B_{2,2})}$, $(B_{6,2}^2)^{(B_{18,2}+B_{3,2})}$, $(B_{8,2}^2)^{(B_{9,2}+B_{4,2})}$ and $(B_{9,2}^2)^{B_{10,2}}$.

(9) If $\iota = 10$, then we have $(C, O) \sim (B_{4,2}^2)^{(B_{32,2}+B_{2,2})}$, $(B_{6,2}^2)^{(B_{23,2}+B_{3,2})}$, $(B_{8,2}^2)^{(B_{14,2}+B_{4,2})}$, $(B_{10,2}^2)^{2B_{5,2}}$, $(B_{11,2}^2)^{B_{6,2}}$, $(B_{12,2}^2)^{2B_{1,2}}$ and $B_{25,4}$.

(b-2) Assume that C_2 is a union of two lines meeting at the origin ($m_2 = 2$) and $T_O C_5$ is $y^2 = 0$.

PROPOSITION 4. *Suppose that the tangent cone $T_O C_5$ is a line with multiplicity* 2 *and C_2 is a union of two lines meeting at the origin. Then the germ (C, O) take one the following singularities.*

(1) *If $(C_5, O) \sim B_{3,2}$, then we have*
 (a) $(C, O) \sim (B_{3,2}^2)^{B_{8,2}}$ *for* $\iota = 4$ *and*
 (b) $(C, O) \sim (B_{3,2}^2)^{B_{13,2}}$ *for* $\iota = 5$.

(2) *If $(C_5, O) \sim B_{4,2}$, then we have*
 (a) $(C, O) \sim (B_{4,2}^2)^{2B_{2,2}}$ *for* $\iota = 4$,
 (b) $(C, O) \sim (B_{4,2}^2)^{2B_{7,2}}$ *for* $\iota = 6$, *and*
 (c) $(C, O) \sim B_{16,2} \circ (B_{2,1}^2)^{B_{7,2}}$ *for* $\iota = 7$.

(3) *If $(C_5, O) \sim B_{5,2}$, then we have*
 (a) $(C, O) \sim B_{10,4}$ *or* $B_{k,2} \circ B_{5,2}$ $(6 \le k \le 15)$ *for* $\iota = 4$,
 (b) $(C, O) \sim (B_{5,2}^2)^{B_{10,2}}$ *for* $\iota = 6$ *and*
 (c) $(C, O) \sim (B_{5,2}^2)^{B_{15,2}}$ *for* $(\iota = 7)$.

(4) *If $(C_5, O) \sim B_{6,2}$, then we have*
 (a) $(C, O) \sim B_{10,4}$ *for* $\iota = 4$ *and*
 (b) $(C, O) \sim (B_{6,2}^2)^{2B_{3,2}}$ *for* $\iota = 6$.

(5) *If $(C_5, O) \sim B_{7,2}$, then we have*
 (a) $(C, O) \sim B_{10,4}$ *for* $\iota = 4$ *and*
 (b) $(C, O) \sim (B_{7,2}^2)^{B_{2,2}}$ *for* $\iota = 6$.

(6) *If $(C_5, O) \sim B_{k,2}$, then we have*
 (a) $(C, O) \sim B_{10,4}$ *for* $\iota = 4$ *and*
 (b) $(C, O) \sim B_{15,4}$ $(8 \le k \le 13)$ *for* $\iota = 6$

PROOF. By taking local coordinates (x, y_1), we can assume

$$f_5(x, y_1) = \alpha y_1^2 + \beta x^\ell + (\text{higher terms}), \quad \alpha, \beta \neq 0.$$

Now we assume that $f_2(x, y_1) = \ell_1(x, y_1)\ell_2(x, y_1)$ where

$$\ell_1 = y_1 + c_\nu x^\nu + (\text{higher terms}), \quad \ell_2 = c_2(y_1 + \gamma x) + (\text{higher terms}), \quad c_\nu, c_2 \neq 0.$$

We put $\iota_1 = I(\ell_1, C_5; O) \le 5$ and $\iota_2 = I(\ell_2, C_5; O)$. As $\gamma \neq 0$, we have $\iota_2 = 2$. Hence $\iota = \iota_1 + 2$ and we have $4 \le \iota \le 7$.

Comparing the Newton boundaries of f_5^2 and f_2^5 and applying a similar argument as in (b-1) of Case II-(b), we get assertions of Proposition 4. □

4.3. Case III: $m_5 = 3$**.** We divide Case III into three cases by the type of $T_O C_5$.

(a) $T_O C_5$ consists of three distinct lines.
(b) $T_O C_5$ consists of a line with multiplicity 2 and another line.
(c) $T_O C_5$ consists of is a single line with multiplicity 3.

First we remark that if C_2 is smooth and C_2 intersects transversely with $T_O C_5$ at the origin ($\iota = 3$), we have $(C, O) \sim B_{6,5}$ by Lemma 2 in §3. So hereafter, we consider the case that C_2 and $T_O C_5$ do not intersect transversely.

Case III-(a): We first consider Case III-(a). We assume that $T_O C_5$ consists of three distinct lines.

PROPOSITION 5. *Under the situation of Case III-(a), we have* $(C, O) \sim B_{6,5}$ *if* $\iota = 3$. *For* $\iota \geq 4$, *we have the following possibilities of* (C, O).

(1) *Assume that* C_2 *is smooth and tangent to* $y = 0$. *Then* (C, O) *can be* $B_{5\iota-14,2} \circ B_{4,3}$ *for* $\iota = 4, \dots, 10$.

(2) *Assume that* C_2 *consists of two distinct lines* ℓ_1, ℓ_2. *Put* $\iota_i = I(\ell_i, C_5; O) \geq 3$ $(i = 1, 2)$. *Then* $(C, O) \sim B_{5\iota_2-9,2} \circ (B_{1,1}^2)^{B_{4,2}} \circ B_{2,5\iota_1-9}$ *with* $\iota_1 + \iota_2 = 6, \dots, 10$.

PROOF. The assertion is immediate from Lemma 3. □

Case III-(b): In this case, we may assume that $T_O C_5$ consists of a line with multiplicity 2 which is defined by $\{y = 0\}$ and a single line $\{x = 0\}$.

(b-1) First we assume that C_2 is smooth ($m_2 = 1$). If $\iota = 3$, we have $(C, O) \sim B_{6,5}$ by Lemma 2. Therefore we consider the case $\iota \geq 4$. The common tangent line of C_2 is either $\{x = 0\}$ or $\{y = 0\}$. When the common tangent line is $\{x = 0\}$, (C, O) is described by Lemma 4.

So we assume that common tangent cone is $\{y = 0\}$. If $\iota = 4$, we have $f_5(x, y) = b_{12}xy^2 + b_{40}x^4 +$ (higher terms) and $\iota = 4$ if and only if $b_{40} \neq 0$. Hence we have $(C, O) \sim B_{8,5}$.

Next we consider the case $\iota \geq 5$ and we take a local coordinate system (x, y_1) so that C_2 is defined by $y_1 = 0$ and we have $f_5(x, y_1) = \beta_{12}xy_1^2 + \beta_{31}x^3y_1 + \beta_{50}x^5 +$ (higher terms) with $\beta_{12} \neq 0$. First we assume that $\iota = 5$. Then $\beta_{50} \neq 0$ and we factor $\mathcal{N}(f; x, y_1)$ as

$$\mathcal{N}(f; x, y_1) = y_1^5 + x^2(\beta_{12}y_1^2 + \beta_{31}x^2y_1 + \beta_{50}x^4)^2 = \prod_{i=1}^{5}(y_1 + \alpha_i x^2)\,.$$

We see that $\Gamma(f; x, y)$ consists of one face with the weight vector $P = {}^t(1, 2)$. Then we have several cases:

(1) $\alpha_1, \dots, \alpha_5$ are all distinct.
(2) $\alpha_1 = \alpha_2$ and $\alpha_3, \alpha_4, \alpha_5$ are mutually distinct and different from α_1.
(3) $\alpha_1 = \alpha_2 = \alpha_3$ and α_4, α_5 are mutually distinct and different from α_1.
(4) $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$ and $\alpha_1 \neq \alpha_3$ and α_5 is different from α_1, α_3.

By an easy computation, we can see that the other cases are not possible. (By a direct computation, we see that if $\mathcal{N}(f; x, y_1) = 0$ has a root with multiplicity 4, $\beta_{50} = 0$ and the intersection number jumps to 6.)

LEMMA 5. *Under the above situation, we further assume that* $\iota = 5$.

(1) *If* $\alpha_1, \dots, \alpha_5$ *are all distinct, then* $(C, O) \sim B_{10,5}$.

(2) *If* $\alpha_1 = \alpha_2$ *and* $\alpha_3, \alpha_4, \alpha_5$ *are mutually distinct and different from* α_1, *then* $(C, O) \sim B_{k,2} \circ B_{6,3}$, $(5 \le k \le 12)$.

(3) *If* $\alpha_1 = \alpha_2 = \alpha_3$ *and* α_4, α_5 *are mutually distinct and different from* α_1, *then* $(C, O) \sim B_{k,3} \circ B_{4,2}$ $(k = 7, \dots, 11)$ *or* $B_{3,1} \circ B_{5,2} \circ B_{4,2}$ *or* $B_{3,1} \circ B_{7,2} \circ B_{4,2}$ *or* $B_{k,2} \circ B_{3,1} \circ B_{4,2}$ $(k = 7, 8, 9)$.

(4) *If* $\alpha_1 = \alpha_2, \alpha_3 = \alpha_4$ *and* $\alpha_1 \ne \alpha_3$ *and* α_5 *is different from* α_1, α_3, *then* $(C, O) \sim B_{k_2+4,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{k_1,2}}$ *where* (k_1, k_2) *moves in the set* $\{(k_1, k_2); 13 - k_2 \ge k_1 \ge k_2 - 4,\ k_2 = 5, \dots, 7\} \cup \{(4, 8), (5, 9)\}$.

