TOKYO J. MATH.
VOL. 24, NO. 1, 2001

Criss-Cross Reduction of the Maslov Index and a Proof of the Yoshida-Nicolaescu Theorem

Bernhelm BOOSS-BAVNBEK, Kenro FURUTANI and Nobukazu OTSUKI

Roskilde University and Science University of Tokyo

(Communicated by K. Komatsu)

Abstract. We consider direct sum decompositions $\beta = \beta_- + \beta_+$ and $L = L_- + L_+$ of two symplectic Hilbert spaces by Lagrangian subspaces with dense embeddings $\beta_- \hookrightarrow L_-$ and $L_+ \hookrightarrow \beta_+$. We show that such criss-cross embeddings induce a continuous mapping between the Fredholm Lagrangian Grassmannians $\mathcal{FL}_{\beta_-}(\beta)$ and $\mathcal{FL}_{L_-}(L)$ which preserves the Maslov index for curves. This gives a slight generalization and a new proof of the Yoshida-Nicolaescu Spectral Flow Formula for families of Dirac operators over partitioned manifolds.

Introduction.

As conjectured in [1], the index, the η-invariant, the determinant, and other spectral invariants of a Dirac operator over a closed partitioned manifold $M = M_0 \cup_\Sigma M_1$ with $M_0 \cap M_1 = \partial M_0 = \partial M_1 = \Sigma$ can be coded by the intersection geometry of the Cauchy data spaces along the partitioning hypersurface Σ. The Yoshida-Nicolaescu Formula belongs to this program. It expresses the spectral flow of a family of Dirac operators with the same principal symbol but continuously varying connections by the Maslov intersection index of the Cauchy data spaces. The Yoshida-Nicolaescu Formula was first proved in [17] in dimension 3. Subsequently it was generalized in [12] and modified by several authors (see e.g. [2], [3], [7], [8]).

The presently available spectral flow formulas differ in the underlying assumptions and the claims made. First of all, the Cauchy data spaces are treated in slightly different ways. On one side, the Cauchy data spaces are established as L^2-closures of smooth sections over the partitioning hypersurface Σ, coming from the restriction to the boundary of all smooth solutions over one of the parts M_j of the partitioned manifold M. In case of the Dirac operator, this Cauchy data space can be represented as the range of the L^2-extension of the Calderón projection and established as a Lagrangian subspace of the symplectic Hilbert space $L^2(-\Sigma)\dot{+}L^2(\Sigma)$.

On the other side, the Cauchy data spaces can be established as subspaces of the symplectic Hilbert space $\beta := D_{\max}/D_{\min}$ of natural boundary values, i.e. the boundary values of sections belonging to the maximal domain $D_{\max}$ of the operator (so in [2], [3], see also [11]). One

Received October 29, 1999

can embed β as a non-closed subspace into the distribution space $H^{-1/2}(-\Sigma)\dot{+}H^{-1/2}(\Sigma)$. This treatment of the Cauchy data spaces is independent of pseudo-differential analysis and has several conceptual and technical advantages: the arguments work for any symmetric elliptic differential operator of first order which satisfies a weak unique continuation property and admits an extension defining a self-adjoint Fredholm operator. Moreover, no product structures are required near the boundary. Finally, the space β is basic and most suitable for a short and general proof of the closedness and the Lagrangian property of the Cauchy data spaces and the continuity of the corresponding transformation from symmetric operators to Lagrangian subspaces. As observed already in [14], establishing this continuity is the crucial step in proving spectral flow formulas.

In [2] it was shown that the treatment of the Cauchy data spaces as subspaces of the natural symplectic Hilbert space β can be fully carried out, be based on standard functional analysis, and it involves only elementary distribution theory. Most important, this approach permits a wider applicability than the usual treatment of the Cauchy data spaces in the L^2-theory. One reason for the disadvantage of the L^2-approach might be that it requires rather deep means like the symbolic calculus and approximation theory which makes it less flexible.

The result of [2] was a rather general spectral flow formula for continuous curves of symmetric elliptic differential operators, not only Dirac operators, without making some of the technical assumptions required in L^2-theory like product structures near the boundary, regularity at the endpoints of the curve of operators, or its differentiability.

The present paper is a continuation of [2]. While a major result of that paper was that a continuous family of symmetric operators induces a continuous family of Cauchy data spaces, we show now, also by standard arguments, that an additional continuous transformation can be obtained, at least when all metric structures are product near the separating (or non-separating) hypersurface: we shall show that the Cauchy data spaces as subspaces of the natural symplectic Hilbert space β can be transformed continuously into the Cauchy data spaces of the 'conventional' L^2-theory. This gives the afore-mentioned slight generalization and a new proof of the L^2-Yoshida-Nicolaescu Formula.

In Section 1 we address a fairly general situation in symplectic functional analysis and prove a 'criss-cross' reduction theorem for the Maslov index. In Section 2 we give a condensed version of the General Spectral Flow Formula of [2]. In Section 3 we connect the two preceding sections and give a new proof of the Yoshida-Nicolaescu Formula. In the Appendix we correct an erroneous description in [2], §§ 1.1–1.2, and add a lemma to be inserted at the end of § 1.2 in [2].

1. Criss-cross reduction of the Maslov index.

Let β and L be symplectic Hilbert spaces with symplectic forms ω_β and ω_L, respectively. Let

$$\beta = \beta_- \dot{+} \beta_+ \quad \text{and} \quad L = L_- \dot{+} L_+ \tag{1.1}$$

be direct sum decompositions by transversal pairs of Lagrangian subspaces. We assume that there exist continuous, injective mappings

$$i_- : \beta_- \to L_- \quad \text{and} \quad i_+ : L_+ \to \beta_+ \tag{1.2}$$

with dense images and which are compatible with the symplectic structures, i.e.

$$\omega_L(i_-(x), a) = \omega_\beta(x, i_+(a)) \quad \text{for all } a \in L_+ \text{ and } x \in \beta_- . \tag{1.3}$$

Let λ_0 be a fixed Lagrangian subspace of β (i.e. a subspace which coincides with its annihilator $(\lambda_0)^0$ with respect to ω_β). We consider the *Fredholm Lagrangian Grassmannian* of λ_0

$$\mathcal{FL}_{\lambda_0}(\beta) := \{\mu \subset \beta \mid \mu \text{ Lagrangian subspace and } (\mu, \lambda_0) \text{ Fredholm pair}\}.$$

Recall that a *Fredholm pair* is a pair of closed subspaces with finite-dimensional intersection and closed sum of finite codimension. The topology of $\mathcal{FL}_{\lambda_0}(\beta)$ is defined by the operator norm of the orthogonal projections onto the Lagrangian subspaces. As shown in [3], its fundamental group is $\mathbf{Z}$, and the mapping of the loops in $\mathcal{FL}_{\lambda_0}(\beta)$ onto $\mathbf{Z}$ is given by the *Maslov index*

$$\mathbf{mas} : \pi_1(\mathcal{FL}_{\lambda_0}(\beta)) \to \mathbf{Z}.$$

It is an intersection index of the loop with the *Maslov cycle*

$$\mathcal{M}_{\lambda_0}(\beta) := \{\mu \in \mathcal{FL}_{\lambda_0}(\beta) \mid \mu \cap \lambda_0 \neq \{0\}\}.$$

