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The Involutions of Compact Symmetric Spaces, V

Tadashi NAGANO and Makiko S. TANAKA

Sophia University and Science University of Tokyo

0. Introduction.

In this part, we will prove that the roots of a symmetric space M defined with the curvature (by using the Jacobi equation) make a root system $R(M)$ and determine their multiplicities. Thus we reestablish the known facts in a more geometric way (including the fact that the roots of a simple Lie algebra make a root system). Conversely, $R^m(M)$, $R(M)$ with the information of the multiplicity, allows one to recover the curvature of M; that is, $R^m(M)$ is a simple and complete description of the curvature tensor (although we will not give a proof).

In our geometric method of determining the multiplicity, we use the classification of the Hopf fibrations by Adams in case the rank $r(M)$ is 1. In case $r(M) = 2$, we use that of the homogeneous isoparametric hypersurfaces of spheres [M2] (or, equivalently, that of the compact Lie groups $K \subset SO(n)$ acting linearly on R^n whose principal orbits have codimen $= 2$ by Hsiang-Lawson [HL] and others). The results in these cases are by and large enough to finish the job in the other case of $r(M) > 2$. To complete the job, we have studied the (adjoint) action of $\boldsymbol{k}(0)$, the centralizer of $\boldsymbol{a}$ in $\boldsymbol{k}$, in 2.2, which will give a deeper isnsight into the symmetric space.

In the final section, we will characterize a (local) Kählerian symmetric space in terms of $R^m(M)$ (Theorem 4.1) and in terms of centriole (see 1.9) (Theorem 4.5).

SYMBOLS. The symmetric spaces are denoted by standard notations (as in [H]) with minor exceptions such as $G_p(K^n)$ denoting the Grassmann manifold of the p dimensional subspaces of a linear space K^n and $AI(n) := SU(n)/SO(n)$, etc. $\mathcal{L}G :=$ the Lie algebra of G. $\boldsymbol{m}^2 := [\boldsymbol{m}, \boldsymbol{m}]$ for a linear subspace of a Lie algebra. See [B] for the symbols and numberings of the roots and weights; in particular, $(\varepsilon_1, \cdots, \varepsilon_n)$ is an orthonormal system.

1. Basic concepts of symmetric spaces.

1.0 DEFINITION. A connected smooth manifold M is *a symmetric space* if (1) there is assigned an involutive diffeomorphism $s_x : M \to M$ of which x is an isolated fixed point and

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(2) there is a linear connection ∇ on M which is left invariant by every point *symmetry* s_x.

1.1 LEMMA. *The symmetry s_x induces $ds_x = -1$ on the tangent space $T_x M$.*

PROOF. s_x permutes the geodesics with affine parameters of ∇. □

1.2 COROLLARY. (i) *The invariant connection ∇ is unique and it has no torsion.* (ii) *If all the point symmetries are isometries of a Riemannian metric g on M, then ∇ is the Levi-Civita connection of g.* (iii) *If M is compact, then M is Riemannian symmetric; i.e. ∇ is the Levi-Civita connection of an invariant metric g.*

1.3 DEFINITION. *A morphism* (or *a homomorphism*): $M \to N$ of a symmetric space M into another one N is a smooth map which commutes with every point symmetry; namely, $f \circ s_x = s_{f(x)} \circ f$ for every $x \in M$.

The symmetric spaces and their morphisms form a category, which is a subcategory of that of the smooth manifolds and their smooth maps.

1.3a DEFINITION. When a smooth manifold M is not connected, we call it *symmetric* if M satisfies (1) and (2) in 1.0 together with the condition: (3) every symmetry s_x is an automorphism of M (in the sense of 1.3).

1.3b REMARK. In case M is connected, the condition (3) follows from (1) and (2). (3) is also satisfied if M is a Lie group with the symmetries s_x defined by $s_x(y) = xy^{-1}x$. Thus the category of the symmetric spaces contains that of Lie groups.

We call *M a subspace* of N if the inclusion map: $M \to N$ is a morphism. For example, the fixed point set $F(a, M)$ of an automorphism $a \in \mathrm{Aut}(M)$ is a subspace along with each connected component $F(a, M)_{(p)}$ (through a point p). For a morphism $f : M \to N$ of a connected M, the image $f(M)$ and the "kernel" $f^{-1}(f(x))$, $x \in M$, are subspaces. And f is the composite of an epimorphism and a monomorphism.

Every geodesic in M is its subspace, as the next lemma states.

1.4 LEMMA. *A smooth map of a euclidean line R into a symmetric space M is a morphism if and only if it is a geodesic in M with an affine parameter. Thus, in case M is connected, a morphism: $M \to N$ is nothing but a totally geodesic map.*

1.5 COROLLARY. (i) *The transformation group generated by the point symmetries of M is transitive on M if M is connected.* (ii) *If M is compact furthermore, then that group contains the identity component of the automorphism group* $\mathrm{Aut}(M)$. (iii) $\mathrm{Aut}(M)$ *is a Lie group.*

1.5a REMARK. The morphisms are affine maps basically. For example, two tori are isomorphic if and only if they have equal dimension, whether or not they are isometric. For another example, $\mathrm{Aut}(E^n)$ of the euclidean space E^n is its affine transformation group.

One knows Cartan's mapping $C : G \to G : b \mapsto b\sigma(b)^{-1}$ defined for a group G and its involution σ and that its image $C(G)$ is a symmetric space; the inclusion map: $C(G) \to G$ is the Cartan embedding. Since C is not a morphism, we proposed in [N] to use the morphism $Q = Q_o : M = G/K \to G : x \mapsto s_x \circ s_o$, which is essentially the same as C as the next theorem states. (Q_o is defined for the pointed space (M, o).)

1.6 THEOREM (cf. 1.9 and 1.12, [N-II]). (i) *C is the composite $Q \circ \pi$ of Q and the projection $\pi : G \to G/K$ if $\mathrm{ad}(s_o) = \sigma$.* (ii) *$C(G)$ is thus a symmetric space $Q(M)$.* (iii) *The "kernel" $C^{-1}(1_G)$ is the fixed point set $F(\sigma, G)$, while the "kernel" $Q^{-1}(1_G)$ is the subspace $\{p \in M \mid s_p = s_o\}$, which is the totality of what we call the poles p of o in M.* (iv) *Therefore Q is a covering morphism and $Q^{-1}(1_G)$ may be regarded as the subspace $F(\sigma, G)/K$.*

PROOF. (i) Given $b \in G$, we write x for $\pi(b)$ and obtain $Q(x) = bs_ob^{-1}s_o = C(b)$. (ii) Since Q is a morphism, $C(G) = Q(M)$ is a symmetric space. (iii) Clearly $C(b) = 1_G$ if and only if $b = \sigma(b)$, while $Q(x) = 1_G$ if and only if $s_x = s_o$. (iv) must be obvious, since the poles are isolated. □

1.6a NOTE. If b satisfies $\sigma(b) = b^{-1}$ (like $Q(x)$), then it does $Q(x) = C(b) = b^2$; this is why we called the morphism Q quadratic.