PROOF. The case (1) is clear. We consider the case (2) and we may assume $\alpha_1 = \alpha_2$. Then we see that the Newton boundary $\Gamma(f; x, y_1)$ has two faces Δ_1 and Δ_2 and f is non-degenerate on Δ_1. Taking a suitable triangular coordinate change, we can make f non-degenerate on Δ_2. Hence this gives the series $(C, O) \sim B_{k,2} \circ B_{6,3}$, $k = 5, \dots, 12$. We can consider the cases (3) and (4) similarly. □

REMARK 2. In (4) of Lemma 5, we have the following symmetry. Let

$$f(x, y_1) = (y_1 + \alpha_1 x^2)^2 (y_1 + \alpha_3 x^2)^2 (y_1 + \alpha_5 x^2) + (\text{higher terms})$$

be a defining polynomial of (C, O). First we take a change of coordinates (x, y_2) with $y_2 = y_1 + \alpha_1 x^2$ and we take further changes of coordinates of type $y_2 \to y_2 + c\,x^j$, $2 \le j \le [k_2/2]$ if necessary and we can assume

$$f(x, y_2) = y_2^2 (y_2 + (\alpha_3 - \alpha_1)x^2)^2 (y_2 + (\alpha_5 - \alpha_1)x^2) + \beta x^{k_2+6} + (\text{higher terms})\,.$$

The Newton boundary consists of two faces Δ_1 and Δ_2 and f is non-degenerate on Δ_2 but degenerate on Δ_1. (Here "higher terms" are linear combinations of monomials above the Newton boundary.) Let $P_1 = {}^t(1, 2)$ and P_2 be the weight vectors corresponding to Δ_1, Δ_2 respectively. To make $\mathrm{Cone}(E_1, P_1)$ regular, we need to put one vertex $T_1 = {}^t(1, 1)$. The subdivision of the cones $\mathrm{Cone}(P_1, P_2)$ and $\mathrm{Cone}(P_2, E_2)$ depends on the parity of k_2 (i.e., either k_2 is even or odd).

For $k_2 = 2m + 1$, we get $P_2 = {}^t(2, 2m + 1)$ and $\mathrm{Cone}(P_1, P_2)$ and $\mathrm{Cone}(P_2, E_2)$ are subdivided into regular fans by adding vertices $\{T_i = {}^t(1, i),\ 3 \le i \le m\}$ and $S = {}^t(1, m+1)$.

For $k_2 = 2m$, we get $P_2 = {}^t(1, m))$ and $\mathrm{Cone}(P_1, P_2)$ is subdivided into a regular fan by adding vertices $\{T_i = {}^t(1, i),\ 3 \le i \le m - 1\}$. The $\mathrm{Cone}(P_2, E_2)$ is already regular. Note that in any case, the corresponding resolution is minimal. In the second case, $\hat{E}(P_2)^2 = -1$ but it intersects with two components of C.

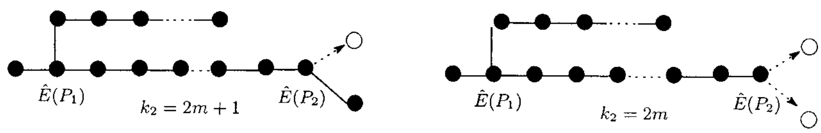


FIGURE 6

After taking the toric modification with respect to the canonical subdivision, we have $(\widetilde{C}, \xi_1) \sim B_{k_1,2}$. Hence $(C, O) \sim B_{k_2,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{k_1,2}}$.

Using the canonical subdivision for the second toric modification, we see that the resolution graph has three branches with the center $\hat{E}(P_1)$: one branch with a single vertex which corresponds to $\hat{E}(T_1)$. The second branch corresponds to the vertices in Cone(P_1, P_2) and Cone(P_2, E_2) (respectively Cone(P_1, P_2)) for k_2 is odd (resp. even). The third branch corresponds to the vertices for the second toric modification.

To see the relation between the second and third branches, we take a change of coordinates $(x, y_3) = (x, y_2 + (\alpha_3 - \alpha_1)x^2)$ from the beginning. After a finite number of triangular changes of coordinates, we arrive at the expression $(C, O) \sim B_{k_2',2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{k_1',2}}$. By an easy calculation and by the minimality of the resolution, we see that $k_1' = k_2 - 4,\ k_2' = k_1 + 4$. Thus $B_{k_2,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{k_1,2}} \sim B_{k_1+4,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{k_2-4,2}}$. Therefore in the classification, we can assume that $k_1 + 4 \geq k_2$.

Next we consider the case $\iota \geq 6$.

LEMMA 6. *Assume the case III-(b) and $\iota = 6$. Then the topological type of (C, O) is generically $B_{9,2} \circ B_{6,3}$ and it can degenerate into $(B_{5,2}^2)^{B_{1,2}} \circ B_{2,1}$ or $B_{9,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{k,2}}$, $(k = 1, \dots, 8)$.*

PROOF. We take a local coordinates system (x, y_1) so that C_2 is defined by $y_1 = 0$ and $f_5(x, y_1) = \beta_{12}xy_1^2 + \beta_{31}x^3y_1 + \beta_{60}x^6 +$ (higher terms). Then $f(x, y_1)$ is written as

$$\begin{aligned} f(x, y_1) &= y_1^5 + (\beta_{12}xy_1^2 + \beta_{31}x^3y_1 + \beta_{60}x^6)^2 + \text{(higher terms)} \\ &= y_1^5 + (\beta_{12}xy_1^2 + \beta_{31}xy_1^3)^2 + (\beta_{31}x^3y_1 + \beta_{60}x^6)^2 - \beta_{31}^2x^6y_1^2 + \text{(higher terms)}\,. \end{aligned}$$

If $\beta_{31} \neq 0$, $\Gamma(f; x, y_1)$ consists of two faces Δ_1, Δ_2 and

$$f_{\Delta_1}(x, y_1) = y_1^5 + x^2y_1^2(\beta_{12}y_1 + \beta_{31}x^2)^2\,, \quad f_{\Delta_2}(x, y) = (\beta_{31}x^3y_1 + \beta_{60}x^6)^2\,.$$

In this case, we first take a triangular change of coordinates of type $(x, y_2) = (x, y_1 + c_3x^3 + c_4x^4)$ so that the face Δ_2 changes into a non-degenerate face Δ_2' (a new face after a change of the coordinate) and

$$f_{\Delta_2'}(x, y_2) = \beta_{31}x^6(y_2^2 + c_3'x^9)\,, \quad c_3 \neq 0\,.$$

If f is non-degenerate on Δ_1, we have $(C, O) \sim B_{9,2} \circ B_{6,3}$.

If f is degenerate on Δ_1 ($\beta_{31} = 4\beta_{12}^3/27,\ \beta_{31} \neq 0$), then

$$f_{\Delta_1}(x, y_1) = \alpha^2y_1^2(9y_1 + \beta_{12}^2x^2)(9y_1 + 4\beta_{12}^2x^2)^2$$

where α is a non-zero constant. To analyze the singularity on Δ_1, we take a toric modification: let $P = {}^t(1, 2)$ be the weight vector corresponding to Δ_1 and we take a toric modification with respect to an admissible regular simplicial cone subdivision Σ^*, $\pi : X \to \mathbf{C}^2$. We may assume that σ =Cone(P_1, T_1) is a cone in Σ^* where $T_1 = {}^t(1,3)$. We take the toric coordinates (u, v) of the chart $\mathbf{C}^2_\sigma$. Then we have $\pi_\sigma(u,v) = (uv, u^2v^3)$ and

$$\pi_\sigma^* f(u,v) = \alpha^2 u^{10} v^{12} \tilde{f}(u,v) = \alpha^2 u^{10} v^{12}\Big((9v + \beta_{12}^2)(9v + 4\beta_{12}^2)^2 + \text{(higher terms)}\Big).$$

The strict transform $\widetilde{C}$ splits into two components. We see that one of the components of $\widetilde{C}$ which correspond to the non-degenerate component of f_{Δ_1} is smooth and intersects transversely with $\hat{E}(P) = \{u = 0\}$. To see the other component of $\widetilde{C}$, we take the translated toric coordinates (u, v_1), $v_1 = 9v + 4\beta_{12}^2$. Then $\tilde{f}(u, v_1) = cv_1^2 + \gamma_1 u +$(higher terms) where c is a non-zero constant. Hence if $\gamma_1 \neq 0$, we get $(C, O) \sim B_{9,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{1,2}}$. If $\gamma_1 = 0$, taking a triangular change of coordinates of the type $(u, v_2) = (u, v_1 + d_1 u + \cdots + d_j u^j)$, $j = [k/2]$, we can easily see that $(C, O) \sim B_{9,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{k,2}}$, $(k = 2, \dots, 8)$.

Next we consider the case $\beta_{31} = 0$. Then

$$\mathcal{N}(f; x, y_1) = y_1^5 + x^2(\beta_{12} y_1^2 + \beta_{60} x^5)^2$$

where $\beta_{60} \neq 0$ since $\iota = 6$ and $\Gamma(f; x, y_1)$ has two faces Δ_1 and Δ_2 and the corresponding face functions are given by $f_{\Delta_1}(x, y_1) = y_1^5 + \beta_{12}^2 x^2 y_1^4$ and $f_{\Delta_2}(x, y_1) = x^2(\beta_{12} y_1^2 + \beta_{60} x^5)^2$. Thus f is non-degenerate on Δ_1 and degenerate on Δ_2. Then taking a toric modification which is the same as (4) of Proposition 3, we get $(C, O) \sim (B_{5,2}^2)^{B_{1,2}} \circ B_{2,1}$. □

For the remaining cases $\iota \geq 7$, we can carry out the classification in the exact same way. So we can summarize the result as follows.

PROPOSITION 6. *Suppose that C_2 is smooth, $m_5 = 3$ and the tangent cone $T_O C_5$ consists of L_1 and a single line L_2 where L_1 is a line with multiplicity 2. If $\iota = 3$, we have $(C, O) \sim B_{6,5}$. If $\iota \geq 4$, then C_2 is tangent to either L_1 or L_2 and we have the following possibilities.*

(I) *Assume that the common tangent cone is L_2.*
Then the germ (C, O) can be of type $B_{3,4} \circ B_{2,5\iota-14}$ for $4 \leq \iota \leq 10$. (Lemma 4).

(II) *Assume that the common tangent cone is L_1.*
Then the germ (C, O) can be of type $B_{2\iota,5}$ for $\iota = 4, 5$ and $B_{5\iota-21,2} \circ B_{6,3}$ for $\iota = 6, \dots, 10$. Further degenerations are given for fixed ι by the following list.

(1) *If $\iota = 5$, then we have*
  - (a) $(C, O) \sim B_{k,2} \circ B_{6,3}$ $(5 \leq k \leq 12)$
  - (b) $(C, O) \sim B_{k,3} \circ B_{4,2}$ $(k = 7, \dots, 11)$, $B_{3,1} \circ B_{5,2} \circ B_{4,2}$, $B_{3,1} \circ B_{7,2} \circ B_{4,2}$ *and* $B_{k,2} \circ B_{3,1} \circ B_{4,2}$ $(k = 7, 8, 9)$.
  - (c) $(C, O) \sim B_{k_2+4,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{k_1,2}}$ *where* (k_1, k_2) *is in* $\{(k_1, k_2); 13 - k_2 \geq k_1 \geq k_2 - 4,\ k_2 = 5, \dots, 7\} \cup \{(4, 8), (5, 9)\}$.