Actually, the Maslov index can be defined for all continuous curves

$$[0, 1] \ni s \mapsto \mu_s \in \mathcal{FL}_{\lambda_0}(\beta)$$

in the following way (see [2] and [3], inspired by [15]):

First we notice that any such μ_s can be obtained as the image of $\lambda_0^\perp$ under a suitable unitary transformation

$$\mu_s = U_s(\lambda_0^\perp).$$

Here we consider the real symplectic Hilbert space β as a complex Hilbert space by the almost complex structure J with $\omega(x, y) = \langle Jx, y\rangle$. Note that U_s is not uniquely determined by μ_s. Actually, from $\beta \cong \lambda_0 \otimes \mathbf{C}$ we obtain a complex conjugation so that we can define the transpose by the following formula

$${}^T U_s = \overline{U_s^*}$$

and obtain a unitary operator $W_s := U_s {}^T U_s$ which can be defined invariantly as the *complex generator* of the Lagrangian space μ_s relative to λ_0. The operator $\mathrm{Id}+W_s$ is a Fredholm operator because (μ_s, λ_0) is a Fredholm pair (see Lemma A.2 in the Appendix). In particular, we have

$$\ker(\mathrm{Id} + W_s) = (\mu_s \cap \lambda_0) \otimes \mathbf{C} = (\mu_s \cap \lambda_0) \dot{+} J(\mu_s \cap \lambda_0). \tag{1.4}$$

To define the Maslov index $\mathbf{mas}(\{\mu_s\}, \lambda_0)$, we count the change of the eigenvalues of W_s near -1 little by little. For example, between $s = 0$ and $s = s'$ we plot the spectrum of the complex generator W_s close to $e^{i\pi}$. In general, there will be no parametrization available of the spectrum near -1. For sufficiently small s', however, we can find barriers $e^{i(\pi+\theta)}$ and $e^{i(\pi-\theta)}$ such that no eigenvalues are lost through the barriers on the interval $[0, s']$. Then we count the number of eigenvalues (with multiplicity) of W_s between $e^{i\pi}$ and $e^{i(\pi+\theta)}$ at the right and left end of the interval $[0, s']$ and subtract. Repeating this procedure over the length of the whole s-interval $[0, 1]$ gives the Maslov intersection index $\mathbf{mas}(\{\mu_s\}, \lambda_0)$ without any assumptions about smoothness of the curve, 'normal crossings' (in the sense of [16]), or non-invertible endpoints.

REMARK 1.1. (a) The Maslov index for curves depends on the specified Maslov cycle $\mathcal{M}_{\lambda_0}(\beta)$. It is worth emphasizing that two *equivalent* Lagrangian subspaces λ_0 and $\hat{\lambda}_0$ (i.e., $\dim \lambda_0/(\lambda_0 \cap \hat{\lambda}_0) < +\infty$) always define the same Fredholm Lagrangian Grassmannian $\mathcal{FL}_{\lambda_0}(\beta) = \mathcal{FL}_{\hat{\lambda}_0}(\beta)$ but may define different Maslov cycles $\mathcal{M}_{\lambda_0}(\beta) \neq \mathcal{M}_{\hat{\lambda}_0}(\beta)$, and the induced Maslov indices may become different

$$\mathbf{mas}(\{\mu_s\}_{s\in[0,1]}, \lambda_0) - \mathbf{mas}(\{\mu_s\}_{s\in[0,1]}, \hat{\lambda}_0) \overset{\text{in general}}{\neq} 0 \tag{1.5}$$

(see [3], Proposition 3.1 and Section 5). However, if the curve is a loop, the Maslov index in infinite dimensions behaves like the Maslov index in finite dimensions and does not depend on the choice of the Maslov cycle. From that property it follows that the difference in (1.5), beyond the dependence on λ_0 and $\hat{\lambda}_0$, depends only on the initial and end points of the path $\{\mu_s\}$ and may be considered as the infinite-dimensional generalization $\sigma_{\text{Hör}}(\mu_0, \mu_1; \lambda_0, \hat{\lambda}_0)$ of the Hörmander index. It is the transition function of the universal covering of the Fredholm Lagrangian Grassmannian.

(b) We refer to [6] for various aspects of the Maslov index and also to [9], [10] for a cohomological treatment.

Our main result is the following

THEOREM 1.2. *Under the assumptions* (1.1), (1.2), *and* (1.3),

(a) *we have a continuous mapping*

$$\tau : \mathcal{FL}_{\beta_-}(\beta) \to \mathcal{FL}_{L_-}(L)$$

(b) *which maps the Maslov cycle $\mathcal{M}_{\beta_-}(\beta)$ of β_- into the Maslov cycle $\mathcal{M}_{L_-}(L)$ of L_- and*

(c) *preserves the Maslov index*

$$\mathbf{mas}(\{\mu_s\}_{s\in[0,1]}, \beta_-) = \mathbf{mas}(\{\tau(\mu_s)\}_{s\in[0,1]}, L_-)$$

for any continuous curve $[0, 1] \ni s \mapsto \mu_s \in \mathcal{FL}_{\beta_-}(\beta)$.

We prove Theorem 1.2 in a series of small lemmata.

1.1. Definition of the mapping τ. We consider the direct sum

$$\mathcal{D} := \beta_+ \oplus L_-,$$

where β and L are identified with subspaces of $\mathcal{D}$. Then we define the mapping τ simply by

$$\tau(\mu) := \mu \cap L \quad \text{for } \mu \in \mathcal{FL}_{\beta_-}(\beta)\,. \tag{1.6}$$

Clearly

$$\tau(\mu) = \{i_-(a) + x \mid a \in \beta_-,\ x \in L_+ \text{ such that } i_+(x) + a \in \mu\}\,. \tag{1.7}$$

To prove that range $(\tau) \subset \mathcal{FL}_{L_-}(L)$ and that τ is continuous, we introduce an alternative description of $\tau(\mu)$ in terms of bounded operators associated to μ. For a given $\mu \in \mathcal{FL}_{\beta_-}(\beta)$, we fix a direct sum decomposition

$$\mu = (\mu \cap \beta_-) \dot{+} \nu$$

with a suitable closed ν. Let $\pi_+ : \beta \to \beta_+$ denote the projection along β_-. Then to claim that (μ, β_-) is a Fredholm pair is equivalent to claiming that the projection $\pi_\mu := \pi_+|_\mu : \mu \to \beta_+$ is a Fredholm operator. So we deduce that $F_\mu := \pi_+(\mu) = \pi_\mu(\nu)$ is closed. For later use we notice that

$$\dim \beta_+/F_\mu = \dim(\beta_+ \dot{+} \beta_-)/(\mu + \beta_-) = \dim \mu \cap \beta_- < +\infty\,. \tag{1.8}$$