Hereafter we assume that M is compact and connected (unless otherwise mentioned). We now recall important subspaces of compact M.

1.7 DEFINITION. Each connected component $F(s_o, M)_{(p)} \ni p$ of the fixed point set $F(s_o, M)$ of the symmetry at o is called *a polar* of o in M, denoted by $M^+(p)$, $M^+(p; o)$ or M^+. A pole p is a polar $M^+(p)$ which is a singleton.

Every polar in M is a subspace of M. Here are a few examples of known theorems which illustrate the relevance of the polars.

1.7a EXAMPLE 1. The harmonic maps of S^2 into a compact simple group G ($\neq E_8, F_4, G_2$) are factored into the product of a constant map and maps into polars in G, as was proved by Uhlenbeck and others [EL].

1.7b EXAMPLE 2. The (simple) H-Kählerian spaces are the polars of 1_G in the 1-connected simple groups G that are the closest to 1_G.

1.7c EXAMPLE 3. The cut locus of $o \in M$ is the union of disc bundles over the polars$\neq o$ in M if M is a 1-connected R-space (see [T79]).

1.8 DEFINITION. *A meridian* $M^-(p)$ is $F(s_o \circ s_p, M)_{(p)}$, the connected component of the fixed point set of $s_o \circ s_p$ for a point p on a polar $M^+(p)$ of o in M. $M^-(p)$ and $M^+(p)$ are said to correspond to each other at p. $M^-(p)$ is *c-orthogonal* to $M^+(p)$ at p; that is, the tangent space $T_pM^-(p)$ is the orthogonal complement of $T_pM^+(p)$ in T_pM, since the involution s_o commutes with s_p.

1.8a M is determined by a pair $(M^-(p), M^+(p))$ of the corresponding polar and meridian (1.15 in [N-II]). One sees $M^+(p) = K/K^+$ and $M^-(p) = G^-/K^+$, where $K(o) = \{o\}$, $G^- := F(\mathrm{ad}(s_o \circ s_p), G)_{(1)}$, and $K^+ := K \cap G^-$.

1.8b The natural homomorphism $\pi_1(M^-) \to \pi_1(M)$ of the fundamental group, induced by the inclusion: $M^- \to M$, is surjective.

1.9 A covering morphism $f : M \to M''$ carries a polar in M onto one in M''. If a polar M''^+ of $f(o)$ in M'' is not the image of one in M and if the covering degree of f is two (which we assume for simplicity without losing generality), then there is a pole p of o, $f(p) = f(o)$, in M, such that some connected component, called *a centriole for* (o, p), of the subspace $\{x \in M \mid s_x(o) = p\}$, called *the centrosome of* (o, p), is carried onto M''^+. The union of the polars of $f(o)$ is the image of that of the polars of o in M and the centrosome of (o, p).

2. The root system $R(M)$.

Let M be a compact connected symmetric space G/K, G locally effective on M and $K(o) = \{o\}$. Then the Lie algebra $\boldsymbol{g} = \mathcal{L}G$ is decomposed into the eigenspaces of $\mathrm{ad}(s_o)$; $\boldsymbol{g} = \boldsymbol{k} + \boldsymbol{m}$. By using the Jacobi equations along geodesics through o in M, we defined the roots (with respect to a Cartan subalgebra $\boldsymbol{a}$) in [N-II]. We recall them briefly.

Given a member H of $\boldsymbol{a}$, consider the geodesic c with $c(0) = o$ and $c'(0) = H = H(o)$. The Jacobi equation along the geodesic c in M

2.1 $$\nabla_H \nabla_H V + R(V, H)H = 0$$

is converted to the ordinary differential equation $V'' + R_H V = 0$ on the pullback of TM by c, where R_H is the linear map: $V \mapsto R(V, H)H$, which is constant by $\nabla R = 0$ and symmetric by the known properties of the curvature tensor R. The eigenvalues $\alpha(H)^2$ are nonnegative by compactness of M. α is a linear form on $\boldsymbol{a}$; α is called *a root* if it is not zero. Since the members of $\boldsymbol{g}$ (restricted to c) are solutions of the Jacobi equation which s_o leaves the equation invariant, one obtains the decompositions

2.2 $$\boldsymbol{m} = \boldsymbol{a} + \sum \boldsymbol{m}(\alpha) \quad \text{and} \quad \boldsymbol{k} = \boldsymbol{k}(0) + \sum \boldsymbol{k}(\alpha),$$

where the summation runs over the roots of M and $\boldsymbol{g}(\alpha) = \boldsymbol{m}(\alpha) + \boldsymbol{k}(\alpha)$ is the eigenspace for $\alpha(H)^2$, $\boldsymbol{a} = \boldsymbol{m}(0)$ and $\boldsymbol{m}(-\alpha) = \boldsymbol{m}(\alpha)$ (except that, in case $\alpha = 0$, $\boldsymbol{g}(0)$ does not equal the eigenspace of R_H for 0). The main difference of M from the compact Lie group is that *the multiplicity* $m(\alpha) := \dim \boldsymbol{m}(\alpha) = \dim \boldsymbol{k}(\alpha)$ is not a universal constant and $\boldsymbol{k}(0)$ also reflects the individuality of M. Thus $m(\alpha)$ and $\boldsymbol{k}(0)$ will be our concern. In terms of the Lie algebra structure of $\boldsymbol{g}$, the decompositions 2.2 are explained as follows. Given an arbitrary member X of $\boldsymbol{m}(\alpha)$, there is a unique member Y of $\boldsymbol{k}(\alpha)$ which satisfies

2.2a $$[H, X] = \alpha(H)Y \quad \text{and} \quad [H, Y] = -\alpha(H)X$$

for every H in $\boldsymbol{a}$. $\boldsymbol{k}(0)$ is the centralizer of $\boldsymbol{a}$ in $\boldsymbol{k}$. Thus $\mathrm{ad}\, H$ is a linear isomorphism of $\boldsymbol{m}(\alpha)$ onto $\boldsymbol{k}(\alpha)$ if $\alpha(H) \neq 0$.

The rank $r(M)$ of M is $r(\boldsymbol{a}) := \dim \boldsymbol{a}$. Let $R^m(M)$ denote the set of the pairs $(\alpha, m(\alpha))$ of the roots and their multiplicities. $r(M)$ and $R^m(M)$ are determined by the curvature R. Conversely, R is recovered by them; indeed, two spaces M and N are locally isomorphic if

$r(M) = r(N)$ and $R^m(M) = R^m(N)$ (see [H] for related facts). For example, if $r(M) = 1$ and $R(M) = \{\pm\alpha\}$ with $m(\alpha) = m$, then M is locally isomorphic with the sphere S^{m+1}; thus $\boldsymbol{k}(0)$ is isomorphic with the Lie algebra $\mathcal{L}O(m)$ acting on $\boldsymbol{m}(\alpha)$ and $\boldsymbol{k}(\alpha)$ as such.