(2) *If $\iota = 6$, then we have*
  - (a) $(C, O) \sim (B_{5,2}^2)^{B_{1,2}} \circ B_{2,1}$

(b) $(C, O) \sim B_{9,2} \circ B_{2,1} \circ (B^2_{2,1})^{B_{k,2}}$ $(k = 1, \dots, 8)$

(3) *If* $\iota = 7$, *then we have*

(a) $(C, O) \sim (B^2_{6,2})^{2B_{1,2}} \circ B_{2,1}$ *and* $B_{13,4} \circ B_{2,1}$.

(b) $(C, O) \sim B_{14,2} \circ B_{2,1} \circ (B^2_{2,1})^{B_{k,2}}$ $(k = 1, \dots, 7)$

(4) *If* $\iota = 8$, *then we have*

(a) $(C, O) \sim (B^2_{6,2})^{B_{6,2}+B_{1,2}} \circ B_{2,1}$ *and* $(B^2_{7,2})^{B_{3,2}} \circ B_{2,1}$

(b) $(C, O) \sim B_{19,2} \circ B_{2,1} \circ (B^2_{2,1})^{B_{k,2}}$ $(k = 1, \dots, 6)$

(5) *If* $\iota = 9$, *then we have*

(a) $(C, O) \sim (B^2_{6,2})^{B_{11,2}+B_{1,2}} \circ B_{2,1}$, $(B^2_{8,2})^{2B_{2,2}} \circ B_{2,1}$, $B_{18,4} \circ B_{2,1}$ *and* $B_{11,2} \circ B_{9,2} \circ B_{2,1}$

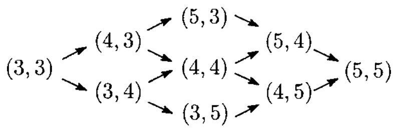
(b) $(C, O) \sim B_{24,2} \circ B_{2,1} \circ (B^2_{2,1})^{B_{k,2}}$, $(k = 1, \dots, 4)$

(6) *If* $\iota = 10$, *then we have*

(a) $(C, O) \sim (B^2_{6,2})^{B_{16,2}+B_{1,2}} \circ B_{2,1}$, $(B^2_{8,2})^{B_{7,2}+B_{2,2}} \circ B_{2,1}$ *and* $(B^2_{9,2})^{B_{5,2}} \circ B_{2,1}$

(b) $(C, O) \sim B_{29,2} \circ B_{2,1} \circ (B^2_{2,1})^{B_{k,2}}$, $(k = 1, \dots, 5,\ k \neq 4)$

(b-2) Assume that C_2 is a union of two lines passing through the origin ($m_2 = 2$) and $T_O C_5$ consists of a line with multiplicity 2 which is defined by $\{y = 0\}$ and a single line is $\{x = 0\}$. Thus we assume that $f_5(x, y) = b_{12}xy^2 + b_{40}x^4 + b_{04}y^4 +$ (higher terms). We assume that two lines of C_2 are defined by $\ell_1 := \{y + \alpha_1 x = 0\}$, $\ell_2 := \{\alpha_2 y + x = 0\}$ and we put $\iota_i = I(\ell_i, C_5; O) \geq 3$ for $i = 1, 2$. Then we have $\iota = \iota_1 + \iota_2 \geq 6$. If $\iota = 6$, then we have $(\iota_1, \iota_2) = (3, 3)$ $(\alpha_1, \alpha_2 \neq 0)$. If $\iota \geq 7$, then we have several possibilities of (ι_1, ι_2):



The above diagram depends only the numbers $(\alpha_1, \alpha_2, b_{40}, b_{04})$.

PROPOSITION 7. *Suppose that* C_2 *is a union of two lines and the tangent cone* $T_O C_5$ *consists of a line with multiplicity* 2 *and a single line* ($m_5 = 3$). *Then we have the following possibilities.*

(1) *If* $\iota = 6$, *then we have*

(a) $(C, O) \sim (B^2_{3,2})^{B_{4,2}} \circ B_{2,6}$,

(b) $(C, O) \sim B_{8,4} \circ B_{2,6}$ *and* $B_{k,2} \circ B_{4,2} \circ B_{2,6}$ $(5 \leq k \leq 12)$.

(2) *If* $\iota = 7$, *then we have two cases:* $(\iota_1, \iota_2) = (4, 3)$ *or* $(3, 4)$.

(a) *If* $(\iota_1, \iota_2) = (4, 3)$, *then we have*

(i) $(C, O) \sim (B^2_{3,2})^{B_{4,2}} \circ B_{2,11}$,

(ii) $(C, O) \sim B_{8,4} \circ B_{2,11}$ *and* $B_{k,2} \circ B_{4,2} \circ B_{2,11}$ $(5 \leq k \leq 10)$.

(b) *If* $(\iota_1, \iota_2) = (3, 4)$, *then we have* $(C, O) \sim (B^2_{3,2})^{B_{9,2}} \circ B_{2,6}$.

(3) *If* $\iota = 8$, *then we have two cases:* $(\iota_1, \iota_2) = (5, 3)$ *or* $(4, 4)$ *or* $(3, 5)$.

(a) *If* $(\iota_1, \iota_2) = (5, 3)$, *then we have* $(C, O) \sim (B^2_{3,2})^{B_{4,2}} \circ B_{2,16}$.

(b) *If* $(\iota_1, \iota_2) = (4, 4)$, *then we have* $(C, O) \sim (B^2_{3,2})^{B_{9,2}} \circ B_{2,11}$.

(c) *If* $(\iota_1, \iota_2) = (3, 5)$, *then we have*

(i) $(C, O) \sim (B^2_{4,2})^{2B_{5,2}} \circ B_{2,6}$,

(ii) $(C, O) \sim (B^2_{5,2})^{B_{6,2}} \circ B_{2,6}$, $(B^2_{6,2})^{2B_{1,2}} \circ B_{2,6}$ *and* $B_{13,4} \circ B_{2,6}$.

(4) *If* $\iota = 9$, *then we have two cases:* $(\iota_1, \iota_2) = (5, 4)$ *or* $(4, 5)$.

(a) *If* $(\iota_1, \iota_2) = (5, 4)$, *then we have* $(C, O) \sim (B^2_{3,2})^{B_{9,2}} \circ B_{2,16}$.

(b) *If* $(\iota_1, \iota_2) = (4, 5)$, *then we have*

(i) $(C, O) \sim (B^2_{4,2})^{2B_{5,2}} \circ B_{2,11}$,

(ii) $(C, O) \sim (B^2_{5,2})^{B_{6,2}} \circ B_{2,11}$, $(B^2_{6,2})^{2B_{1,2}} \circ B_{2,11}$ *and* $B_{13,4} \circ B_{2,11}$.

If $\iota = 10$, *then we have*

(a) $(C, O) \sim (B^2_{4,2})^{2B_{5,2}} \circ B_{2,16}$,

(b) $(C, O) \sim (B^2_{5,2})^{B_{6,2}} \circ B_{2,16}$, $(B^2_{6,2})^{2B_{1,2}} \circ B_{2,16}$ *and* $B_{13,4} \circ B_{2,16}$.

We omit the proof as it is parallel to that of Proposition 3.

Case III-(c): In this case, we may assume that $T_O C_5$ is defined by $y^3 = 0$.

(c-1) Assume that $m_2 = 1$. If $\iota = 3$, we have $(C, O) \sim B_{6,5}$ by Lemma 2. Therefore we consider the case $\iota \geq 4$. If $\iota = 4$, we have $(C, O) \sim B_{8,5}$ as in case (b-1). If $\iota \geq 5$, we get the following possibilities.

PROPOSITION 8. *Suppose that* C_2 *is smooth*, $m_5 = 3$ *and the tangent cone* $T_O C_5$ *is a line with multiplicity* 3. *Then the germ* (C, O) *can be of type* $B_{2\iota,5}$ *for* $\iota = 3, 4$ *and if* $\iota \geq 5$, *we have the following possibilities.*

(1) *If* $\iota = 5$, *then we have* $(C, O) \sim B_{5,10}$ *and* $B_{k,2} \circ B_{6,3}$ $(k = 5, \dots, 12)$.

(2) *If* $\iota = 6$, *then we have* $(C, O) \sim B_{9,2} \circ B_{6,3}$ *and* $B_{12,5}$.

(3) *If* $\iota = 7$, *then we have* $(C, O) \sim B_{14,2} \circ B_{6,3}$, $B_{7,2} \circ B_{8,3}$ *and* $B_{14,5}$.

(4) *If* $\iota = 8$, *then we have* $(C, O) \sim B_{19,2} \circ B_{6,3}$, $B_{12,2} \circ B_{8,3}$ *and* $B_{16,5}$.

(5) *If* $\iota = 9$, *then we have* $(C, O) \sim B_{24,2} \circ B_{6,3}$, $B_{17,2} \circ B_{8,3}$, $B_{10,2} \circ B_{10,3}$ *and* $B_{18,5}$.

(6) *If* $\iota = 10$, *then we have* $(C, O) \sim B_{29,2} \circ B_{6,3}$, $B_{22,2} \circ B_{8,3}$, $B_{15,2} \circ B_{10,3}$ *and* $B_{20,5}$.

We omit the proof as it is parallel to Lemma 5 and Lemma 6.

(c-2) Assume that C_2 is a union of two lines passing through the origin and $T_O C_5$ is defined by $\{y^3 = 0\}$. Then we have $\iota \geq 6$ and we can list the possibilities as in the following proposition.

PROPOSITION 9. *Suppose that* C_2 *is a union of two lines passing through the origin and the tangent cone* $T_O C_5$ *is defined by* $\{y^3 = 0\}$. *Then we have*:

(1) *If* $\iota = 6$, *then we have* $(C, O) \sim (B^2_{4,3})^{B_{6,2}}$, $B_{4,2} \circ (B^2_{3,2})^{B_{2,2}}$, $B_{10,6}$ *or* $B_{6,3} \circ B_{5,3}$.

(2) *If* $\iota = 7$ *then we have* $(C, O) \sim (B^2_{4,3})^{B_{11,2}}$.

(3) *If* $\iota = 8$ *then we have* $(C, O) \sim B_{9,2} \circ (B^2_{3,2})^{B_{7,2}}$ *or* $(C, O) \sim (B^2_{5,3})^{B_{10,2}}$.

The proof is parallel to the previous computations.

4.4. Case IV: $m_5 = 4$. We divide Case IV into five subcases by the type of $T_O C_5$.

(a) $T_O C_5$ consists of four distinct lines.
(b) $T_O C_5$ consists of a line with multiplicity 2 and two distinct lines.
(c) $T_O C_5$ consists of a line with multiplicity 3 and another line.
(d) $T_O C_5$ consists of a line with multiplicity 4.
(e) $T_O C_5$ consists of two lines with multiplicity 2.

First remark that if C_2 is smooth and C_2 intersects transversely with $T_O C_5$ at the origin ($\iota = 4$), we have $(C, O) \sim B_{8,5}$ by Lemma 2. So hereafter, we consider the case C_2 and $T_O C_5$ does not intersect transversely. First we prepare the following Lemma.