By the injectivity of $\pi_+|_\nu$ we can write ν as the graph of a uniquely determined bounded operator

$$f_\nu : F_\mu \to \beta_-\,.$$

Then we rewrite

$$\tau(\mu) = i_-(\mu \cap \beta_-) + \mathrm{graph}(\varphi_\mu)\,, \tag{1.9}$$

where

$$\begin{array}{cccc} \varphi_\mu : & i_+^{-1}(F_\mu) & \to & L_- \\ & x & \mapsto & i_- \circ f_\nu \circ i_+(x) \end{array}.$$

Since φ_μ is bounded and its domain is closed in L, its graph is also closed in L, and so is $\tau(\mu)$ which differs from graph(φ_μ) only by a space of finite dimension. More precisely, we have:

LEMMA 1.3. *For each $\mu \in \mathcal{FL}_{\beta_-}(\beta)$, we have $\tau(\mu) \in \mathcal{FL}_{L_-}(L)$.*

PROOF. Regarding the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & F_\mu \arrow[r, hook] & \beta_+ \arrow[r, "p"] & \beta_+/F_\mu \arrow[r] & 0 \\ & & L_+ \arrow[u, "i_+"] & & \end{tikzcd}$$

we know by the assumptions that the range of i_+ is dense in β_+ and $\dim \beta_+/F_\mu < +\infty$. So, the map $p \circ i_+$ is surjective. We therefore have

$$\beta_+ = F_\mu + i_+(L_+) \tag{1.10}$$

and $\ker(p \circ i_+) = i_+^{-1}(F_\mu)$. It follows that

$$L_+/i_+^{-1}(F_\mu) \cong (F_\mu + i_+(L_+))/F_\mu = \beta_+/F_\mu\,. \tag{1.11}$$

This implies that the closed subspace $i_+^{-1}(F_\mu)$ of L_+ is of finite codimension.

Now let $x, y \in i_+^{-1}(F_\mu)$ and set $a := i_+(x)$ and $b := i_+(y)$. Then

$$\begin{aligned}\omega_L(x+\varphi_\mu(x), y+\varphi_\mu(y)) &= \omega_L(x, \varphi_\mu(y)) + \omega_L(\varphi_\mu(x), y)\\ &= \omega_\beta(a, f_\nu(b)) + \omega_\beta(f_\nu(a), b)\\ &= \omega_\beta(a+f_\nu(a), b+f_\nu(b)) = 0\end{aligned}$$

by the compatibility condition (1.3) and the isotropy of ν. So graph φ_μ is isotropic. Moreover, we have

$$\dim(\operatorname{graph}(\varphi_\mu)^0/\operatorname{graph}(\varphi_\mu)) = 2\dim(L_+/i_+^{-1}(F_\mu)) = 2\dim(\beta_+/F_\mu)\,.$$

Here $\operatorname{graph}(\varphi_\mu)^0$ denotes the annihilator of $\operatorname{graph}(\varphi_\mu)$ with respect to the symplectic form ω_L. Clearly,

$$i_-(\mu\cap\beta_-) \subset \operatorname{graph}(\varphi_\mu)^0 \quad\text{and}\quad i_-(\mu\cap\beta_-)\cap\operatorname{graph}(\varphi_\mu) = \{0\}\,.$$

Further, we have

$$\dim i_-(\mu\cap\beta_-) = \dim(\mu\cap\beta_-) = \dim(\beta_+/F_\mu)\,,$$

since the Fredholm index of the Lagrangian pair (μ, β_-) vanishes. From this dimension examination we see that adding $i_-(\mu\cap\beta_-)$ to $\operatorname{graph}(\varphi_\mu)$ makes $\tau(\mu)$ a Lagrangian subspace of L. It also follows that $\tau(\mu)\cap L_- = i_-(\mu\cap\beta_-)$ is of finite dimension and that $L_- + \tau(\mu) = L_- + \operatorname{graph}(\varphi_\mu)$ is of finite codimension, hence $(\tau(\mu), L_-)$ is a Fredholm pair in L. $\square$

REMARK 1.4. From equation (1.9) it particularly follows that

$$\tau(\mathcal{M}_{\beta_-}(\beta)) \subset \mathcal{M}_{L_-}(L)\,.$$

1.2. The continuity of τ. We fix a closed $W \subset L_-$ with $\dim L_-/W < +\infty$ and choose a Lagrangian subspace θ of β with $\theta \pitchfork \beta_+$ and $W_\beta \subset \theta \subset W_\beta^0$, where $W_\beta := i_-^{-1}(W)$. Here "$\pitchfork$" means that the two subspaces intersect transversally. We notice that

$$W^0 = L_- \dot{+} (L_+ \cap W^0)$$

and, correspondingly,

$$W_\beta^0 = \beta_- \dot{+} (\beta_+ \cap W_\beta^0)\,.$$

Next we exploit that the injection $i_- : \beta_- \to L_-$ has a dense range. By regarding a similar exact sequence as in the proof of Lemma 1.3

$$\begin{array}{ccccccc} 0 \to & W & \hookrightarrow & L_- & \xrightarrow{q} & L_-/W & \to 0 \\ & & & \uparrow{\scriptstyle i_-} & & & \\ & & & \beta_- & & & \end{array},$$

we deduce

$$L_- = W + i_-(\beta_-) \quad\text{and}\quad \dim(\beta_-/W_\beta) = \dim(L_-/W)\,.$$

Moreover, we have

$$i_+(W^0\cap L_+) = W_\beta^0 \cap \beta_+\,. \tag{1.12}$$

To show the inclusion "$\subset$", we consider any $x \in W^0 \cap L_+$. Then $\omega_L(x, y) = 0$ for all $y \in W$, and hence also for $y = i_-(z)$ for any $z \in W_\beta = i_-^{-1}(W)$. Now, by (1.3),

$$0 = \omega_L(x, i_-(z)) = \omega_\beta(i_+(x), z).$$

So,

$$i_+(x) \in W_\beta^0 \cap \beta_+.$$

We deduce the identity "$=$" by the following dimension examination:

$$2\dim(W^0 \cap L_+) = \dim(W^0/W) = \dim(W_\beta^0/W_\beta) = 2\dim(W_\beta^0 \cap \beta_+).$$

Equation (1.12) permits us to rewrite

$$W_\beta^0 = i_+(L_+ \cap W^0) \dot{+} \beta_-$$

and to define

$$\begin{array}{cccc} i_W : & W_\beta^0 & \rightarrow & W^0 \\ & i_+(x) + z & \mapsto & x + i_-(z) \end{array}$$

for $x \in L_+ \cap W^0$ and $z \in \beta_-$. We obtain a new splitting of the symplectic Hilbert spaces:

PROPOSITION 1.5. *The space $\eta := i_W(\theta) + W$ is a Lagrangian subspace of L and the mapping $i_W|_\theta : \theta \rightarrow \eta$ has a dense image. Further, we have new direct sum decompositions*

$$\beta = \beta_+ \dot{+} \theta \quad \text{and} \quad L = L_+ \dot{+} \eta$$

which are compatible with regard to the symplectic forms ω_L and ω_β (similar to (1.3)*).*

PROOF. Clearly

$$W \subset \eta = i_W(\theta) + W \subset W^0 \quad \text{and} \quad \dim W^0/W < +\infty,$$

hence η is also closed and isotropic. We notice that the mapping $q \circ i_W|_\theta$ is surjective with $\ker q \circ i_W|_\theta = W_\beta$, where $q : \eta \rightarrow \eta/W$ denotes the projection. So,

$$W_\beta^0/W_\beta \supset \theta/W_\beta \cong \eta/W \subset W^0/W.$$

Then, from the dimension examination

$$\dim(W_\beta^0/W_\beta) = \dim(W^0/W),$$

it follows that η is a Lagrangian subspace of L.