As to $\boldsymbol{k}(0)$, its rank $r(\boldsymbol{k}(0))$ equals $r(\boldsymbol{g}) - r(M)$. $\boldsymbol{k}(0)$ was determined in every case by Tamaru [Tm]. We will find its action on each $\boldsymbol{g}(\alpha)$, which is relevant to determining the multiplicity.

2.3 THEOREM. *If $M = G/K$ is semisimple (i.e. G is semisimple), then the set $R(M)$ of the roots (with respect to $\boldsymbol{a}$) is a root system in $\boldsymbol{a}$; that is,* (i) *$R(M)$ does not contain* 0 *but space $\boldsymbol{a}$ (which is identified with the dual space $\boldsymbol{a}^*$ with the metric)*; (ii) *$R(M)$ is a finite set and the number*

$$n(\alpha, \beta) := 2\langle\alpha, \beta\rangle/\langle\beta, \beta\rangle$$

is an integer for any $\alpha, \beta \in R(M)$; and (iii) *$R(M)$ is mirror-symmetric: the reflection: $\boldsymbol{a} \to \boldsymbol{a} : \alpha \mapsto \alpha - n(\alpha, \beta)\beta$ preserves $R(M)$ for every root β.*

PROOF. (i) must be obvious. We will prove that $n(\alpha, \beta)$ is an integer. One sees that X and Y in 2.2b generate a subalgebra $\boldsymbol{g}_X \cong \mathcal{L}O(3)$ of $\boldsymbol{g}$; in fact, the inner product $\langle H, [X, Y]\rangle = \langle [H, X], Y\rangle = \langle \alpha(H)Y, Y\rangle = \alpha(H)\|Y\|^2$, which implies $[X, Y] = \|Y\|^2\alpha$ ($\in \boldsymbol{a}$). Thus S^2 is a covering space of $\operatorname{Exp}\boldsymbol{g}_X$ of degree 1 or 2. We normalize the basis (X, Y, H) for $\boldsymbol{g}_X$ so that $\|X\|^2 = \|Y\|^2 = \|\alpha\|^{-2}$ and $H = \|\alpha\|^{-2}\alpha$. Then we have $[H, X] = Y$, $[H, Y] = -X$, and $[X, Y] = H$. With this, the geodesic $c : t \mapsto \operatorname{Exp}(tH)$ satisfies $c(2\pi) = c(0)$. Hence every member of $\boldsymbol{k}(\beta)$, a solution of the Jacobi equation along c, vanishes at $t = 2\pi$. Therefore $n(\beta, \alpha) = 2\beta(H)$ is an integer. Now ad $\exp(\pi Y)$ acts trivially on the vectors $H' \in \boldsymbol{a}$ satisfying $\alpha(H') = 0$ and carries α into $-\alpha$. Thus ad $\exp(\pi Y)$ acts on $\boldsymbol{a}$ as the reflection: $\beta \mapsto \beta - n(\beta, \alpha)\alpha$ on $\boldsymbol{a}$, while the automorphism ad $\exp(\pi Y)$ which stabilizes $\boldsymbol{a}$ permutes the roots. □

2.3a **The root systems.** We recall them for every simple M. First fix an orthonormal basis $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r)$ for a metric vector space. $D_r = R(\mathcal{L}O(2r))$ consists of $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leqq i < j \leqq r$, $r \geqq 3$. The lower index r in D_r denotes the rank of the root system D_r as is always. $A_{r-1} = R(\mathcal{L}SU(r))$ consists of $\pm(\varepsilon_i - \varepsilon_j)$, $1 \leqq i < j \leqq r$, $r \geqq 2$; that is, the subsystem consisting of those which are orthogonal to $\varpi_r(D_r) = (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r)/2$ (by the method to obtain a meridian in $SO(2r)$, [N-II] 2.15). $B_r = R(\mathcal{L}O(2r + 1))$ is the union $D_r \cup (\pm\varepsilon_j \mid 1 \leqq j \leqq r)$, which is the minimal mirror-symmetric set that contains $D_r \cup \{\varpi_1(D_r) = \varepsilon_1\}$, denoted by $MSS(D_r \cup \{\varpi_1(D_r)\})$. $C_r = R(\mathcal{L}Sp(r))$ is dual to B_r; i.e. it is $MSS(D_r \cup \{2\varpi_1(D_r)\})$. Formally, this gives the correct $B_2 \cong C_2$. $BC_r = B_r \cup C_r$; i.e. BC_r consists of $\pm\varepsilon_i \pm \varepsilon_j$, ε_k and $2\varepsilon_k$, $1 \leqq i < j \leqq r$, $1 \leqq k \leqq r$, $r \geqq 1$. $E_8 = MSS(D_8 \cup \{\varpi_8(D_8)\})$. $E_7 = MSS(A_7 \cup \{\varpi_4(A_7)\})$. $E_6 = MSS(D_5 \cup \{\varpi_5(D_5) + (\varepsilon_6 + \varepsilon_7 - \varepsilon_8)/2\})$. $F_4 = MSS(B_4 \cup \{\varpi_4(B_4)\}$. And $G_2 = MSS(A_2 \cup \{\varpi_2(A_2)\})$. The lower indices in $D_r, \cdots, G_2$ mean the rank of each root system.

2.3b We owe the concept of the mirror-symmetric set to O. Ikawa and H. Tasaki. (Later they found that C. T. C. Wall had used a similar concept.)

2.3c Our root system $R(M)$ happens to coincide with the restricted root system ([H]) essentially, which was proved in [OS] to be a root system. Since our category of symmetric spaces contains that of compact semisimple Lie groups as a subcategory, Theorem 2.3 gives another proof of the existence of the root system in the subcategory.

3. The root system with multiplicity.

In this section, we will determine the possible multiplicity of each root of the compact simple space M, beginning with more basic spaces. (Certain partial works were done in [N-II], but we will start from scratches, observing $\boldsymbol{k}(0)$.)

3.0 PROPOSITION. *The multiplicity $m(\alpha) \geqq m(\beta)$ if the length $\|\alpha\| \leqq \|\beta\|$ and M is simple* ([N-II], 2.4c).

A proof will be given later (3.2b after the Case 2°).

3.0a As a matter of notation, in the symbols $F_4(a, b)$, $BC_r(a, b, c)$, etc. the multiplicities of shorter roots precede those of longer ones; that is, $a = m(\alpha)$, $b = m(\beta)$ and $c = m(\gamma)$ are the multiplicites of the roots with length in the order $\|\alpha\| < \|\beta\| < \|\gamma\|$.