LEMMA 7. *Suppose that the conic C_2 is smooth and let (x, y_1) be a local coordinate system so that C_2 is defined by $y_1 = 0$. We put $f_5(x, y_1) = y_1(y_1 + c_1 x)(y_1 + c_2 x)(y_1 + c_3 x) + (\text{higher terms})$. Then*

(1) *If $c_i \neq 0$ for $i = 1, 2, 3$, then $(C, O) \sim B_{2\iota,5}$ for $\iota = 4, 5$ and if $\iota \geq 5$, we have two series $B_{k,2} \circ B_{6,3}$, $5 \leq k \leq 10$ for $\iota = 5$ and $B_{5\iota-21,2} \circ B_{6,3}$ for $\iota = 6, \dots, 10$.*
(2) *If $c_1 = 0$ and $c_i \neq 0$ for $i = 2, 3$, then we have $(C, O) \sim B_{2\iota,5}$ for $\iota = 4, 5, 6$ and $(C, O) \sim B_{5\iota-28,2} \circ B_{8,3}$ for $\iota = 7, \dots, 10$.*
(3) *If $c_1 = c_2 = 0$ and $c_3 \neq 0$, then we have $(C, O) \sim B_{2\iota,5}$ for $\iota = 4, \dots, 8$ and $(C, O) \sim B_{5\iota-35,2} \circ B_{10,3}$ for $\iota = 9, 10$.*
(4) *If $c_i = 0$, $(i = 1, 2, 3)$, then we have $(C, O) \sim B_{2\iota,5}$ for $\iota = 4, \dots, 10$.*

PROOF. We consider the Newton boundary $\mathcal{N}(f, x, y_1)$. We have

$$\mathcal{N}(f, x, y_1) = y_1^5 + c^2 x^6 y_1^2 + 2c\beta_{50} x^8 y_1 + \beta_{50}^2 x^{10}, \qquad \text{where } c = c_1 c_2 c_3$$

and $\Gamma(f; x, y_1)$ consists of one face Δ with the weight vector ${}^t(1, 2)$. We can see that the discriminant R of the face function $f_\Delta(x, y_1) = \mathcal{N}(f, x, y_1)$ in y_1 can be written as $R = \beta_{50}^5 \alpha x^{40}$ where α is a polynomial of c and β_{50}. Then we have $(C, O) \sim B_{10,5}$ if $R \neq 0$, $\iota = 5$ ($\iota = 5$ if and only if $\beta_{50} \neq 0$). We observe that $R \neq 0$, if $c = 0$ and $\beta_{50} \neq 0$.

We first consider the case (1): $c_i \neq 0$, $i = 1, 2, 3$. Note that $\alpha = 0$ and $\beta_{50} = 0$ is impossible. If $\beta_{50}\alpha \neq 0$, then we have $\iota = 5$ and $(C, O) \sim B_{10,5}$ as observed above. Thus we consider the case $\alpha = 0$ or $\beta_{50} = 0$. In both cases, by taking a suitable triangular change of coordinates, we can get a non-degenerate singularity. If $\alpha = 0$, then we have $(C, O) \sim B_{k,2} \circ B_{6,3}$, $5 \leq k \leq 10$. If $\beta_{50} = 0$, then we have $B_{5\iota-21,2} \circ B_{6,3}$ for $\iota = 6, \dots, 10$. Hence we have assertion (1).

To consider the cases (2) $\sim$ (4), we may assume $\beta_{50} = 0$. Then we can write

$$f_5(x, y_1) = y_1^4 + (c_2 + c_3) x y_1^3 + c_2 c_3 x^2 y_1^2 + \beta_{41} x^4 y_1 + \beta_{60} x^6 + (\text{higher terms})$$

and we have $\mathcal{N}(f; x, y_1) = y_1^5 + \beta_{60}^2 x^{12}$ and note that $\beta_{60} \neq 0$ if and only if $\iota = 6$. If $\beta_{60} \neq 0$, then we have $(C, O) \sim B_{12,5}$ and if $\beta_{60} = 0$, then we have $\iota \geq 7$. Secondly we consider that (2): $c_1 = 0, c_i \neq 0$, $(i = 2, 3)$. Then

$$f_5(x, y_1) = y_1^4 + (c_2 + c_3) x y_1^3 + c_2 c_3 x^2 y_1^2 + \beta_{41} x^4 y_1 + \beta_{70} x^7 + (\text{higher terms})$$

and we have $\mathcal{N}(f; x, y_1) = y_1^5 + x^8(\beta_{41}y_1 + \beta_{70}x^3)^2$. The assertion (2) follows easily by the Newton boundary argument. For the assertions (3) and (4), we can consider them similarly. □

Case IV-(a): Now we classify the singularities in this case. We have the following.

PROPOSITION 10. *Suppose that the tangent cone $T_O C_5$ consists of four distinct lines.*

(1) *If the conic C_2 is smooth, then we have the following possibilities.*

If $\iota = 4$, then we have $(C, O) \sim B_{2\iota,5}$.

If $\iota = 5$, then we have $(C, O) \sim B_{2\iota,5}$ or $B_{k,2} \circ B_{6,3}$ $(5 \le k \le 10)$.

If $\iota \ge 6$, then we have $B_{5\iota-21,2} \circ B_{6,3}$ for $\iota = 6, \dots, 10$.

(2) *Assume that the conic C_2 is a union of two lines ℓ_1, ℓ_2. Putting $\iota_i = I(\ell_i, C_5; O)$ for $i = 1, 2$, we have $(C, O) \sim B_{5\iota_1-16,2} \circ (B_{2,2}^2)^{2B_{2,2}} \circ B_{2,5\iota_2-16}$ with $\iota_1 + \iota_2 = \iota$.*

PROOF. The assertion (1) immediately follows from Lemma 7. The assertion (2) follows from the Lemma 3. □

Case VI-(b): In this case, we denote components of $T_O C_5$ by L_1, L_2 and L_3 where L_1 is a line with multiplicity 2.

(b-1) Assume that C_2 is smooth $(m_2 = 1)$. Then we have the following.

PROPOSITION 11. *Suppose that C_2 is smooth and the tangent cone $T_O C_5$ consists of a line with multiplicity 2 and two distinct lines. The germ $(C, O) \sim B_{8,5}$ for $\iota = 4$. If $\iota \ge 5$, we have the following possibilities for (C, O).*

(1) *We assume that C_2 is tangent to L_2 or L_3. Then $(C, O) \sim B_{5,10}$ or $(C, O) \sim B_{3,6} \circ B_{2,k}$, $(5 \le k \le 10)$ for $\iota = 5$ and $B_{3,6} \circ B_{2,5\iota-21}$ for $\iota = 6, \dots, 10$.*

(2) *We assume that C_2 is tangent to L_1. Then $(C, O) \sim B_{2\iota,5}$ for $\iota = 5, 6$ and $(C, O) \sim B_{5\iota-28,2} \circ B_{8,3}$ for $\iota = 7, \dots, 10$.*

PROOF. The assertions (1) and (2) are immediate by Lemma 4 and Lemma 7. □

(b-2) Assume that C_2 is a union of two lines passing through the origin $\ell_i := \{a_i x + b_i y = 0\}$, $(i = 1, 2)$ and we assume that $L_1 = \{y^2 = 0\}$, $L_2 = \{x = 0\}$ and $L_3 = \{y + cx = 0\}$.

PROPOSITION 12. *Suppose that C_2 is a union of two lines and the tangent cone $T_O C_5$ consists of a line with multiplicity 2 and two distinct lines. Then*

$$\iota = 8: \quad (C, O) \sim B_{6,4} \circ (B_{1,1}^2)^{B_{2,2}} \circ B_{2,4}, \quad B_{4,2} \circ B_{3,2} \circ (B_{1,1}^2)^{B_{2,2}} \circ B_{2,4}.$$

$$\iota = 9: \quad (C, O) \sim (B_{3,2}^2)^{B_{5,2}} \circ (B_{1,1}^2)^{B_{2,2}} \circ B_{2,4}, \quad \ell_1 = L_1 \qquad or$$

$$B_{6,4} \circ (B_{1,1}^2)^{B_{2,2}} \circ B_{2,9}, \quad B_{4,2} \circ B_{3,2} \circ (B_{1,1}^2)^{B_{2,2}} \circ B_{2,9}, \quad \ell_1 = L_2.$$

$$\iota = 10: \quad (C, O) \sim (B_{3,2}^2)^{B_{5,2}} \circ (B_{1,1}^2)^{B_{2,2}} \circ B_{2,9},\ \ell_1 = L_1,\ \ell_2 = L_2,$$

$$B_{6,4} \circ (B_{1,1}^2)^{B_{7,2}} \circ B_{2,9},\ B_{4,2} \circ B_{3,2} \circ (B_{1,1}^2)^{B_{7,2}} \circ B_{2,9}, \quad \ell_1 = L_2,\ \ell_2 = L_3.$$

We omit the proof as it follows from an easy calculation.

Case IV-(c): We assume that $T_O C_5$ consists of L_1 and a simple L_2 where L_1 is a line with multiplicity 3.

(c-1) Assume that C_2 is smooth. Then we have the following.

PROPOSITION 13. *Suppose that C_2 is smooth and C_5 is as above. Then $(C, O) \sim B_{8,5}$ for $\iota = 4$.*

Assume that $\iota \geq 5$. Then the possibilities of (C, O) are:

(1) *If C_2 is tangent to L_2, then we have $(C, O) \sim B_{5,10}$, or $B_{3,6} \circ B_{2,k}$, $(5 \leq k \leq 10)$ for $\iota = 5$ and $(C, O) \sim B_{3,6} \circ B_{2,5\iota-21}$ for $\iota = 6, \dots, 10$.*

(2) *If C_2 is tangent to L_1, then we have $(C, O) \sim B_{2\iota,5}$ for $\iota = 4, \dots, 8$ and $(C, O) \sim B_{5\iota-35,2} \circ B_{10,3}$ for $\iota = 9, 10$.*

The proofs of (1) and (2) are immediate from Lemma 4 and Lemma 7.

(c-2) Assume that C_2 is a union of two lines ℓ_i, $i = 1, 2$ passing through the origin and put $\ell_i := \{a_i x + b_i y = 0\}$ for $i = 1, 2$. We assume that $L_1 = \{y^3 = 0\}$ and $L_2 = \{x = 0\}$. Then we have $\iota \geq 8$.