To see that the range of $i_W|_\theta$ is dense in η we recall that the mapping $i_W : W_\beta^0 \rightarrow W^0$ is an isomorphism $W_\beta^0 \cap \beta_+ \cong W^0 \cap L_+$ on the first finite-dimensional component, and it is equal to the dense embedding $i_- : \beta_- \rightarrow L_-$ on the second component, hence its restriction to θ has a dense range in η.

The new direct sum decompositions and the compatibility of the symplectic forms follow at once. □

The preceding proposition permits a further simplification of the graph representation of $\tau(\mu)$, obtained in (1.9):

COROLLARY 1.6. *For any* $\mu \in \mathcal{FL}_{\beta_-}(\beta)$ *with* $\mu \pitchfork \theta$, *we have*

$$\tau(\mu) = \operatorname{graph}(i_W \circ f_\mu \circ i_+),$$

where $f_\mu : \beta_+ \to \theta$ *such that* graph $f_\mu = \mu$.

Now we can prove the continuity of τ and the invariance of the Maslov index under τ. We set

$$\mathcal{FL}_{W_\beta}(\beta)^{(0)} := \{\nu \in \mathcal{FL}_{W_\beta}(\beta) \mid \nu \cap W_\beta = \{0\}\}.$$

Then we have an open covering

$$\bigcup_W \mathcal{FL}_{W_\beta}(\beta)^{(0)} = \mathcal{FL}_{\beta_-}(\beta),$$

where the union is taken over all closed subspaces $W \subset L_-$ of finite codimension. It follows that $\tau(\mathcal{FL}_{W_\beta}(\beta)^{(0)}) \subset \mathcal{FL}_W(L)^{(0)}$. Thus, we have established restrictions

$$\tau_W : \mathcal{FL}_{W_\beta}(\beta)^{(0)} \to \mathcal{FL}_W(L)^{(0)}$$

of τ for each W.

We fix a W. We denote the space of Lagrangian subspaces in a finite-dimensional symplectic space by Lag$(\cdot)$ and define the reduction map

$$\begin{array}{rccc} \rho_{W_\beta} : & \mathcal{FL}_{W_\beta}(\beta)^{(0)} & \to & \operatorname{Lag}(W_\beta^0/W_\beta) \\ & \mu & \mapsto & (\mu \cap W_\beta^0 + W_\beta)/W_\beta \end{array}.$$

Then, for each Lagrangian subspace θ in β with $\theta \supset W_\beta$ and $\theta \pitchfork \beta_+$ (as before), the set

$$U(W_\beta, \theta) := \{\Lambda \in \operatorname{Lag}(W_\beta^0/W_\beta) \mid \Lambda \pitchfork \theta/W_\beta\}$$

is open in $\operatorname{Lag}(W_\beta^0/W_\beta)$ and we have an open covering

$$\bigcup_{\theta \pitchfork \beta_+} \rho_{W_\beta}^{-1}(U(W_\beta, \theta)) = \mathcal{FL}_{W_\beta}(\beta)^{(0)}.$$

By Corollary 1.6, the mapping τ is continuous on each $\rho_{W_\beta}^{-1}(U(W_\beta, \theta))$, and so it is continuous on the whole $\mathcal{FL}_{\beta_-}(\beta)$. This concludes the proof of (a) of Theorem 1.2.

Moreover, we have

$$\tau(\rho_{W_\beta}^{-1}(U(W_\beta, \theta))) \subset \rho_W^{-1}(U(W, \tau_W(\theta))),$$

where the reduction map $\rho_W : \mathcal{FL}_W(L)^{(0)} \to \operatorname{Lag}(W^0/W)$ is defined corresponding to ρ_{W_β}. We denote by

$$\overline{\tau_W} : \operatorname{Lag}(W_\beta^0/W_\beta) \to \operatorname{Lag}(W^0/W)$$

the mapping naturally induced by τ_W. Then we have the following finite-dimensional symplectic reduction:

PROPOSITION 1.7. *The diagram*

$$
\begin{tikzcd}
\mathcal{FL}_{\beta_-}(\beta) \arrow[d, "\tau"'] & \mathcal{FL}_{W_\beta}(\beta)^{(0)} \arrow[l, hook', "\hookleftarrow"'] \arrow[r, "\rho_{W_\beta}"] \arrow[d, "\tau_W"'] & \mathrm{Lag}(W_\beta^0/W_\beta) \arrow[d, "\cong \overline{\tau_W}"] \\
\mathcal{FL}_{L_-}(L) & \mathcal{FL}_W(L)^{(0)} \arrow[l, hook'] \arrow[r, "\rho_W"'] & \mathrm{Lag}(W^0/W)
\end{tikzcd}
\tag{1.13}
$$

is commutative.

For any continuous curve $\{\mu_s\}_{s\in[0,1]} \in \mathcal{FL}_{\beta_-}(\beta)$, we can find a closed subspace $W \subset L_-$ of finite codimension such that the whole curve is contained in $\mathcal{FL}_{W_\beta}(\beta)^{(0)}$, respectively $\{\tau(\mu_s)\}_{s\in[0,1]}$ is contained in $\mathcal{FL}_W(L)^{(0)}$. By finite symplectic reduction we obtain at once:

COROLLARY 1.8. *The Maslov index coincides under the transformations τ, τ_W, and $\overline{\tau_W}$ for loops and also for paths.*

NOTE. For paths, the Maslov index depends on the choice of the Maslov cycle as pointed out in Remark 1.1.a. In $\mathrm{Lag}(W^0/W)$, the Maslov index is taken with respect to the Maslov cycle $\mathcal{M}_{L_-/W}(W^0/W)$. Correspondingly, the Maslov index in $\mathrm{Lag}(W_\beta^0/W_\beta)$ is taken with respect to the Maslov cycle $\mathcal{M}_{\beta_-/W_\beta}(W_\beta^0/W_\beta)$. Note that the specified Maslov cycle in $\mathrm{Lag}(W_\beta^0/W_\beta)$ is mapped onto the specified Maslov cycle in $\mathrm{Lag}(W^0/W)$ by $\overline{\tau_W}$.