3.0b REMARK. The following fact will be used below explicitly (in the proof of Corollary 3.6a) or implicitly. If M^- is a meridian in a simple space M, then $R^m(M)$ contains $R^m(M^-)$, the multiplicity map: $\alpha \mapsto m(\alpha)$ restricts to $R(M^-)$ and the Dynkin diagram of $R(M^-)$ is obtained either from that of $R(M)$ by removing α_j for $n^j = 1$ or from the extended Dynkin diagram of $R(M)$ by removing α_j for $n^j = 2$, where n^j is the coefficient which appears in $\alpha^\sim = \sum n^j \alpha_j$ for the highest root $\alpha^\sim$. The converse is true if M is the bottom space $M^{\%}$ (cf. Theorem 2.15 in [N-II]).

Case 1° $R(M) = A_1$. $R^m(M) = A_1(m)$. *Then M is a space of positive constant curvature* such as the sphere S^{m+1}. One has $\boldsymbol{m} = \boldsymbol{a} + \boldsymbol{m}(\alpha)$ and $\boldsymbol{k} = \boldsymbol{k}(0) + \boldsymbol{k}(\alpha)$ (2.2), $\dim \boldsymbol{a} = 1$ and $\dim \boldsymbol{m}(\alpha) = m$. Thus $\boldsymbol{k}(0) = \boldsymbol{m}(\alpha)^2 := [\boldsymbol{m}(\alpha), \boldsymbol{m}(\alpha)] \cong \mathcal{L}O(m)$ acts on $\boldsymbol{m}(\alpha)$ as $\mathcal{L}O(m)$. Also one sees $\boldsymbol{k} \cong \mathcal{L}O(m+1)$. This is true in a more general case as the next lemma states (cf. Lemma 2.25 in [N-II]).

3.1 LEMMA. *If α is a root in $R(M)$ but 2α is not, then $R\alpha + \boldsymbol{m}(\alpha)$ is the tangent space to a subspace as in case* 1°, *denoted by $S(\alpha)$. In particular, $\boldsymbol{m}(\alpha)^2 \subset \boldsymbol{k}(0)$ acts on $\boldsymbol{m}(\alpha)$ as $\mathcal{L}O(m(\alpha))$ (just as $\boldsymbol{k}(0)$ itself does). And $\boldsymbol{m}(\alpha)^2 = \boldsymbol{k}(\alpha)^2$ is an ideal in $\boldsymbol{k}(0)$. Moreover $\boldsymbol{k}\{\alpha\}^2 := \boldsymbol{k}(\alpha) + \boldsymbol{k}(\alpha)^2$ is isomorphic with $\mathcal{L}O(m(\alpha) + 1)$ and acts on $R\alpha + \boldsymbol{m}(\alpha)$ as such. Hence $S(\alpha)$ has a positive constant curvature.*

3.1a REMARK and a DEFINITION. If the length of α equals that of another root β, then $S(\alpha)$ is congruent with $S(\beta)$; i.e. a member of G will carry one onto the other. If α is the longest, then we call $S(\alpha)$ *a Helgason sphere*, abbreviated to *an H-sphere*, which is a maximal connected subspace with the greatest sectional curvature in M. $S(\alpha)$ is isomorphic with a sphere (unless M is a real projective space), as Helgason discovered [H]. If the simple M is 1-connected, a Helgason sphere $S(\alpha)$ has the smallest volume among the submanifolds in the non-zero homology class which $S(\alpha)$ represents (whether or not the class is a torsion) [L]. One also knows that M is $(m(\alpha)-1)$-connected then.

Case 2° $R(M) = BC_1$. *If* $R^m(M) = BC_1(a,b)$ *for some space* M, *then one has* $(a,b) = (pq, p-1)$, *where* $p = 2, 4$ *or* 8 *and, in case* $p = 8$, $q = 1$ (otherwise q can be any positive integer). Geometrically, M is a complex (or quaternionic) projective space or the Cayley projective plane.

PROOF. By $r(M) = 1$ (2.1a), the H-spheres are determined by any one of its tangent vectors $\neq 0$, and the isotropy subgroup K at o is transitive on the geodesics through o. Thus any one, c, of them meets the unique polar M^+ of o at a unique point x. Hence the map $c \mapsto x \mapsto M^-(x) \equiv$the H-sphere $S(2\alpha)$ is bijective and one has a fibration $S^{p-1} \to S^{n-1} \to M^+$, $n := \dim M$ and $p :=$the dimension of $\boldsymbol{m}_- := \boldsymbol{a} + \boldsymbol{m}(2\alpha)$; $S(2\alpha) \cong S^p$. One can rotate a unit tangent vector in $T_o(S(2\alpha))$ around o into $T_o(\boldsymbol{m}_+)$, $\boldsymbol{m}_+ := \boldsymbol{m}(\alpha)$, to see the whole $S(2\alpha)$ is carried onto an H-sphere S in $\mathrm{Exp}(\boldsymbol{m}_+)$ which is congruent with the polar M^+ by a parallel translation (or displacement) along the geodesic $\mathrm{Exp}(\boldsymbol{a})$.

Assume $S = \mathrm{Exp}(\boldsymbol{m}_+)(\equiv M^+)$ for the moment. To visualize a projective plane, call an H-sphere a line. Then two distinct points lie on a single line. Dually, two distinct lines meet at a single point, as one easily sees by using the double fibration $S \leftarrow M \to M^+$ (cf. [NT-III]). Hence M is a projective plane. Alternatively, one verifies the conclusion by using Adams' theorem: the (Hopf) fibration $S^{p-1} \to S^{2p-1} \to S^p$ exists only for $p = 1, 2, 4$ or 8. For another proof, one can observe that S^{p-1} is parallelizable to apply another theorem of Adams. One could also use the classification of the compact transitive Lie groups acting on the spheres.

Back to the general case, one observes $R(M^+) = BC_1$ and moreover $R^m(M^+) = BC_1(p(q-1), p-1)$ by $M^+ \supset S$ and induction on q. One now knows $q = 1$ for $p = 8$, since the Cayley algebra is not associative and hence the Desargues theorem does not obtain, for example. □

SUMMARY. If $R(M) = BC_1$, M is a projective space whose lines through o are the meridians corresponding to the polar M^+ of o. And the polar M^+ is the dual hyperplane to o. The lines are exactly the H-spheres. Finally, M is globally determined by $R^m(M)$ in this case 2°; M is $G_1(\mathbf{C}^{2+q})$, $G_1(\mathbf{H}^{2+q})$ or FII.