PROPOSITION 14. *Suppose that C_2 is a union of two lines ℓ_1, ℓ_2 and $T_O C_5$ consists of L_1 and a simple line L_2 where L_1 is a line with multiplicity 3. Then*

(1) *If $\iota = 8$, then we have $(C, O) \sim B_{8,6} \circ B_{2,4}$ and $B_{5,3} \circ B_{4,3} \circ B_{2,4}$.*

(2) *Suppose that $\iota = 9$.*

  (a) *If $\ell_1 = L_1$, then we have $(C, O) \sim (B_{4,3}^2)^{B_{5,2}} \circ B_{2,4}$ and $B_{5,3} \circ B_{4,3} \circ B_{2,9}$.*

  (b) *If $\ell_2 = L_2$, then we have $(C, O) \sim B_{8,6} \circ B_{2,9}$.*

(3) *If $\iota = 10$, then we have $\ell_1 = L_1$, $\ell_2 = L_2$ and $(C, O) \sim (B_{4,3}^2)^{B_{5,2}} \circ B_{2,9}$.*

Case VI-(d):

PROPOSITION 15. *Suppose that the tangent cone $T_O C_5$ is a line with multiplicity 4.*

(1) *If C_2 is smooth, then we have $(C, O) \sim B_{2\iota,5}$ for $\iota = 4, \dots, 10$.*

(2) *Suppose C_2 consists of two lines.*

  (a) *If $\iota = 8$, then we have $(C, O) \sim B_{10,8}$ and $B_{2,1} \circ B_{4,3} \circ B_{5,4}$.*

  (b) *If $\iota = 9$, then we have $(C, O) \sim (B_{5,4}^2)^{B_{5,2}}$.*

Note that when $\iota = 10$, C is a linear torus curve. See §6.

Case VI-(e): In this case, we denote the lines with multiplicity 2 of $T_O C_5$ by L_1 and L_2.

(e-1) Assume that C_2 is smooth.

PROPOSITION 16. *Suppose that C_2 is smooth and the tangent cone $T_O C_5$ consists of two lines with multiplicity 2. If $\iota = 4$, then the germ $(C, O) \sim B_{8,5}$. If $\iota \geq 5$, then we have the following possibilities of (C, O).*

(1) *If C_2 is tangent to L_1, then we have $(C, O) \sim B_{2\iota,5}$ for $\iota = 5, 6$ and $B_{5\iota-28,2} \circ B_{8,3}$ for $\iota = 7, \dots, 10$.*

(2) *If C_2 is tangent to L_2, then we have $(C, O) \sim B_{5,2\iota}$ for $\iota = 5, 6$ and $B_{3,8} \circ B_{2,5\iota-28}$ for $\iota = 7, \dots, 10$.*

(e-2) Assume that C_2 is a union of two lines.

PROPOSITION 17. *Suppose that C_2 is the union of two lines and the tangent cone $T_O C_5$ consists of two lines with multiplicity 2. Then*

(1) *If $\iota = 8$, then we have $(C, O) \sim B_{6,4} \circ B_{4,6}$, $B_{4,2} \circ B_{3,2} \circ B_{4,6}$ and $B_{4,2} \circ B_{3,2} \circ B_{2,3} \circ B_{2,4}$.*

(2) *If $\iota = 9$, then we have $(C, O) \sim B_{6,4} \circ (B_{2,3}^2)^{B_{5,2}}$ and $B_{4,2} \circ B_{3,2} \circ (B_{2,3}^2)^{B_{5,2}}$.*

(3) *If $\iota = 10$, then we have $(C, O) \sim (B_{3,2}^2)^{B_{5,2}} \circ (B_{2,3}^2)^{B_{5,2}}$*

4.5. Case V: $m_5 = 5$. In this case, we have $\iota \geq 5$. Similarly we can also divide this case depending either C_2 is smooth or a union of two lines.

PROPOSITION 18. *Suppose that the multiplicity of the quintic C_5 is 5, i.e., C_5 consists of five line components. Then we have the following possibilities of (C, O).*

(1) *If C_2 is an irreducible conic, then we have $(C, O) \sim B_{2\iota,5}$ for $\iota = 5, \dots, 10$.*

(2) *If C_2 consists of two lines, then f is a homogeneous polynomial of degree 10 and therefore $(C, O) \sim B_{10,10}$, i.e., C consists of 10 line components.*

5. The classification

Now we have the following local classification.

THEOREM 1. *Let $C = \{f = f_2^5 + f_5^2 = 0\}$ is a (2,5)-torus curve. We assume that $C_2 = \{f_2 = 0\}$ is a reduced conic. The topological type of (C, O) is equivalent to one of the following where $\dagger(C, O)$ denotes that it has the degenerate series.*

I. *If C_5 is smooth, then we have $(C, O) \sim B_{5\iota,2}$, $\iota = 1, \dots, 10$.*

II. *Assume that C_5 is singular.*

(II-1) *Assume first that C_2 is an irreducible conic. Then we have:*

ι	(C, O)
2	$B_{3,2} \circ B_{2,3}$, $B_{5,4}$
3	$B_{8,2} \circ B_{2,3}$, $(B_{3,2}^2)^{B_{3,2}}$, $B_{6,5}$
4	$B_{13,2} \circ B_{2,3}$, $B_{6,2} \circ B_{4,3}$, $B_{8,5}$, $\dagger(B_{4,2}^2)^{2B_{2,2}}$
5	$B_{18,2} \circ B_{2,3}$, $B_{11,2} \circ B_{4,3}$, $\dagger B_{10,5}$, $\dagger(B_{4,2}^2)^{B_{7,2}+B_{2,2}}$
6	$B_{23,2} \circ B_{2,3}$, $B_{16,2} \circ B_{4,3}$, $\dagger B_{9,2} \circ B_{6,3}$, $B_{12,5}$, $\dagger(B_{4,2}^2)^{B_{12,2}+B_{2,2}}$
7	$B_{28,2} \circ B_{2,3}$, $B_{21,2} \circ B_{4,3}$, $\dagger B_{14,2} \circ B_{6,3}$, $\dagger(B_{4,2}^2)^{B_{17,2}+B_{2,2}}$, $B_{7,2} \circ B_{8,3}$, $B_{14,5}$
8	$B_{33,2} \circ B_{2,3}$, $B_{26,2} \circ B_{4,3}$, $\dagger B_{19,2} \circ B_{6,3}$, $\dagger(B_{4,2}^2)^{B_{22,2}+B_{2,2}}$, $B_{12,2} \circ B_{8,3}$, $B_{16,5}$

9	$B_{38,2}\circ B_{2,3}$, $B_{31,2}\circ B_{4,3}$, $\dagger B_{24,2}\circ B_{6,3}$, $\dagger(B_{4,2}^2)^{B_{27,2}+B_{2,2}}$ $B_{17,2}\circ B_{8,3}$, $B_{10,2}\circ B_{10,3}$, $B_{18,5}$
10	$B_{43,2}\circ B_{2,3}$, $B_{36,2}\circ B_{4,3}$, $\dagger B_{29,2}\circ B_{6,3}$, $\dagger(B_{4,2}^2)^{B_{32,2}+B_{2,2}}$ $B_{22,2}\circ B_{8,3}$, $B_{15,2}\circ B_{10,3}$, $B_{20,5}$

The singularities with † have further degenerations as is indicated below.

† $(B_{4,2}^2)^{2B_{2,2}}$: $B_{10,4}$, $B_{k,2}\circ B_{2,5}$ $(6\le k\le 15)$

† $(B_{4,2}^2)^{B_{7,2}+B_{2,2}}$: $(B_{5,2}^2)^{B_{5,2}}$

† $(B_{4,2}^2)^{B_{12,2}+B_{2,2}}$: $(B_{6,2}^2)^{2B_{3,2}}$, $(B_{7,2}^2)^{B_{2,2}}$, $B_{15,4}$

† $(B_{4,2}^2)^{B_{17,2}+B_{2,2}}$: $(B_{6,2}^2)^{B_{8,2}+B_{3,2}}$, $(B_{7,2}^2)^{B_{7,2}}$

† $(B_{4,2}^2)^{B_{22,2}+B_{2,2}}$: $(B_{6,2}^2)^{B_{13,2}+B_{3,2}}$, $(B_{8,2}^2)^{2B_{4,2}}$, $(B_{9,2}^2)^{B_{5,2}}$
$B_{20,4}$, $B_{k,2}\circ B_{10,2}$ $(k=11,12)$

† $(B_{4,2}^2)^{B_{27,2}+B_{2,2}}$: $(B_{6,2}^2)^{B_{18,2}+B_{3,2}}$, $(B_{8,2}^2)^{B_{9,2}+B_{4,2}}$, $(B_{9,2}^2)^{B_{10,2}}$

† $(B_{4,2}^2)^{B_{32,2}+B_{2,2}}$: $(B_{6,2}^2)^{B_{23,2}+B_{3,2}}$, $(B_{8,2}^2)^{B_{13,2}+B_{4,2}}$, $(B_{10,2}^2)^{2B_{5,2}}$, $(B_{11,2}^2)^{B_{6,2}}$, $(B_{12,2}^2)^{2B_{1,2}}$, $B_{25,4}$

† $B_{10,5}$: $B_{k,2}\circ B_{6,3}$ $(5\le k\le 12)$, $B_{k,3}\circ B_{4,2}$ $(7\le k\le 11)$, $B_{3,1}\circ B_{5,2}\circ B_{4,2}$, $B_{3,1}\circ B_{7,2}\circ B_{4,2}$, $B_{k,2}\circ B_{3,1}\circ B_{4,2}$ $(k=7,8,9)$, $B_{k_2+4,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{k_1,2}}$, $(k_1,k_2)\in\{(k_1,k_2)\mid k_2-4\le k_1\le 13-k_2,\ 5\le k_2\le 7\}\cup\{(4,8),(5,9)\}$.