Proposition 1.7 and Corollary 1.8 together with Remark 1.4 give the proof of (b) and (c) of Theorem 1.2.

2. The general spectral flow formula.

Let $\mathcal{H}$ be a real separable Hilbert space and A an (unbounded) closed symmetric operator defined on the domain $D_{\min}$ which is supposed to be dense in $\mathcal{H}$. Let A^* denote its adjoint operator with domain $D_{\max}$. We have that $A^*|_{D_{\min}} = A$ and that A^* is the maximal closed extension of A in $\mathcal{H}$.

We form the space β of *boundary values* with the *natural trace map* γ in the following way:

$$
\begin{array}{ccc}
D_{\max} & \xrightarrow{\gamma} & D_{\max}/D_{\min} =: \beta \\
x & \mapsto & \gamma(x) = [x] := x + D_{\min}\,.
\end{array}
$$

The space β becomes a symplectic Hilbert space with the scalar product induced by the graph norm

$$(x, y)_G := (x, y) + (A^*x, A^*y) \tag{2.1}$$

and the symplectic form given by Green's form

$$\omega([x],[y]) := (A^*x, y) - (x, A^*y) \quad \text{for } [x],[y] \in \beta\,. \tag{2.2}$$

We define the "*Cauchy data space*" $\Lambda := \gamma(\ker A^*)$. It is a Lagrangian subspace of β under the assumption that A admits at least one self-adjoint Fredholm extension A_D.

Actually, we shall assume that A has a self-adjoint extension A_D with compact resolvent. Then $(\Lambda, \gamma(D))$ is a Fredholm pair of subspaces of β, i.e. $\Lambda \in \mathcal{FL}_{\gamma(D)}(\beta)$.

We consider a continuous curve $\{C_s\}_{s\in[0,1]}$ in the space of bounded self-adjoint operators on $\mathcal{H}$. From [2] we recall general definition of the spectral flow of the family $\{A_D + C_s\}_{s\in I}$. First we apply the transformation

$$\begin{array}{rccc} \mathcal{R}: & C\hat{\mathcal{F}} & \to & \hat{\mathcal{F}} \\ & A & \mapsto & \mathcal{R}(A) := A\sqrt{\mathrm{Id} + A^2}^{-1}, \end{array} \tag{2.3}$$

where $C\hat{\mathcal{F}}$ denotes the space of (not necessarily bounded) self-adjoint Fredholm operators. We define the convergence in $C\hat{\mathcal{F}}$ by the *gap* metric, i.e. the convergence of the orthogonal projection operators onto the graphs of the operators. In [2], the continuity was established of the composed map

$$C \mapsto A_D + C \mapsto \mathcal{R}(A_D + C)$$

from $\hat{\mathcal{B}}$ to $\hat{\mathcal{F}}$. Here $\hat{\mathcal{B}}$ denotes the space of bounded self-adjoint operators on $\mathcal{H}$ and $\hat{\mathcal{F}}$ denotes the space of bounded self-adjoint Fredholm operators from $\mathcal{H}$ to $\mathcal{H}$.

Next, exploiting [15], we define the *spectral flow* by

$$\mathbf{sf}(\{A_D + C_s\}) = \mathbf{sf}(\{\mathcal{R}(A_D + C_s)\}) = \sum_{j=1}^{N} k(s_j, \varepsilon_j) - k(s_{j-1}, \varepsilon_j)$$

with

$$k(t, \varepsilon_j) := \sum_{0\le\theta<\varepsilon_j} \dim\ker(\mathcal{R}(A_D + C_s) - \theta) \quad \text{for } s_{j-1} \le s \le s_j\,,$$

where the horizontal and vertical spacings $(s_0, \cdots, s_N)$, $(\varepsilon_1, \cdots, \varepsilon_N)$ are chosen such that

$$\ker(\mathcal{R}(A_D + C_s) - \varepsilon_j) = \{0\} \quad \text{and} \quad \dim\ker(\mathcal{R}(A_D + C_s) - \theta) < \infty \tag{2.4}$$

for $s_{j-1} \le s \le s_j$ and $0 \le |\theta| < \varepsilon_j$.

It is possible to choose a vertical and horizontal spacing which satisfies (2.4).

Now we assume that the operators $A^* + C_s - r$ have no 'inner solutions', i.e. satisfy the *weak unique continuation property* (UCP)

$$\ker(A^* + C_s - r) \cap D_{\min} = \{0\}$$

for $s \in [0, 1]$ and $|r| < \varepsilon_0$ with $\varepsilon_0 > 0$.

Clearly, the domains $D_{\max}$ and $D_{\min}$ are unchanged by the perturbation C_s for any s. So, β does not depend on the parameter s. Moreover, the symplectic form ω is invariantly defined on β and so also independent of s. It follows (see [2], Theorem 3.9) that the curve $\{\Lambda_s := \gamma(\ker(A^* + C_s))\}$ is continuous in $\mathcal{FL}_{\gamma(D)}(\beta)$.

Given this, the family $\{A_D + C_s\}$ can be considered at the same time in the spectral theory of self-adjoint operators, defining a spectral flow, and in the symplectic category, defining a Maslov index. Under the preceding assumptions, the main result obtainable at that level is the following general spectral flow formula (proved in [2], Theorem 5.1):

THEOREM 2.1. *Let A_D be a self-adjoint extension of A with compact resolvent and let $\{A_D + C_s\}$ be a family satisfying the UCP assumption. Then*

$$\mathbf{sf}\{A_D + C_s\} = \mathbf{mas}(\{\Lambda_s\}, \gamma(D)) .$$

3. A proof of the Yoshida-Nicolaescu formula.

Let M be a closed connected smooth Riemannian manifold and let $\Sigma \subset M$ be a hypersurface. Let

$$\{\mathrm{A}_s := \mathrm{A}_0 + C_s : C^\infty(M; S) \to C^\infty(M; S)\}_{0 \le s \le 1}$$

be a continuous family of symmetric elliptic differential operators of first order with the same principal symbol, acting on sections of a real bundle S over M. The variation of the fixed operator A_0 is given by a continuous family $\{C_s\}_{s \in [0,1]}$ of smooth symmetric bundle homomorphisms. We assume that the normal bundle of the hypersurface Σ is orientable and that all metric structures of M and S are *product* in a collar neighbourhood $\mathcal{N} = (-1, 1) \times \Sigma$ of Σ. We have

$$\mathrm{A}_s = \sigma \left(\frac{\partial}{\partial \tau} + B_s \right) \quad \text{on } \mathcal{N}, \tag{3.1}$$

where τ denotes the normal coordinate, σ is orthogonal (assuming $\sigma^2 = -\mathrm{Id}$), $\sigma B_s = -B_s \sigma$, and B_s is a self-adjoint elliptic differential operator over Σ, called the *tangential operator*. Here the point of the product structure is that then σ and B_s do not depend on the normal variable.