3.2 As to $\boldsymbol{k}(0)$ and its action of on $\boldsymbol{m}(\alpha)$ in 2°, one observes $\boldsymbol{m}(\alpha)$ is a module with inner product over the complex, quaternion or Cayley algebra according as $p = 2, 4$ or 8. Since $r(M) = 1$, the principal orbits $K/K(0)$, $\mathcal{L}K(0) = \boldsymbol{k}(0)$, are spheres S^{n-1}, $n := \dim M$, which admit generalized Hopf-fibrations $S^{p-1} \to S^{n-1} \to M^+$. Thus $\boldsymbol{k}(0)$ is isomorphic with $\mathcal{L}U(q)$, $\mathcal{L}(Sp(1) \times Sp(q))$ or $\mathcal{L}O(7)$ respectively and $\boldsymbol{k}(0)$ acts on $\boldsymbol{m}(\alpha)$ as such; $\mathcal{L}O(7)$ acts on it through the spin representation. We will improve Corollary 2.26a in [N-II] and correct an error in it. Easy computations show $\boldsymbol{m}(\alpha)^2 + \boldsymbol{k}(\alpha)^2 = \boldsymbol{k}(0) + \boldsymbol{k}(2\alpha)$ and $\boldsymbol{m}(\alpha)^2 \cap \boldsymbol{k}(\alpha)^2 = \mathcal{L}SU(q)$ or $\mathcal{L}Sp(q)$ for $p = 2$ or 4. For $p = 8$, one sees $\boldsymbol{m}(\alpha)^2 = \boldsymbol{k}(\alpha)^2 = \boldsymbol{k}(0) + \boldsymbol{k}(2\alpha) \cong \mathcal{L}O(8)$, since there is no intermediate Lie algebra in the sequence $\boldsymbol{k}(0) \cong \mathcal{L}O(7) \subset \mathcal{L}O(8) \subset$

$\mathcal{L}O(9) \cong \boldsymbol{k}$. Note that $\mathcal{L}O(8)$ acts on $\boldsymbol{m}_-$, $\boldsymbol{m}(\alpha)$ and $\boldsymbol{k}(\alpha)$ through the three inequivalent representations ϖ_1, ϖ_3 and ϖ_4, since otherwise it is impossible to have the bracket product of any two of $\boldsymbol{m}_-$, $\boldsymbol{m}(\alpha)$ and $\boldsymbol{k}(\alpha)$ equal to the third (cf. 3.3). Therefore q must equal 1 if $p = 8$ (for a second proof). The monomorphisms $G_1(\mathbf{C}^{2+q}) \to G_1(\mathbf{H}^{2+q})$ and $G_1(\mathbf{H}^3) \to FII$ are $\boldsymbol{k}(0)$-equivariant (for the corresponding $\boldsymbol{k}(0)$); these maps are obtained by finding that each space is the fixed point set of an involution (an analog of the complex conjugation) of the target space (as well as the monomorphisms to be mentioned below).

3.2a LEMMA. *Assume three roots α, β and γ satisfy these conditions*: (1) $\alpha+\gamma = \beta$, (2) $\alpha + 2\gamma$ *is not a root*, (3) $\gamma \wedge \beta \neq 0$, *and* (4) *the inner product* $\langle\gamma, \beta\rangle \neq 0$. (*See examples below for these conditions.*) *Then* $[Z, X] \neq 0$ *for every nonzero* $Z \in \boldsymbol{k}(\gamma)$ *and every nonzero* $X \in \boldsymbol{m}(\beta)$. *That is, the map* $\operatorname{ad} Z : \boldsymbol{m}(\beta) \to \boldsymbol{m}(\alpha)$ *is injective.*

PROOF. Assume some pair (Z, X) satisfies the equality $[Z, X] = 0$. Choose H in $\boldsymbol{a}$ such that $\gamma(H) = 0$ but $\beta(H) \neq 0$. Then one has $0 = [H, [Z, X]] = [Z, [H, X]] = [Z, \beta(H)Y]$, $Y \in \boldsymbol{k}(\beta)$. Thus $0 = [Z, Y]$; hence $0 = [Z, [X, Y]] = [Z$, nonzero scalar multiple of $\beta]$ =nonzero member of $\boldsymbol{m}(\gamma)$, a contradiction. □

EXAMPLES. The assumption is satisfied if the pair (α, γ) is any one of $(\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3)$, $(\varepsilon_1, \varepsilon_2)$ and $(\varepsilon_1 - \varepsilon_2, -\varepsilon_2 + \varepsilon_3)$ in A_2, B_2 and G_2 respectively.

3.2b PROOF OF 3.0. We may assume $\|\alpha\| < \|\beta\|$, since the Weyl group acts transitively on the roots of equal length. 3.0 is obvious in the case of BC_1 from the result in the Case 2°; $pq > p - 1$. We may asume $\alpha + \gamma = \beta$ for some root γ by using the Weyl group, while $\alpha + 2\gamma$ is not a root. If $R(M)$ is not BC_r, these roots satisfy the assumption of 3.2a, from which 3.0 follows immediately. In case $R(M)$ is BC_r, β can happen to equal 2α; otherwise the above argument works. If $\beta = 2\alpha$ and $r > 1$, then one can find a fourth root δ such that one can apply the above argument for (α, γ) to the cases of (α, δ) and $(\alpha + \delta, \alpha - \delta)$ to reach the conclusion. □

For the spaces of rank 2, the multiplicities of the roots are immediately determined with a theorem of Hsiang-Lawson [HL] to verify the next proposition; see [TT] or [Y] for its more detailed proofs. A topological proof is found in [As].

3.2c PROPOSITION. *Let M be a simple space of rank* 2. *Then $R^m(M)$ is one of the following*: (1) $A_2(m)$ *with* $m = 1, 2, 4$ *or* 8; (2) $B_2(a, b)$ *with* $(a, b) = (m, 1), m \geqq 1, (2, 2)$ *or* $(4, 3)$ (cf. Case 4°); (3) $BC_2(a, b, c)$ *with* $(a, b, c) = (2m, 2, 1), m \geqq 1, (4m, 4, 3), m \geqq 1$, $(4, 4, 1)$ *or* $(8, 6, 1)$; *and* (4) $G_2(a, b)$ *with* $(a, b) = (1, 1)$ *or* $(2, 2)$.

PROOF. We make the isotropy subgroup K act on the tangent space T_oM, $K(o) = \{o\}$ and $M = G/K$; the so-called s-representation is considered. Then the principal orbits have codimension= 2 in T_oM by the formulas 2.2. Conversely, under the assumption on codimension= 2 of the principal orbits of a compact Lie group$\subset SO(n)$ acting on R^n, Theorem 5 in [HL] asserts that M is exactly one of the known symmetric spaces (described above). And a geometric proof is finished. □

Case 3° $R(M) = A_2$. $R^m(M) = A_2(m)$. *Then* m *is* 1, 2, 4 *or* 8.