† $B_{9,2}\circ B_{6,3}$: $(B_{5,2}^2)^{B_{1,2}}\circ B_{2,1}$, $B_{9,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{k,2}}$ $(1\le k\le 8)$

† $B_{14,2}\circ B_{6,3}$: $(B_{5,2}^2)^{B_{1,2}}\circ B_{2,1}$, $B_{13,2}\circ B_{2,1}$, $B_{14,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{k,2}}$ $(1\le k\le 7)$

† $B_{19,2}\circ B_{6,3}$: $B_{12,2}\circ (B_{3,1}^2)^{B_{1,2}}\circ B_{2,1}$, $(B_{7,2}^2)^{B_{4,2}}\circ B_{2,1}$, $B_{19,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{k,2}}$ $(1\le k\le 6)$

† $B_{24,2}\circ B_{6,3}$: $B_{17,2}\circ (B_{3,1}^2)^{B_{1,2}}\circ B_{2,1}$, $(B_{8,2}^2)^{2B_{2,2}}\circ B_{1,2}$, $B_{18,4}\circ B_{2,1}$,
$B_{24,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{k,2}}$ $(1\le k\le 5)$

† $B_{29,2}\circ B_{6,3}$: $B_{22,2}\circ (B_{3,1}^2)^{B_{1,2}}\circ B_{2,1}$, $B_{22,2}\circ (B_{4,1}^2)^{B_{2,2}}\circ B_{2,1}$, $(B_{9,2}^2)^{B_{5,2}}\circ B_{2,1}$,
$B_{29,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{k,2}}$ $(1\le k\le 5,\ k\ne 4)$

(II-2) *Next we assume that* C_2 *consists of the two distinct lines. Then we have*:

ι	(C, O)
4	$B_{8,2} \circ B_{2,8}$, $\dagger(B^2_{3,2})^{B_{8,2}}$
5	$B_{13,2} \circ B_{2,8}$, $(B^2_{3,2})^{B_{13,2}}$
6	$B_{18,2} \circ B_{2,8}$, $B_{13,2} \circ B_{2,13}$, $\dagger(B^2_{4,2})^{2B_{7,2}}$ $B_{6,2} \circ (B^2_{1,1})^{B_{4,2}} \circ B_{2,6}$, $\dagger(B^2_{3,2})^{B_{4,2}} \circ B_{2,6}$, $\dagger(B^2_{4,3})^{B_{6,2}}$
7	$B_{23,2} \circ B_{2,8}$, $B_{18,2} \circ B_{2,13}$, $\dagger B_{16,2} \circ (B^2_{2,1})^{B_{7,2}}$ $B_{11,2} \circ (B^2_{1,1})^{B_{4,2}} \circ B_{2,6}$, $(B^2_{3,2})^{B_{9,2}} \circ B_{2,6}$, $\dagger(B^2_{3,2})^{B_{4,2}} \circ B_{2,11}$, $(B^2_{4,3})^{B_{11,2}}$
8	$B_{23,2} \circ B_{2,13}$, $B_{18,2} \circ B_{2,18}$, $B_{11,2} \circ (B^2_{1,1})^{B_{4,2}} \circ B_{2,11}$, $B_{6,4} \circ B_{6,4}$, $B_{10,8}$ $B_{16,2} \circ (B^2_{1,1})^{B_{4,2}} \circ B_{2,6}$, $(B^2_{3,2})^{B_{9,2}} \circ B_{2,11}$, $(B^2_{3,2})^{B_{4,2}} \circ B_{2,16}$, $\dagger(B^2_{4,2})^{2B_{5,2}} \circ B_{2,6}$ $B_{4,2} \circ (B^2_{2,2})^{2B_{2,2}} \circ B_{2,4}$, $\dagger B_{6,4} \circ (B^2_{1,1})^{2B_{2,2}} \circ B_{2,4}$, $\dagger B_{8,6} \circ B_{2,4}$, $B_{4,2} \circ (B^2_{3,2})^{B_{7,2}}$
9	$B_{23,2} \circ B_{2,18}$, $B_{16,2} \circ (B^2_{1,1})^{B_{4,2}} \circ B_{2,11}$ $(B^2_{3,2})^{B_{9,2}} \circ B_{2,16}$, $\dagger(B^2_{4,2})^{2B_{5,2}} \circ B_{2,11}$, $B_{9,2} \circ (B^2_{2,2})^{2B_{2,2}} \circ B_{2,4}$ $\dagger B_{6,4} \circ (B^2_{1,1})^{B_{2,2}} \circ B_{2,9}$, $(B^2_{3,2})^{B_{5,2}} \circ (B^2_{1,1})^{B_{2,2}} \circ B_{2,4}$, $\dagger B_{6,4} \circ (B^2_{1,1})^{B_{7,2}} \circ B_{2,4}$ $\dagger(B^2_{4,3})^{B_{5,2}} \circ B_{2,4}$, $(B^2_{5,4})^{B_{5,2}}$, $\dagger B_{6,4} \circ (B^2_{2,3})^{B_{5,2}}$, $B_{8,6} \circ B_{2,9}$
10	$B_{23,2} \circ B_{2,23}$, $B_{16,2} \circ (B^2_{1,1})^{B_{4,2}} \circ B_{2,16}$ $\dagger(B^2_{4,2})^{2B_{5,2}} \circ B_{2,16}$, $B_{9,2} \circ (B^2_{2,2})^{2B_{2,2}} \circ B_{2,9}$, $\dagger B_{6,4} \circ (B^2_{1,1})^{B_{7,2}} \circ B_{2,9}$ $(B^2_{3,2})^{B_{5,2}} \circ (B^2_{1,1})^{B_{2,2}} \circ B_{2,9}$, $B_{6,4} \circ (B^2_{1,1})^{B_{7,2}} \circ B_{2,4}$ $(B^2_{4,3})^{B_{5,2}} \circ B_{2,9}$, $(B^2_{3,2})^{B_{5,2}} \circ (B^2_{2,3})^{B_{5,2}}$

The singularities with † have further degenerations as is indicated below.

$\dagger\ (B^2_{3,2})^{B_{8,2}}\ :\ (B^2_{4,2})^{2B_{2,2}},\ B_{10,4},\ B_{k,2} \circ B_{2,5}\ (6 \le k \le 15)$

$\dagger\ (B^2_{4,2})^{2B_{7,2}}\ :\ (B^2_{5,2})^{B_{10,2}},\ (B^2_{6,2})^{2B_{3,2}},\ (B^2_{7,2})^{B_{2,2}},\ B_{15,4}$

$\dagger\ B_{16,2} \circ (B^2_{2,1})^{B_{7,2}}\ :\ (B^2_{5,2})^{B_{15,2}}$

$\dagger\ (B^2_{3,2})^{B_{4,2}} \circ B_{2,6},\quad B_{8,4} \circ B_{2,6},\quad B_{k,2} \circ B_{4,2} \circ B_{2,6}\ (5 \le k \le 12)$

$\dagger\ (B^2_{3,2})^{B_{4,2}} \circ B_{2,11}\ :\ B_{8,4} \circ B_{2,11},\quad B_{k,2} \circ B_{4,2} \circ B_{2,11}\ (5 \le k \le 10)$

$\dagger\ (B^2_{4,2})^{2B_{5,2}} \circ B_{2,6}\ :\ (B^2_{5,2})^{B_{6,2}} \circ B_{2,6},\ (B^2_{6,2})^{2B_{1,2}} \circ B_{2,6},\quad B_{13,4} \circ B_{2,6}$

$\dagger\ (B^2_{4,2})^{2B_{5,2}} \circ B_{2,11}\ :\ (B^2_{5,2})^{B_{6,2}} \circ B_{2,11},\ (B^2_{6,2})^{2B_{1,2}} \circ B_{2,11},\quad B_{13,4} \circ B_{2,11}$

$\dagger\ (B^2_{4,2})^{2B_{5,2}} \circ B_{2,16}\ :\ (B^2_{5,2})^{B_{6,2}} \circ B_{2,16},\ (B^2_{6,2})^{2B_{1,2}} \circ B_{2,16},\quad B_{13,4} \circ B_{2,16}$

$\dagger\ (B^2_{4,3})^{B_{6,2}}\ :\ B_{10,6},\quad B_{6,3} \circ B_{5,3}$

$\dagger\ B_{6,4} \circ (B^2_{1,1})^{B_{2,2}} \circ B_{2,9}\ :\ B_{4,3} \circ B_{3,2} \circ (B^2_{1,1})^{B_{2,2}} \circ B_{2,9}$

$\dagger\ B_{6,4} \circ (B^2_{1,1})^{B_{7,2}} \circ B_{2,4}\ :\ B_{4,3} \circ B_{3,2} \circ (B^2_{1,1})^{B_{7,2}} \circ B_{2,4}$

$\dagger\ B_{6,4} \circ (B^2_{1,1})^{B_{7,2}} \circ B_{2,9}\ :\ B_{4,3} \circ B_{3,2} \circ (B^2_{1,1})^{B_{7,2}} \circ B_{2,9}$

† $B_{8,6} \circ B_{2,4}$: $B_{5,4} \circ B_{4,3} \circ B_{2,4}$
† $B_{8,6} \circ B_{2,9}$: $B_{5,4} \circ B_{4,3} \circ B_{2,9}$
† $B_{6,4} \circ B_{4,6}$: $B_{4,2} \circ B_{3,2} \circ B_{2,6}$, $B_{4,2} \circ B_{3,2} \circ B_{2,3} \circ B_{2,4}$
† $B_{6,4} \circ (B_{2,3}^2)^{B_{5,2}}$: $B_{4,2} \circ B_{3,2} \circ (B_{2,3}^2)^{B_{5,2}}$

6. Linear torus curve of type (2, 5)

DEFINITION 3. Let C be a torus curve of type (2, 5) which has a defining polynomial f which can be written as $f(x, y) = f_2(x, y)^5 + f_5(x, y)^2$ where $\deg f_j = j$ $(j = 2, 5)$. If $f_2(x, y) = \ell(x, y)^2$ for some linear form ℓ, then C is called *a linear torus curve.*

A linear torus curve of degree 10 is a union of two quintics.

6.1. Local Classification. In this section we determine the local singularity of linear torus curve type (2, 5).

For the determination of (C, O), we divide it into five cases as in the previous sections. If C_5 is smooth, we have already determined the singularity (C, O) by Lemma 1. Hence we consider the case that the multiplicity of C_5 is greater than or equal to 2.

6.2. Case L-II: $m_5 = 2$**.** In this case, $T_O C_5$ has two types and we have $\iota = 4, 6, 8, 10$.

PROPOSITION 19. *Suppose* $m_5 = 2$*.*
(1) *If the tangent cone* $T_O C_5$ *consists of two distinct lines, then we have* $(C, O) \sim B_{5\iota-12,2} \circ B_{2,8}$ *for* $\iota = 4, 6, 8, 10$.
(2) *Assume that the tangent cone* $T_O C_5$ *is a line with multiplicity* 2.
  (a) *If* $(C_5, O) \sim B_{3,2}$*, then we have*
    (i) $(C, O) \sim (B_{3,2}^2)^{B_{8,2}}$ *for* $\iota = 4$ *and*
    (ii) $(C, O) \sim (B_{3,2}^2)^{B_{18,2}}$ *for* $\iota = 6$.
  (b) *If* $(C_5, O) \sim B_{4,2}$*, then we have*
    (i) $(C, O) \sim (B_{4,2}^2)^{2B_{2,2}}$ *for* $\iota = 4$,
    (ii) $(C, O) \sim (B_{4,2}^2)^{2B_{12,2}}$ *for* $\iota = 8$ *and*
    (iii) $(C, O) \sim (B_{4,2}^2)^{B_{22,2}+B_{12,2}}$ *for* $\iota = 10$.
  (c) *If* $(C_5, O) \sim B_{5,2}$*, then we have*
    (i) $(C, O) \sim B_{10,4}$ *and* $B_{k,2} \circ B_{2,5}$ $(6 \le k \le 15)$ *for* $\iota = 4$
    (ii) $(C, O) \sim (B_{5,2}^2)^{B_{20,2}}$ *for* $\iota = 8$.
    (iii) $(C, O) \sim (B_{5,2}^2)^{B_{30,2}}$ *for* $\iota = 10$.
  (d) *If* $(C_5, O) \sim B_{6,2}$*, then we have* $(C, O) \sim (B_{6,2}^2)^{2B_{8,2}}$ *for* $\iota = 8$.
  (e) *If* $(C_5, O) \sim B_{7,2}$*, then we have* $(C, O) \sim (B_{7,2}^2)^{B_{12,2}}$ *for* $\iota = 8$.