We cut the manifold at Σ and attach a copy of Σ to each side. So, we obtain a new manifold $M_\sharp$ with boundary $\Sigma_0 \sqcup \Sigma_1 = (-\Sigma) \sqcup \Sigma$. Then the β-space of $M_\sharp$, being a $C^\infty(\partial M_\sharp)$-module, splits according to the connected components of $\partial M_\sharp$ and we obtain two symplectic Hilbert spaces

$$\beta := \beta_0 \dot{+} \beta_1 \quad \text{and} \quad L := L_0 \dot{+} L_1$$

with

$$\gamma = \gamma_0 \oplus \gamma_1 : D_{\max}(M_\sharp) \to \beta_0 \dot{+} \beta_1$$

and $L_j = L^2(\pm\Sigma; S|_\Sigma)$ for $j = 0, 1$. We note that L_0 and L_1 are naturally identified as Hilbert spaces, but their symplectic forms have opposite signs. In β we have a Lagrangian subspace

$$\Delta_\beta := \{(x, x) \in \beta_0 \dot{+} \beta_1 \mid \gamma_0(u) = x = \gamma_1(u) \text{ for } u \in H^1(M)\}, \tag{3.2}$$

where $H^1(M)$ denotes the first order Sobolev space. Since M is closed, the operator A_s on $H^1(M)$ is a self-adjoint Fredholm operator. It can be identified with the operator $A^\sharp_{s;D} = A^\sharp_{0,D} + C_s$ over the new manifold $M_\sharp$ with domain

$$D := \{u \in D_{\max}(M_\sharp) \mid \gamma_0(u) = \gamma_1(u)\},$$

i.e. it can be considered a global self-adjoint boundary problem with $\gamma(D) = \Delta_\beta$. (Actually, D coincides with $H^1(M)$).

We set $\beta_- := \Delta_\beta$. In L, we have a Lagrangian subspace

$$L_- := \text{the diagonal of } L^2(-\Sigma; S|_\Sigma) \dot{+} L^2(\Sigma; S|_\Sigma)\,. \tag{3.3}$$

Clearly, the embedding $\beta_- \hookrightarrow L_-$ is bounded and dense.

To define suitable complementary Lagrangian subspaces β_+ and L_+, we must fall back on a spectral resolution $\{\varphi_k, \mu_k\}$ of $L^2(\Sigma)$ by eigensections of B_0. (Here and in the following we do not mention the bundle S). For simplicity, we assume $\ker B_0 = \{0\}$. Otherwise we must decompose the finite-dimensional symplectic vector space $\ker B_0$ into two Lagrangian subspaces and add these spaces to the half parts defined by the spectral cut at 0.

Following [3], Proposition 7.15 (for related results see also [5] and [11]), we decompose

$$\beta_0 = \beta_-^0 \dot{+} \beta_+^0 \quad \text{and} \quad \beta_1 = \beta_-^1 \dot{+} \beta_+^1\,,$$

where

$$\beta_-^1 := \overline{[\{\varphi_k\}_{k<0}]}^{H^{\frac{1}{2}}(\Sigma)} \quad \text{and} \quad \beta_+^1 := \overline{[\{\varphi_k\}_{k>0}]}^{H^{-\frac{1}{2}}(\Sigma)}$$

and

$$\beta_-^0 := \overline{[\{\varphi_k\}_{k<0}]}^{H^{-\frac{1}{2}}(\Sigma)} \quad \text{and} \quad \beta_+^0 := \overline{[\{\varphi_k\}_{k>0}]}^{H^{\frac{1}{2}}(\Sigma)}\,.$$

In a similar way, we decompose

$$L_0 = L_-^0 \dot{+} L_+^0 \quad \text{and} \quad L_1 = L_-^1 \dot{+} L_+^1$$

with

$$L_-^1 := \overline{[\{\varphi_k\}_{k<0}]}^{L^2(\Sigma)} \quad \text{and} \quad L_+^1 := \overline{[\{\varphi_k\}_{k>0}]}^{L^2(\Sigma)}$$

and

$$L_-^0 := \overline{[\{\varphi_k)_{k<0}]}^{L^2(\Sigma)} \quad \text{and} \quad L_+^0 := \overline{[\{\varphi_k\}_{k>0}]}^{L^2(\Sigma)}\,.$$

Rewriting

$$\beta = \beta_0 \dot{+} \beta_1 = \beta_-^0 \dot{+} \beta_+^0 \dot{+} \beta_-^1 \dot{+} \beta_+^1\,, \tag{3.4}$$

we obtain

$$\beta_- = \Delta_\beta = \{(a,b,a,b) \mid (a,b) = \gamma_0(u) = \gamma_1(u) \text{ with } u \in H^1(M)\}\,.$$

Correspondingly, we define

$$\beta_+ := \{(x,0,0,y) \mid x \in \beta_-^0,\ y \in \beta_+^1\}$$

and

$$L_+ := \{(w,0,0,v) \mid w \in L_-^0,\ v \in L_+^1\}\,.$$

Clearly, we have $L = L_- \dot{+} L_+$, a dense bounded embedding $L_+ \hookrightarrow \beta_+$, and $\Delta_\beta \cap \beta_+ = \{0\}$. From the decomposition (3.4) we have also that $\Delta_\beta + \beta_+ = \beta$. Then, as explained above in Section 2, the β-theory gives the continuity of the family of Cauchy data spaces $\{\Lambda_s\}$ (of the continuous operator family $\{A_s\}$, considered over $M_\sharp$). They are all Lagrangian subspaces of β and make Fredholm pairs with $\beta_- = \Delta_\beta = \gamma(H^1(M))$. From the General Spectral Flow Formula (here Theorem 2.1) and by the criss-cross reduction of the Maslov index (Theorem 1.2), we obtain

$$\mathbf{sf}\{A_0 + C_s\} = \mathbf{sf}\{A^{\sharp}_{0,D} + C_s\} \tag{3.5}$$

$$\stackrel{\text{Th.2.1}}{=} \mathbf{mas}(\{\Lambda_s\}, \Delta_\beta) \tag{3.6}$$

$$\stackrel{\text{Th.1.2}}{=} \mathbf{mas}(\{\Lambda_s \cap L\}, L_-)\,. \tag{3.7}$$

Note that here

$$\Lambda_s \cap L = \{u|_{(-\Sigma)\sqcup\Sigma} \mid u \in H^{\frac{1}{2}}(M_\sharp) \text{ and } A_s(u) = 0 \text{ in } M_\sharp \setminus \partial M_\sharp\}\,,$$

i.e. it coincides with the L^2-definition of the Cauchy data spaces.