ANOTHER PROOF. Assume $m > 1$. Decompose $\boldsymbol{m}$ into the direct sum $\boldsymbol{m}_+ \oplus \boldsymbol{m}_-$ of $\boldsymbol{m}_+ := \boldsymbol{m}(\alpha) + \boldsymbol{m}(\gamma)$ and $\boldsymbol{m}_- := \boldsymbol{a} + \boldsymbol{m}(\beta)$, $\alpha + \gamma = \beta$. And $\boldsymbol{m}_+$ is the tangent space to a subspace $M_+ = \operatorname{Exp} \boldsymbol{m}_+$ at o; $[\boldsymbol{m}(\alpha), \boldsymbol{m}(\gamma)] = \boldsymbol{k}(\beta)$. M_+ has rank= 1 by 3.1 and 3.2a. Since $m > 1$, one sees $R(M_+) = BC_1$. $R^m(M_+) = BC_1(m, m-1)$ with $m = 2, 4$ or 8, in view of 2°. □

3.3 The action of $\boldsymbol{k}(0)$ in 3°; $R^m(M) = A_2(m)$ for $m = 1, 2, 4$ and 8. One knows $\boldsymbol{k}(0) = \{0\}$, $\mathcal{L}U(1)^2$, $\mathcal{L}Sp(1)^3$, and $\mathcal{L}0(8)$ respectively. If $m = 2$, it acts on each root space as the Lie algebras of $SO(2)$ and sort of Spin(2). In case $m = 4$, $\boldsymbol{k}(0)$ acts on $\boldsymbol{m}(\varepsilon_j - \varepsilon_k)$ as $S_j + S_k$, where $S_j \cong S_k \cong \mathcal{L}Sp(1)$; $\boldsymbol{k}(0) = \sum S_j$. Thus the natural map: $\boldsymbol{k}(\alpha) \otimes \boldsymbol{k}(\gamma) \to [\boldsymbol{k}(\alpha), \boldsymbol{k}(\gamma)] = \boldsymbol{k}(\beta)$ is just $H \otimes_H H \cong H \to H$. (These facts are valid for $A_r(4)$, $r \geqq 1$, that is, for every $AII(n)$, $n \geqq 2$.) In case $m = 8$, $\boldsymbol{k}(0) = \mathcal{L}0(8)$ acts on $\boldsymbol{k}(\alpha)$, $\boldsymbol{k}(\gamma)$ and $\boldsymbol{k}(\beta)$ through the three representations with the highest weights, say, ϖ_3, ϖ_4 and ϖ_1 of D_4 respectively; then $\boldsymbol{k}(\alpha) \otimes \boldsymbol{k}(\gamma)$ is the direct sum of the $\mathcal{L}0(8)$-module for $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varpi_3 + \varpi_4$ and the one for ϖ_1. Thus it is crucial that $\mathcal{L}0(8)$ admits an outer automorphism of order three, suggesting that there could not exist any M with $R^m(M) = A_r(8)$ for $r > 2$ (cf. 5°).

3.3a The monomorphisms $SU(3) \subset SU(3) \times SU(3) \subset SU(6) \subset E_6$ restrict to the inclusions $AI(3) \subset SU(3) \subset AII(3) \subset EIV$. This sequence cannot be continued without changing the root system as we have just proved. Restricting them to their polars one obtains the sequence of projective planes (2°) plus the polar $G_1(R^3)$ in $AI(3)$.

3.3b COROLLARY. *One sees* $R^m(M) = F_4(a, 1)$ *only for a is* 1, 2, 4 *or* 8.

These numbers are possible as shown by the inclusions $FI \subset EII \subset EVI \subset EIX$ of H-Kaehlerian spaces.

Case 4° $R(M) = B_2 = C_2$. $R^m(M) = B_2(a, b)$ meaning $R(M)$ consists of ε_1, ε_2 and $\varepsilon_1 \pm \varepsilon_2$ with $m(\varepsilon_j) = a$ and $m(\varepsilon_1 \pm \varepsilon_2) = b$. *Then* $(a, b) = (m, 1)$, $m \geqq 1$, $(2, 2)$ *or* $(4, 3)$; let us notice $B_2(4, 3) = R^m(G_2(H^4))$.

PART OF ANOTHER PROOF. One has

3.4 $\quad [\boldsymbol{k}(\varepsilon_1 \pm \varepsilon_2), \boldsymbol{m}(\varepsilon_j)] = \boldsymbol{m}(\varepsilon_k)$, $j + k = 3$, by 3.2a, and

3.4a $\quad [\boldsymbol{k}(\varepsilon_1), \boldsymbol{m}(\varepsilon_2)] \subset \boldsymbol{m}(\varepsilon_1 + \varepsilon_2) + \boldsymbol{m}(\varepsilon_1 - \varepsilon_2)$.

$\boldsymbol{k}(0)$ acts on $\boldsymbol{k}(\varepsilon_1 \pm \varepsilon_2)$ as $\mathcal{L}0(b)$ and on $\boldsymbol{m}(\varepsilon_j)$ as $\mathcal{L}0(a)$ by Lemma (3.1).

Assume $b > 1$. Observe the action of $\mathcal{L}(0(b) \times 0(a))$ on $R^b \otimes R^a$.

If $a > b$, then the action is irreducible generally (for example, provided both $0(b)$ and $0(a)$ are simple), contrary to 3.4. $(a, b) = (4, 3)$ is thus the only possibility.

Now suppose $a = b$. First consider the case in which a single $\mathcal{L}0(a) \subset \boldsymbol{k}(0)$ acts on both $\boldsymbol{k}(\varepsilon_1 + \varepsilon_2)$ and $\boldsymbol{m}(\varepsilon_j)$ as such. The natural module $\mathbf{R}^a \otimes \mathbf{R}^a$ is the direct sum $\mathbf{R}^a \vee \mathbf{R}^a \oplus \mathbf{R}^a \wedge \mathbf{R}^a$ of the symmetric and the alternating products, and $\mathbf{R}^a \vee \mathbf{R}^a$ is the direct sum of $\mathbf{R}$ and its orthogonal complement $\mathbf{R}^\perp$ which is simple. $\mathbf{R}^a \wedge \mathbf{R}^a$ is simple unless $a = 1, 2$ or 4. Thus

3.4 cannot be valid unless $a \leqq 3$. In case $a = 3$, we do not have a good proof and leave it to Proposition 3.2c. □

Case 5° *$R(M) = A_3$. $R^m(M) = A_3(m)$. Then $m = 1, 2$ or 4.*

PROOF. M contains a polar M^+ with $R^m(M^+) = C_2(m, m-1)$ if $m > 1$. Thus one has the conclusion in view of 3° and 4°. Alternatively, one can use 3.3 again to exclude the case $m = 8$. □

Case 6° *$R(M) = B_3$. $R^m(M) = B_3(a, b)$. Then $(a, b) = (a, 1)$, $a \geqq 1$, or $(2, 2)$.*

PROOF. By 4°, $(a, b) = (a, 1)$, $a \geqq 1$, $(2, 2)$ or $(4, 3)$. In case (a, b) is $(4, 3)$, M contains a subspace with $R^m(N) = A_2(3)$, contrary to 5°. □