(f) *If* $(C_5, O) \sim B_{8,2}$, *then we have* $(C, O) \sim (B_{8,2}^2)^{2B_{4,2}}$ *for* $\iota = 8$.

(g) *If* $(C_5, O) \sim B_{9,2}$, *then we have* $(C, O) \sim (B_{9,2}^2)^{B_{4,2}}$ *for* $\iota = 8$.

(h) *If* $(C_5, O) \sim B_{10,2}$, *then we have* $(C, O) \sim B_{20,5}$ *and* $B_{k,2} \circ B_{10,3}$ $(11 \le k \le 13)$ *for* $\iota = 8$.

PROOF. The assertion (1) is shown by the Newton boundary argument (See Lemma 3). The proof of the assertion (2) is mainly computational. See the proof of Proposition 3. □

6.3. Case L-III: $m_5 = 3$**.** In this case, $T_O C_5$ has three types and we have $\iota = 2k$, $(k = 3, 4, 5)$.

PROPOSITION 20. *Suppose* $m_5 = 3$.

(1) *If the tangent cone* $T_O C_5$ *consists of three distinct lines, then we have* $(C, O) \sim (B_{3,3}^2)^{3B_{4,2}}$ *and* $B_{16,2} \circ (B_{2,2}^2)^{2B_{4,2}}$ *and* $B_{26,2} \circ (B_{2,2}^2)^{2B_{4,2}}$ *for* $\iota = 6, 8, 10$.

(2) *Suppose the tangent cone* $T_O C_5$ *is a line with multiplicity* 2 *and a single line.*

(a) *If* $\iota = 6$, *then we have* $(C, O) \sim B_{6,2} \circ (B_{2,3}^2)^{B_{4,2}}$, $B_{8,4} \circ B_{2,6}$ *and* $B_{k,2} \circ B_{4,2} \circ B_{2,6}$, $(5 \le k \le 10)$.

(b) *If* $\iota = 8$, *then we have the following.*

(i) *If the tangent cone* $T_O C_5$ *is* $\{xy^2 = 0\}$, *then we have* $(C, O) \sim (B_{3,2}^2)^{B_{14,2}} \circ B_{2,6}$.

(ii) *If the tangent cone* $T_O C_5$ *is* $\{x^2 y = 0\}$, *then we have* $(C, O) \sim B_{16,2} \circ (B_{2,3}^2)^{B_{4,2}}$, $B_{16,2} \circ B_{4,8}$ *and* $B_{16,2} \circ B_{2,4} \circ B_{2,k}$, $(5 \le k \le 10)$.

(c) *If* $\iota = 10$, *then we have the following.*

(i) *If the tangent cone* $T_O C_5$ *is* $\{xy^2 = 0\}$, *then we have* $(C, O) \sim (B_{4,2}^2)^{2B_{10,2}} \circ B_{2,6}$, $(B_{5,2}^2)^{B_{16,2}} \circ B_{2,6}$, $(B_{6,2}^2)^{2B_{6,2}} \circ B_{2,6}$, $(B_{8,2}^2)^{2B_{2,2}} \circ B_{2,6}$, $B_{18,2} \circ B_{2,6}$ *and* $B_{19,2} \circ B_{2,6}$.

(ii) *If the tangent cone* $T_O C_5$ *is* $\{x^2 y = 0\}$, *then we have* $(C, O) \sim B_{26,2} \circ (B_{2,3}^2)^{B_{4,2}}$, $B_{26,2} \circ B_{4,8}$ *and* $B_{26,2} \circ B_{2,4} \circ B_{2,k}$, $(5 \le k \le 9)$.

(3) *Suppose that the tangent cone* $T_O C_5$ *is a line with multiplicity 3.*

(a) *If* $\iota = 6$, *then we have* $(C, O) \sim (B_{4,3}^2)^{B_{6,2}}$, $B_{4,2} \circ (B_{3,2}^2)^{B_{2,2}}$, $B_{10,6}$ *and* $B_{3,6} \circ B_{5,3}$.

(b) *If* $\iota = 8$, *then we have* $(C, O) \sim (B_{4,3}^2)^{B_{16,2}}$.

(c) *If* $\iota = 10$, *then we have* $(C, O) \sim B_{4,2} \circ (B_{3,2}^2)^{B_{12,2}}$ *and* $(B_{5,3}^2)^{B_{20,2}}$.

6.4. Case L-IV: $m_5 = 4$**.**

PROPOSITION 21. *Suppose that the multiplicity of* (C_5, O) *is* 4.

(1) *Assume that the tangent cone* $T_O C_5$ *is four distinct lines. Then*

(a) *If* $\iota = 8$, *then we have* $(C, O) \sim (B_{4,4}^2)^{4B_{2,2}}$.

(b) *If* $\iota = 10$, *then we have* $(C, O) \sim B_{14,2} \circ (B_{3,3}^2)^{3B_{2,2}}$.

(2) *Assume that the tangent cone* $T_O C_5$ *is a line with multiplicity* 2 *and two distinct lines.*

(a) *If* $\iota = 8$, *then we have* $(C, O) \sim (B_{2,2}^2)^{2B_{2,2}} \circ B_{4,6}$ *and* $(B_{2,2}^2)^{2B_{2,2}} \circ B_{2,3} \circ B_{2,4}$

(b) *Suppose* $\iota = 10$.

(i) *If the tangent cone* $T_O C_5$ *is* $\{x^2 y(y + cx) = 0\}$, *then we have* $(C, O) \sim B_{14,2} \circ (B_{1,1}^2)^{B_{2,2}} \circ B_{4,6}$ *and* $B_{14,2} \circ (B_{1,1}^2)^{B_{2,2}} \circ B_{2,3} \circ B_{2,4}$.

(ii) *If the tangent cone* $T_O C_5$ *is* $\{xy^2(y + cx) = 0\}$, *then we have* $(C, O) \sim (B_{3,2}^2)^{B_{10,2}} \circ (B_{2,2}^2)^{2B_{2,2}} \circ B_{2,4}$.

(3) *Assume that the tangent cone* $T_O C_5$ *is a line with multiplicity* 3 *and another line.*

(a) *If* $\iota = 8$, *then we have* $(C, O) \sim B_{4,2} \circ B_{6,8}$ *and* $B_{4,2} \circ B_{3,4} \circ B_{2,3} \circ B_{1,2}$.

(b) *Suppose* $\iota = 10$.

(i) *If the tangent cone* $T_O C_5$ *is* $\{x^3 y = 0\}$, *then we have* $B_{14,2} \circ B_{6,8}$ *and* $B_{14,2} \circ B_{3,4} \circ B_{2,3} \circ B_{1,2}$.

(ii) *If the tangent cone* $T_O C_5$ *is* $\{xy^3 = 0\}$, *then we have* $(B_{4,3}^2)^{B_{10,2}} \circ B_{2,4}$.

(4) *Assume that the tangent cone* $T_O C_5$ *is a line with multiplicity* 4.

(a) *If* $\iota = 8$, *then we have* $(C, O) \sim B_{8,10}$ *and* $B_{4,5} \circ B_{3,4} \circ B_{1,2}$.

(b) *If* $\iota = 10$, *then we have* $(C, O) \sim (B_{4,5}^2)^{B_{10,2}}$.

(5) *Assume that the tangent cone* $T_O C_5$ *consists of two lines with multiplicity* 2.

(a) *If* $\iota = 8$, *then we have* $(C, O) \sim (B_{2,2}^4)^{2B_{10,2}}$, $(B_{1,1}^4)^{B_{10,2}} \circ B_{4,6}$ *and* $(B_{1,1}^4)^{B_{10,2}} \circ B_{2,3} \circ B_{2,4}$.

(b) *If* $\iota = 10$, *then we have* $(C, O) \sim (B_{3,2}^2)^{B_{10,2}} \circ B_{4,6}$ *and* $(B_{3,2}^2)^{B_{10,2}} \circ B_{2,3} \circ B_{2,4}$.

Putting together the above classifications, we have:

ι	(C, O)
2	$B_{10,2}$
4	$B_{20,2}$, $B_{8,2} \circ B_{2,8}$, $\dagger(B_{3,2}^2)^{B_{8,2}}$
6	$B_{30,2}$, $B_{18,2} \circ B_{2,8}$, $\dagger(B_{3,2}^2)^{B_{18,2}}$ $\dagger B_{6,2} \circ (B_{2,3}^2)^{B_{4,2}}$, $\dagger(B_{4,3})^{B_{6,2}}$, $\dagger(B_{3,3}^2)^{3B_{4,2}}$
8	$B_{40,2}$, $B_{28,2} \circ B_{2,8}$, $\dagger(B_{4,2}^2)^{2B_{12,2}}$, $\dagger B_{8,10}$, $(B_{4,3}^2)^{B_{16,2}}$, $(B_{4,4}^2)^{4B_{2,2}}$ $(B_{3,2}^2)^{B_{14,2}} \circ B_{2,6}$, $\dagger B_{16,2} \circ (B_{2,3}^2)^{B_{4,2}}$, $\dagger(B_{2,2}^2)^{2B_{2,2}} \circ B_{4,6}$ $\dagger B_{4,2} \circ B_{6,8}$, $\dagger(B_{2,2}^4)^{2B_{10,2}}$, $B_{16,2} \circ (B_{2,2}^2)^{2B_{4,2}}$
10	$B_{50,2}$, $B_{38,2} \circ B_{2,8}$, $\dagger(B_{4,2}^2)^{B_{22,2}+B_{12,2}}$, $(B_{4,3}^2)^{B_{10,2}} \circ B_{2,4}$ $\dagger(B_{4,2}^2)^{2B_{10,2}} \circ B_{2,6}$, $B_{26,2} \circ (B_{2,3}^2)^{B_{4,2}}$, $B_{10,10}$, $\dagger B_{14,2} \circ (B_{1,1}^2)^{B_{2,2}} \circ B_{4,6}$ $\dagger B_{4,2} \circ (B_{3,2}^2)^{B_{12,2}}$, $B_{14,2} \circ (B_{3,3}^2)^{3B_{2,2}}$, $(B_{4,5}^2)^{B_{10,2}}$, $\dagger B_{14,2} \circ B_{6,8}$ $\dagger(B_{3,2}^2)^{B_{10,2}} \circ (B_{2,2}^2)^{2B_{2,2}} \circ B_{2,4}$, $B_{26,2} \circ (B_{2,2}^2)^{2B_{4,2}}$, $(B_{4,5}^2)^{B_{10,2}}$