In the particular case of a *partitioned* manifold

$$M = M_0 \cup_\Sigma M_1 \text{ with } \Sigma = \partial M_0 = \partial M_1 = M_0 \cap M_1\,,$$

not only β and L split but also the Cauchy data spaces split

$$\Lambda_s = \Lambda_s^0 \dot{+} \Lambda_s^1\,,$$

according to the splitting of M into two parts. So, we obtain from (3.7) the Yoshida-Nicolaescu Formula, though without any assumptions about the regularity at the endpoints or the differentiability of the curve:

THEOREM 3.1.

$$\begin{aligned}\mathbf{sf}\{A_0 + C_s\} &= \mathbf{mas}(\{\Lambda_s^0 \cap L^2(-\Sigma) \dot{+} \Lambda_s^1 \cap L^2(\Sigma)\}, L_-)\\ &=: \mathbf{mas}(\{\Lambda_s^0 \cap L^2(-\Sigma)\}, \{\Lambda_s^1 \cap L^2(\Sigma)\})\,,\end{aligned}$$

where the last expression is given by the formula of the Maslov index of Fredholm pairs of two curves.

REMARK 3.2. (a) We notice that the subspaces β_- and L_- are defined independently of a reference tangential operator, here B_0, but the choice of any other reference operator B_s would have given a different decomposition, though not a different result.

(b) In the literature on the Yoshida-Nicolaescu formula (see e.g. [7], [8], [12]) one always assumes a product structure near Σ, whereas the general spectral flow formulas, as proved in [2] and expressed in [3] in $\beta \subset H^{-1/2}(\Sigma)$, do not require product structures near Σ. However, we also need product structures near Σ to apply our criss-cross reduction theorem and to transform the general spectral flow formulas, expressed in distribution spaces, into L^2-formulas. Possibly, the criss-cross reduction theorem may provide one explanation for the need of product structures for L^2-formulas.

(c) We also want to point to a misprint in [3], p. 74 (after Equation (7.13)), where it must read that '$\gamma(S) \cap L^2(\hat{\Sigma})$ is not closed in $L^2(\hat{\Sigma})$' instead of '$\gamma(S) \cap L^2(\Sigma)$ is not closed in $L^2(\Sigma)$'.

Appendix. Corrections and Addendum to [2].

In this Appendix we shall correct a statement which was made in [2] but is not valid in general. We also shall explain why the main results of [2] remain valid.

Let $\mathcal{L}$ denote the space of all Lagrangian subspaces of a real separable symplectic Hilbert space $\mathcal{H}$, and let $\mathcal{L}^{\mathbf{C}}$ denote the space of all complex Lagrangian subspaces ($L^{\perp} = J \otimes \mathrm{Id}(L)$) of $\mathcal{H} \otimes \mathbf{C}$. The full group $\mathcal{U}(\mathcal{H})$ of unitary operators of $\mathcal{H}$ acts on $\mathcal{L}$ but *not* on $\mathcal{FL}_{\lambda_0}(\mathcal{H})$, whereas the reduced group $\mathcal{U}_c(\mathcal{H})$ does. It consists of unitary operators of the form $\mathrm{Id} + K$, where K is a compact operator. This is a very interesting and useful group. But it is too small to generate the whole Fredholm Lagrangian Grassmannian $\mathcal{FL}_{\lambda_0}(\mathcal{H})$.

In [2], below on p. 5, we claimed that the reduced group acts transitively on $\mathcal{FL}_{\lambda_0}(\mathcal{H})$, i.e. the mapping

$$\begin{array}{rccc} \rho: & \mathcal{U}_c(\mathcal{H}) & \to & \mathcal{FL}_{\lambda_0}(\mathcal{H}) \\ & U & \mapsto & U(\lambda_0^{\perp}) \end{array}$$

is surjective. This claim is not correct in general, since the difference of the orthogonal projections onto a Fredholm pair of closed subspaces of $\mathcal{H}$ is generally not of the form $\mathrm{Id} + K$. It can become an arbitrary Fredholm operator as proved in [3], Appendix.

If we replace $\mathcal{U}_c(\mathcal{H})$ by $\mathcal{U}(\mathcal{H})$, $\mathcal{FL}_{\lambda_0}(\mathcal{H})$ by $\mathcal{L}, \mathcal{U}_c(\mathcal{H} \otimes \mathbf{C})$ by $\mathcal{U}(\mathcal{H} \otimes \mathbf{C})$, and $\mathcal{FL}^{\mathbf{C}}_{\lambda_0 \otimes \mathbf{C}}(\mathcal{H} \otimes \mathbf{C})$ by $\mathcal{L}^{\mathbf{C}}$, the remainder of [2], §§ 1.1–1.2 remains valid. In the following, the mappings and diagrams of §§ 1.1–1.2 of [2] are corrected where necessary. A new Lemma A.2 is added which is of independent interest.

The group $\mathcal{U}(\mathcal{H})$ (resp. $\mathcal{U}(\mathcal{H} \otimes \mathbf{C})$) acts transitively on $\mathcal{L}$ (resp. on $\mathcal{L}^{\mathbf{C}}$). So let

$$\begin{array}{rccc} \rho: & \mathcal{U}(\mathcal{H}) & \to & \mathcal{L} \\ & U & \mapsto & U(\lambda_0^{\perp}) \end{array} \tag{A.1}$$

and

$$\begin{array}{rccc} \rho^{\mathbf{C}}: & \mathcal{U}(\mathcal{H} \otimes \mathbf{C}) & \to & \mathcal{L}^{\mathbf{C}} \\ & g & \mapsto & g(\lambda_0^{\perp} \otimes \mathbf{C}) \end{array} \tag{A.2}$$

denote the mappings defined by these actions. We obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{U}(\mathcal{H}) & \xrightarrow{\tilde{\tau}} & \mathcal{U}(\mathcal{H} \otimes \mathbf{C}) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho^{\mathbf{C}}} \\ \mathcal{L} & \xrightarrow{\tau} & \mathcal{L}^{\mathbf{C}}, \end{array} \tag{A.3}$$

where τ and $\tilde{\tau}$ denote the complexification. Recall that the space $\mathcal{H} \otimes \mathbf{C}$ splits into a direct sum of the two eigenspaces E_-, E_+ of $J \otimes \mathrm{Id}$ for the eigenvalues $\mp\sqrt{-1}$.

Corresponding to [2], Proposition 1.3 we have

PROPOSITION A.1. *The mapping*

$$\rho^{\mathbf{C}} \circ \Phi : \mathcal{U}(\mathcal{H}) \to \mathcal{L}^{\mathbf{C}}$$

is a homeomorphism, where Φ *is the mapping*

$$\begin{array}{rccc} \Phi: & \mathcal{U}(\mathcal{H}) & \to & \mathcal{U}(\mathcal{H} \otimes \mathbf{C}) \\ & U & \mapsto & \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & U \end{pmatrix}. \end{array}$$

Here Id *operates on* E_- *and* U *on* E_+.