3.6 COROLLARY. *(i) If $R^m(M) = B_r(a, b)$, $r \geqq 3$, then $(a, b) = (a, 1)$, $a \geqq 1$, or $(2, 2)$. (ii) If $R^m(M) = F_4(a, b)$, then $(a, b) = (2, 2)$ or $(a, 1)$, for $a = 1, 2, 4$ or 8 (see 3.3b and 3°).*

3.6a COROLLARY. *(i) If $R^m(M) = C_r(b, c)$, $r \geqq 3$, then either $R^m(M) = C_3(8, 1)$ or (b, c) is one of $(1, 1)$, $(2, 2)$, $(2, 1)$, $(4, 1)$ and $(4, 3)$. (ii) If $R^m(M) = BC_r(a, b, c)$, $r \geqq 3$, then (a, b, c) is one of $(2m, 2, 1)$, $(4, 4, 1)$ and $(4m, 4, 3)$, $m \geqq 1$.*

PROOF. The root system $BC_r(a, b, c)$ contains $C_r(b, c)$ (corresponding to a meridian), which contains $A_{r-1}(b)$ (corresponding to the simple factor of a meridian). And $C_r(b, c)$ contains $C_{r-1}(b, c)$ similarly. Thus (i) follows from the facts in the cases 3°, 4° and 5°. And so does (ii) from (i) and Proposition 3.2c. □

3.6b REMARK. Although C_r contains D_r, the multiplicity map: $\alpha \mapsto m(\alpha)$ does not restrict; for example, $C_r(2p, 1)$ contains $D_r(p)$ but not $D_r(2p)$ for $p = 1, 2$ or 4 (the case $p = 4$ occurs only if $r = 3$).

Case 7° *$R(M) = D_4$. $R^m(M) = D_4(m)$. Then $m = 1$ or 2.*

PROOF. D_4 consists of the roots $\{\varepsilon_j \pm \varepsilon_k) \mid 1 \leqq j < k \leqq 4\}$. It contains three root systems $\cong A_3$ with bases $(\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4)$, $(\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 + \varepsilon_4)$, and $(\varepsilon_3 - \varepsilon_4, \varepsilon_2 - \varepsilon_3, \varepsilon_3 + \varepsilon_4)$. Hence we have only to exclude the case $m = 4$, in view of 5°. Suppose $m = 4$. Then $\boldsymbol{k}(0)$ acts on each root space $\boldsymbol{m}(\alpha)$ as $\mathcal{L}O(4) \cong \boldsymbol{m}(\alpha)^2$ by 3.1. By 3.3 applied to the above root systems, one sees that S_3 in $\boldsymbol{m}(\varepsilon_3 - \varepsilon_4)^2$ must act nontrivially on $\boldsymbol{m}(\varepsilon_3 + \varepsilon_4)$, contrary to the fact that these roots are strongly orthogonal to each other. □

3.7 COROLLARY. *(i) If $R^m(M) = D_r(m)$, $r \geqq 4$, then $m = 1$ or 2. (ii) If $R^m(M) = E_r(m)$, $6 \leqq r \leqq 8$, then $m = 1$ or 2.*

Case 8° *$R(M) = G_2$. $R^m(M) = G_2(a, b)$. Then $(a, b) = (1, 1)$ or $(2, 2)$.*

ANOTHER PROOF. A principal orbit of K acting on T_oM is an isoparametric hypersurface of the sphere which contains it (see [M]). Since $R(M)$ has 6 roots, one can apply Abresch's theorem [A] to conclude that $a = b$ and equals 1 or 2; the theorem is proven by topological methods in [A]. A few years earlier, a topological proof appeared in [As], however. □

3.8 REMARK. A simple algebraic proof was given by O. Ikawa before us.

We have determined the multiplicities of the roots of all the simple spaces.

4. The Kählerian space.

First we will show a new characterization of the Kählerian space in terms of R^m, the root system with multiplicity, and 1-connectedness (Theorem 4.1). Secondly we will do it in terms of a geometric position, i.e. as a centriole in a group (Theorem 4.5). This is analogous to the characterization of a simple H-(quaternionic) Kählerian space as the nearest polar of 1_G of a simple 1-connected group G.

Here is a well known fact which characterizes the Kählerian space (Theorem 6.1, p. 381 in [H], e.g.):

4.0 FACT. A simple space G/K has a Kählerian structure if and only if K has a one-dimensional normal subgroup (and G is simple). We do not need this in proving our theorems but we rather give another proof thereof.

4.1 THEOREM. *A simple space M is Kählerian if and only if* (i) *the root system $R(M)$ is BC_r or C_r, $r \geqq 1$ ($C_1 := A_1$)*; (ii) *the multiplicity $m(\alpha^{\sim})$ of the highest root equals one*; *and* (iii) *M is 1-connected.*

PROOF. We first prove the if-part. The idea is to use what we call *the Helgason product* $P := \prod_j S(2\varepsilon_j) = S(2\varepsilon_1) \times \cdots \times S(2\varepsilon_r) \cong S^2 \times \cdots \times S^2$ = a product of r Riemann spheres (in this case) to define an invariant complex structure on M in such a way that the monomorphism of P into M is holomorphic. By (i) and (ii), P is locally embedded into M. P contains a maximal torus $A \ni o$ of M.

If $R(M) = C_r$, then the geodesic c from o in the direction of $\sum_j \varepsilon_j$ is the shortest one to the pole o' of o in M which lies on P as the shortest one to the remotest pole of o in A. Thus the local monomorphism of P into $M = G/K$ extends to a global one.

To define an appropriate complex structure on P, we use the action of the Weyl group W of (M, A); we consider the normalizer W' of A in K, $K(o) = o$. W' acts on A as W which is, by (i), known to be generated by the permutations of $(\varepsilon_1, \cdots, \varepsilon_j, \cdots, \varepsilon_r)$ and the transformations which carry this into $(\pm\varepsilon_1, \cdots, \pm\varepsilon_j, \cdots, \pm\varepsilon_r)$. Therefore P has a unique (up to ± 1) W'-invariant complex structure J_p by (ii) and by the fact that an H-sphere is determined by any one of its tangent vector $\neq 0$; J_p carries $\boldsymbol{a}$ onto its c-orthogonal complement in T_oP.

If a member $b \in K$ preserves P, then the composite $a \circ b$ preserves A as well for some $a \in K_p$, the isotropy subgroup for $P = G_p/K_p$ at o, since K_p acts transitively on the unit sphere in T_oP and K_p is naturally a subgroup of K. Since $a \circ b \in W'$ is holomorphic on P, so is b. Thus we have defined a K-invariant complex structure J on T_oM. In terms of the Lie algebra, J is a member of $\sum_j \boldsymbol{k}(2\varepsilon_j)$ which defines a complex structure on $T_oP =$

$\boldsymbol{a}+\sum_j \boldsymbol{m}(2\varepsilon_j)$ through the s-representation and which has turned out to be K-invariant. Note that J may be thought of as a member of the group K also.