THEOREM 2. *Let C be a linear torus curve of type* $(2,5)$. *The* (C,O) *is described as follows.*

† $(B_{3,2}^2)^{B_{8,2}}$: $B_{10,4}$, $(B_{4,2}^2)^{2B_{2,2}}$, $B_{k,2}\circ B_{5,2}$ $(6\le k\le 15)$

† $(B_{3,2}^2)^{B_{18,2}}$: $(B_{5,2}^2)^{B_{20,2}}$, $(B_{6,2}^2)^{2B_{8,2}}$, $(B_{7,2}^2)^{B_{12,2}}$, $(B_{8,2}^2)^{2B_{4,2}}$, $(B_{9,2}^2)^{B_{4,2}}$, $B_{20,5}$, $B_{k,2}\circ B_{10,3}$ $(1\le k\le 13)$

† $(B_{4,2}^2)^{B_{22,2}+B_{12,2}}$: $(B_{5,2}^2)^{B_{30,2}}$

† $B_{6,2}\circ(B_{2,3}^2)^{B_{4,2}}$: $B_{8,4}\circ B_{2,6}$, $B_{k,2}\circ B_{4,2}\circ B_{2,6}$ $(5\le k\le 10)$

† $B_{16,2}\circ(B_{2,3}^2)^{B_{4,2}}$: $B_{16,2}\circ B_{4,8}$, $B_{16,2}\circ B_{2,4}\circ B_{2,k}$ $(5\le k\le 10)$

† $B_{26,2}\circ(B_{2,3}^2)^{B_{4,2}}$: $B_{26,2}\circ B_{4,8}$, $B_{26,2}\circ B_{2,4}\circ B_{2,k}$ $(5\le k\le 9)$

† $(B_{4,2}^2)^{2B_{10,2}}\circ B_{2,6}$: $(B_{5,2}^2)^{B_{16,2}}\circ B_{2,6}$, $(B_{6,2}^2)^{2B_{6,2}}\circ B_{2,6}$, $(B_{8,2}^2)^{2B_{2,2}}\circ B_{2,6}$, $B_{18,2}\circ B_{2,6}$, $B_{19,2}\circ B_{2,6}$

† $(B_{4,3}^2)^{B_{6,2}}$: $B_{4,2}\circ(B_{3,2}^2)^{B_{2,2}}$, $B_{10,6}$, $B_{6,3}\circ B_{5,3}$

† $B_{4,2}\circ(B_{3,2}^2)^{B_{12,2}}$: $(B_{5,3}^2)^{B_{20,2}}$

† $(B_{2,2}^2)^{2B_{2,2}}\circ B_{4,6}$: $B_{2,4}\circ B_{2,3}\circ(B_{2,2}^2)^{2B_{2,2}}\circ B_{2,3}\circ B_{2,4}$

† $B_{14,2}\circ(B_{1,1}^2)^{B_{2,2}}\circ B_{4,6}$: $B_{14,2}\circ(B_{1,1}^2)^{B_{2,2}}\circ B_{2,3}\circ B_{2,4}$

† $B_{4,2}\circ B_{6,8}$: $B_{4,2}\circ B_{3,4}\circ B_{2,3}\circ B_{1,2}$.

† $B_{14,2}\circ B_{6,8}$: $B_{14,2}\circ B_{3,4}\circ B_{2,3}\circ B_{1,2}$

† $B_{8,10}$: $B_{4,5}\circ B_{3,4}\circ B_{1,2}$

† $(B_{2,2}^4)^{2B_{10,2}}$: , $(B_{1,1}^4)^{B_{10,2}}\circ B_{4,6}$, $(B_{1,1}^4)^{B_{10,2}}\circ B_{2,3}\circ B_{2,4}$

† $(B_{3,2}^2)^{B_{10,2}}\circ B_{4,6}$: $(B_{3,2}^2)^{B_{10,2}}\circ B_{2,3}\circ B_{2,4}$

7. Appendix

In this section, we give some examples of singularities which were obtained in the previous sections.

EXAMPLE 1 (Case I). We assume that the quintic C_5 is smooth at O. Then we have $(C,O)\sim B_{5\iota,2}$, $1\le\iota\le 10$. The following example

$$C:\ f(x,y)=(y^5+y+x^2)^2+(y+x^2)^5$$

corresponds to $\iota=10$ and $(C,O)\sim A_{49}$ and the Milnor number μ is 49 ([1]).

EXAMPLE 2 (Case II-(a)). We assume that T_OC_5 consists of two distinct lines.

(a-1) $C: f(x,y)=(y-x^2)^5+(y^5+xy-x^3)^2$, $(C,O)\sim B_{43,2}\circ B_{2,3}$, $\iota=10$, $\mu=51$.

(a-2) $C: f(x,y)=x^5y^5+(y^5+yx+x^5)^2$, $(C,O)\sim B_{23,2}\circ B_{2,23}$, $\iota=10$, $\mu=51$.

EXAMPLE 3 (Case II-(b)). We assume that $T_O C_5$ consists of a line with multiplicity 2.

(b-1) $C_t : f(x, y) = (y+x^2)^5+(-y^5+2xy^3+(2x^3+1)y^2+(2+t)x^2y+(1+t)x^4)^2$,
$(C_t, O) \sim (B_{4,2}^2)^{B_{32,2}+B_{2,2}}$, $\iota = 10$, $\mu = 55$.

This singularity degenerate into:

$$C_0 : f(x, y) = (y + x^2)^5 + (-y^5 + 2xy^3 + (2x^3 + 1)y^2 + 2x^2y + x^4)^2,$$
$$(C_0, O) \sim B_{25,4}, \iota = 10, \ \mu = 72.$$

Note that C_0 is a rational curve.

(b-2) $C : f(x, y) = (y^2+xy)^5+(y^2+x^5)^2$, $(C, O) \sim (B_{5,2}^2)^{B_{15,2}}$, $\iota = 7$, $\mu = 42$.

EXAMPLE 4 (Case III-(a)). We assume that $T_O C_5$ consists of three distinct lines.

(a-1) $C : f(x, y) = (y + x^2)^5 + (y^5 + xy^3 + (x^3 + x)y^2 + (x^3 + x^2)y + x^4)^2$
$(C, O) \sim B_{36,2} \circ B_{4,3}$, $\iota = 10$, $\mu = 56$.

(a-2) $C : f(x, y) = x^5y^5 + (y^5 + xy^2 + x^2y + x^5)^2$
$(C, O) \sim B_{16,2} \circ (B_{1,1}^2)^{B_{4,2}} \circ B_{2,16}$, $\iota = 10$, $\mu = 57$.

EXAMPLE 5 (Case III-(b)). We assume that $T_O C_5$ consists of a line with multiplicity 2 and a single line.

(b-1) $C : f(x, y) = (y + x^2)^5 + (y^5 + y^2x - x^5)^2$, $(C, O) \sim B_{29,2} \circ B_{6,3}$, $\iota = 10$, $\mu = 61$.

(b-2) $C : f(x, y) = x^5y^5 + (y^5+ y^2x + x^5)^2$, $(C, O) \sim (B_{4,2}^2)^{2B_{5,2}} \circ B_{2,16}$, $\iota = 10$, $\mu = 61$.

EXAMPLE 6 (Case III-(c)). We assume that $T_O C_5$ consists of a line with multiplicity 3.

(c-1) $C : f(x, y) = (y + x^2)^5 + (y^5 + y^3 + y^2x^2 + yx^3 + x^5)^2$,
$(C, O) \sim B_{29,2} \circ B_{6,3}$, $\iota = 10$, $\mu = 61$.

(c-2) $C : f(x, y) = x^5y^5 + (y^3 + x^5)^2$, $(C, O) \sim (B_{5,3}^2)^{B_{10,2}}$, $\iota = 8$, $\mu = 55$.

EXAMPLE 7 (Case IV-(a)). We assume that $T_O C_5$ consists of four distinct lines.

(a-1) $C : f(x, y) = (y^5 + y^4x + xy^3 + 2y^2x^3 + x^3y + x^5)^2 + (y^2 + y + x^2)^5$,
$(C, O) \sim B_{29,2} \circ B_{6,3}$, $\iota = 10$, $\mu = 61$.

(a-2) $C : f(x, y) = x^5y^5 + (y^5 + y^3x + yx^3 + x^5)^2$,
$(C, O) \sim B_{14,2} \circ (B_{2,2}^2)^{2B_{2,2}} \circ B_{2,14}$, $\iota = 10$, $\mu = 67$.

EXAMPLE 8 (Case IV-(b)). We assume that $T_O C_5$ consists of a line with multiplicity 2 and two distinct lines.

(b-1) $C : f(x, y) = (y^5 + y^4 + x^2y^3 + x^2y^2 + x^4y)^2 + (y + x^2)^5$,
$(C, O) \sim B_{22,2} \circ B_{6,3}$, $\iota = 10$, $\mu = 66$.

(b-2) $C : f(x, y) = (2y^5 + (x^2 + x)y^3 - y^2x^2 + 2x^5)^2 + (yx - x^2)^5$,
$(C, O) \sim B_{6,4} \circ (B_{1,1}^2)^{B_{7,2}} \circ B_{2,9}$, $\iota = 10$, $\mu = 69$.

EXAMPLE 9 (Case IV-(c)). We assume that $T_O C_5$ consists of a line with multiplicity 3 and a single line.

(c-1) $C: f(x,y) = (2y^5 + y^4x + (x^2+x)y^3 + 3y^2x^3)^2 + (y^2 + (x+1)y + 3x^2)^5$
$(C, O) \sim B_{15,2} \circ B_{10,3}, \quad \iota = 10, \ \mu = 71.$

(c-2) $C: f(x,y) = (y^5 + y^3x + x^5)^2 + y^5x^5, \ (C,O) \sim (B^2_{4,3})^{B_{5,2}} \circ B_{2,9}, \ \iota = 10,$
$\mu = 71.$

EXAMPLE 10 (Case IV-(d)). We assume that $T_O C_5$ consists of a line with multiplicity 4.

(d-1) $C: f(x,y) = (2y^5 + (x+1)y^4 + x^2y^3)^2 + (y^2 + (x+1)y + x^2)^5$
$(C,O) \sim B_{20,5}, \quad \iota = 10, \ \mu = 76.$

(d-2) $C: f(x,y) = (y^4+x^5)^2 + (y^2+yx)^5, \quad (C,O) \sim (B^2_{5,4})^{B_{5,2}}, \quad \iota = 9, \ \mu = 68.$

EXAMPLE 11 (Case IV-(e)). We assume that $T_O C_5$ consists of two lines with multiplicity 2.

$$C: f(x,y) = (y^5 + y^2x^2 + x^5)^2 + y^5x^5, \ (C,O) \sim (B^2_{3,2})^{B_{5,2}} \circ (B^2_{2,3})^{B_{5,2}},$$
$$\iota = 10, \mu = 71.$$

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