Now we must add the following lemma to [2], following Definition 1.4:

LEMMA A.2. *Let* $\mu \in \mathcal{L}$ *and* $\mu = U(\lambda_0^\perp)$. *Then we have that*

$$\mu \in \mathcal{FL}_{\lambda_0}(\mathcal{H}) \Leftrightarrow U\bar{U}^{-1} + \mathrm{Id} \text{ is a Fredholm operator}.$$

PROOF. By identifying $\mathcal{H} \cong \lambda_0 \otimes \mathbf{C} \cong \lambda_0 \oplus \sqrt{-1}\lambda_0$, we represent U as

$$U = X + \sqrt{-1}Y$$

with $X, Y : \lambda_0 \to \lambda_0$. Since

$$U\bar{U}^{-1} + \bar{U}\bar{U}^{-1} = (U + \bar{U})\bar{U}^{-1} = 2X\bar{U}^{-1},$$

we have that $U\bar{U}^{-1} + \mathrm{Id}$ is a Fredholm operator, if and only if X is a Fredholm operator. (Notice that it is not sufficient to assume that $X = \mathrm{Id} + K$ with K compact operator).

First we assume $\mu \in \mathcal{FL}_{\lambda_0}(\mathcal{H})$. Then we know already by [2], Equation (1.1) that

$$\ker X = \mu \cap \lambda_0, \tag{A.4}$$

and we have also

$$\mu + \lambda_0 = \{-Y(x) + \sqrt{-1}X(x) + y \mid x, y \in \lambda_0\}. \tag{A.5}$$

So

$$\mathcal{H}/(\mu + \lambda_0) \cong \lambda_0/\mathrm{range}(X), \tag{A.6}$$

hence $\mathrm{range}(X)$ is closed and of finite codimension in λ_0.

Conversely, if X is a Fredholm operator, then also by (A.4), (A.5), and (A.6) we have $\mu \in \mathcal{FL}_{\lambda_0}(\mathcal{H})$. □

In the preceding proof we split unitary operators on $\mathcal{H}$ into a real and an imaginary part, regarding a fixed Lagrangian subspace λ_0. Let us denote the subspace of unitary operators which have a Fredholm operator as real part by $\mathcal{U}(\mathcal{H})^{\mathrm{Fred}}$. This is the total space of a principal fibre bundle over the Fredholm Lagrangian Grassmannian $\mathcal{FL}_{\lambda_0}(\mathcal{H})$ as base space and with the orthogonal group $\mathcal{O}(\mathcal{H})$ as structure group. The projection is given by the restriction of the trivial bundle $\rho : \mathcal{U}(\mathcal{H}) \to \mathcal{L}$ of (A.3). The new bundle

$$\mathcal{U}(\mathcal{H})^{\mathrm{Fred}} \xrightarrow{\rho} \mathcal{FL}_{\lambda_0}(\mathcal{H})$$

is also trivial as a principal fibre bundle, but it may be considered as the infinite-dimensional generalization of the well-studied bundle $\mathrm{U}(n) \to \mathrm{Lag}(\mathbf{R}^{2n})$ for finite n.

References

[1] B. BOJARSKI, The abstract linear conjugation problem and Fredholm pairs of subspace, *In Memoriam I.N. Vekua*, Tbilisi Univ. (1979), 45–60 (Russian).

[2] B. BOOSS-BAVNBER and K. FURUTANI, The Maslov Index—a functional analytical definition and the spectral flow formula, Tokyo J. Math. **21** (1998), 1–34.

[3] B. BOOSS-BAVNBEK and K. FURUTANI, Symplectic functional analysis and spectral invariants, (eds. B. BOOSS-BAVNBEK and K. P. WOJCIECHOWSKI), *Geometric Aspects of Partial Differential Equations*, Contemporary Mathematics **242** (1999), Amer. Math. Soc., 53–83.
[4] B. BOSS-BAVNBEK and K. P. WOJCIECHOWSKI, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser (1993).
[5] J. BRÜNING and M. LESCH, Spectral theory of boundary value problems for Dirac type operators, (eds. B. BOOSS-BAVNBEK and K. P. WOJCIECHOWSKI), *Geometric Aspects of Partial Differential Equations*, Contemporary Mathematics **242** (1999), Amer. Math. Soc., 203–215.
[6] S. E. CAPPELL, R. LEE and E. Y. MILLER, On the Maslov index, Comm. Pure Appl. Math. **47** (1994), 121–186.
[7] S. E. CAPPELL, R. LEE and E. Y. MILLER, Selfadjoint elliptic operators and manifold decompositions Part I: Low eigenmodes and stretching, Comm. Pure Appl. Math. **49** (1996), 825–866/Part II: Spectral flow and Maslov index, Comm. Pure Appl. Math. **49** (1996), 869–909/Part III: Determinant line bundles and Lagrangian intersection, Comm. Pure Appl. Math. **52** (1999), 543–611.
[8] M. DANIEL and P. KIRK, with an appendix by K. P. WOJCIECHOWSKI, A general splitting formula for the spectral flow, Bloomington preprint (1999).
[9] M. DE GOSSON, The structure of q-symplectic geometry, J. Math. Pures Appl. **71** (1992), 429–453.
[10] M. DE GOSSON, *Maslov Classes, Metaplectic Representation and Lagrangian Quantization*, Mathematical Research **95** (1997), Akademie Verlag.
[11] M. LESCH and N. PEYERIMHOFF, On index formulas for manifolds with metric horns, Comm. Partial Differential Equations **23** (1998), 649–684.
[12] L. NICOLAESCU, The Maslov index, the spectral flow, and decomposition of manifolds, Duke Math. J. **80** (1995), 485–533.
[13] L. NICOLAESCU, Generalized Symplectic Geometries and the Index of Families of Elliptic Problems, Mem. Amer. Math. Soc. **609** (1997).
[14] N. OTSUKI and K. FURUTANI, Spectral flow and Maslov index arising from Lagrangian intersections, Tokyo J. Math. **14** (1991), 135–150.
[15] J. PHILLIPS, Self-adjoint Fredholm operators and spectral flow, Canad. Math. Bull. **39** (1996), 460–467.
[16] J. ROBBIN and D. SALAMON, The Maslov index for paths, Topology **32** (1993), 827–844.
[17] T. YOSHIDA, Floer homology and splittings of manifolds, Ann. of Math. **134** (1991), 277–323.

Present Address:
BERNHELM BOOSS-BAVNBEK
INSTITUT FOR MATEMATIK OF FYSIK, ROSKILDE UNIVERSITY,
DK-4000 ROSKILDE, DENMARK.
e-mail: booss@mmf.ruc.dk

KENRO FURUTANI
DEPARTMENT OF MATHEMATICS, SCIENCE UNIVERSITY OF TOKYO,
NODA, CHIBA, 278–8510 JAPAN.
e-mail: furutani@ma.noda.sut.ac.jp

NOBUKAZU OTSUKA
DEPARTMENT OF MATHEMATICS, SCIENCE UNIVERSITY OF TOKYO,
NODA, CHIBA, 278–8510 JAPAN.
e-mail: nobukazu@ma.noda.sut.ac.jp