The complex structure J extends to a G-invariant almost complex structure J_M on M, which defines a Kählerian structure by the known fact, or alternatively, J_M is invariant under the point symmetries ($s_o = J^2$ on T_oP, e.g.) and hence $\nabla J_M = 0$.

Finally we take care of the case of $R(M) = BC_r$. Then M contains a meridian M^- with $R(M^-) = C_r$. P in M^- is a Helgason product in M which shares the Weyl group with this M^-. Hence the constructed invariant complex structure extends to one on M. (Let us add that, in case $R(M) = BC_r$, $J \in \boldsymbol{k}$ is not contained in $\sum_j \boldsymbol{k}(2\varepsilon_j)$ but in $\boldsymbol{k}(0) + \sum_j \boldsymbol{k}(2\varepsilon_j)$.)

In passing, note that M has exactly r polars ($\neq o$) of o, $r = r(M)$.

Now we will prove the converse by inducting on $\dim M$. We assume that M is Kählerian. If M has $\dim M = 2$, then M is the Riemann sphere, which satisfies (i) through (iii). Assume $\dim M > 2$. The meridian M^- is of course Kählerian. Thus M^- satisfies (i), (ii) and (iii) by the induction assumption. Hence M is 1-connected by 1.8b. If M has the rank $r(M) = 1$, then M^- is S^2 and M is a complex projective space satisfying (i) and (ii) (Case 2°). If $r(M) > 1$, $R(M)$ contains $2\varepsilon_j$ and $\varepsilon_j \pm \varepsilon_k$; that is, $R(M)$ is one of BC_r, C_r and F_4. If $R(M)$ were F_4, M would have 4 polars ($\neq o$) as the reader has noted in the above, while M does have 2 of them. (Alternatively, if $R(M) = F_4$, then M contains a meridian M^- with $R(M^-) = B_4$, contrary to the induction assumption. For a third proof, we note that every meridian contains a Helgason product $\prod_j S(2\varepsilon_j)$ of rank $= r(M)$; the point is that the roots $\varepsilon_1, \cdots, \varepsilon_j, \cdots, \varepsilon_r$ are strongly orthogonal and hence F_4 is excluded.) Therefore M satisfies (i) and (ii). □

4.2 COROLLARY. *A simple Kählerian space M contains $CI(r)$ as a subspace if $r(M)$ is r. There is a one-to-one correspondence between the polars of o in $CI(r) \ni o$ and those in M, which is defined by the inclusion map. Thus every simple connected Kählerian space M of rank r has exactly r polars $\neq o$ of o in M.*

4.3 HISTORICAL NOTE. Takeuchi [T68] used the Helgason product to determine the L-orbits in a Kählerian space M, where L is the group for the noncompact form L/K of M. (Since L/K is an open subspace of M, L acts on M.)

4.4 REMARK. A maximal torus A in a Kählerian simple symmetric space M is not a Kählerian subspace, but its complexification is exactly a Helgason product $P = \prod_j S(2\varepsilon_j)$. P is thus the minimal Kählerian subspace that contains A. Thus P could play a fundamental role in the category of the semisimple Kählerian symmetric spaces (whose morphisms are holomorphic besides) as A does in the category of the compact symmetric spaces. For instance, if M has rank r, then M has exactly r polars $\neq \{o\}$ of o. If M is simple furthermore, M contains $CI(r)$. In this subcategory a morphism of M into another one, N, $r(N) = r(M)$, thus gives rise to a bijection of the polars of o in M with those of the image of o in N which assigns each of them the polar it contains. And so forth.

4.5 THEOREM. (i) *Every simple Kählerian space M is a subspace of a group L, contained as a centriole for a pair $(1_L, p)$ which meets a minimum geodesic from 1_L to the pole*

p in L. (ii) *Every centriole in L is, conversely, a simple Kählerian space if it has this minimum property.*

PROOF. (i) First assume $R(M) = C_r$. The complex structure J_o at o is regarded as a member of $\sum_j \boldsymbol{k}(2\varepsilon_j)$. J_o and the member $\sum_j \varepsilon_j$ of $\boldsymbol{a}$ generate a 3-dimensional subalgebra $\mathcal{L}S$ of g, $S \cong SU(2)$ in $G = G^{\sim}$, $o = 1_G$. $G^{\sim}$ contains M (by the lifted Cartan immersion). The tangent vector $\sum_j \varepsilon_j$ at 1_G is tangent to a minimal geodesic c to the pole p of 1_G in S, hence in M too. p is also the pole in M. Since the group C_J generated by J_o is the center of the isotropy subgroup $K \subset G$ and $p \in S$ lies on C_J also, p is a pole of 1_G in $G^{\sim}$. Therefore the midpoint m of the segment $[1_G, p]$ on C_J carries M into the desired position, say, by the left translation; the position is the orbit $\mathrm{ad}(G^{\sim})(m)$ through m. We are done if we just write L for $G^{\sim}$. Now we assume $R(M) = BC_r$, the other case in view of 4.1. M is then 1° $G_p(\mathbf{C}^n)$, 2° $DIII(2r+1)$ or 3° $EIII$. In the case 1°, M is a centriole in $L := U(n)$ which is the $\mathrm{ad}(L)$-orbit through the midpoint of the minimal geodesic from 1_L to the pole with the initial tangent $J_o \in \mathcal{L}U(n)$ whose eigenvalues are $\pm i$, $i^2 = -1$, the multiplicity of $-i$ being p. In 2°, M is a centriole in $L := SO(4r+2)$ which is the $\mathrm{ad}(L)$-orbit through the midpoint of a minimal geodesic from 1_L to the pole. In 3°, M is a subspace of the Kählerian space $EVII$, $R(EVII) = C_3$. $EVII$ is a centriole in E_7 as shown above. Its c-orthogonal space is $U(1) \cdot E_6$, which meets $EVII$ at a single point, its pole in $EVII$ and two copies of the centriole $EIII$ in $U(1) \cdot E_6$. We are done for 3°, omitting certain details; although $EIII$ is not the nearest polar in E_6, it is in $U(1) \cdot E_6$. (ii) The converse must be obvious; the basic reason is that a minimal geodesic arc c to the pole (which extends to a subgroup C_J of L) lies in the center of the isotropy subgroup $K \subset \mathrm{ad}(L)$ at the midpoint m of c, making Kählerian the orbit $\mathrm{ad}(L)(m)$ through m. □

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Present Addresses:
TADASHI NAGANO
KYODO 2–31–20–201, SETAGAYA-KU, TOKYO, 156–0052 JAPAN.
e-mail: ANB64234@nifty.com

MAKIKO S. TANAKA
DEPARTMENT OF MATHEMATICS, SCIENCE UNIVERSITY OF TOKYO,
NODA, CHIBA, 278–0022 JAPAN.
e-mail: makiko@ma.noda.sut.ac.jp