TOKYO J. MATH.
VOL. 32, NO. 1, 2009

A Diophantine Approximation of $e^{1/s}$ in Terms of Integrals

Takao KOMATSU

Hirosaki University

(Communicated by K. Shinoda)

Abstract. Let p_n/q_n be the n-th convergent of the continued fraction expansion of a real number α. It is known that $|p_n - q_n\alpha|$ is very small tending to 0 as n tends to infinity. In this paper we establish a method how to express $p_n - q_n\alpha$ in terms of integrals when α is an e-type real number and its continued fraction expansion is quasi-periodic.

1. Introduction

$\alpha = [a_0; a_1, a_2, \dots]$ denotes the simple continued fraction expansion of a real α, where

$$\alpha = a_0 + 1/\alpha_1\,, \qquad a_0 = \lfloor \alpha \rfloor\,,$$
$$\alpha_n = a_n + 1/\alpha_{n+1}\,, \qquad a_n = \lfloor \alpha_n \rfloor \quad (n \ge 1)\,.$$

The n-th convergent of the continued fraction expansion is denoted by $p_n/q_n = [a_0; a_1, \dots, a_n]$, and p_n and q_n satisfy the recurrence relation:

$$p_n = a_n p_{n-1} + p_{n-2} \quad (n \ge 0)\,, \qquad p_{-1} = 1\,, \qquad p_{-2} = 0\,,$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad (n \ge 0)\,, \qquad q_{-1} = 0\,, \qquad q_{-2} = 1\,.$$

An irrational number α is well approximated by its n-th convergent p_n/q_n. Namely, for $n \ge 0$

$$\frac{1}{q_{n+1} + q_n} < |p_n - q_n\alpha| < \frac{1}{q_{n+1}}$$

([4] p. 20). Precisely speaking, by using the algorithm mentioned above, the error can be expressed as

$$p_n - q_n\alpha = \frac{(-1)^{n+1}}{\alpha_{n+1} q_n + q_{n-1}}$$

([1] Lemma 5.4).

Osler [8] gave a remarkable proof of the simple continued fraction

$$e^{1/s} = [1; \overline{(2k-1)s - 1, 1, 1}]_{k=1}^{\infty} \quad (s \ge 2)$$

Received January 24, 2008

by expressing this error explicitly in terms of integrals. Namely, when p_n/q_n is the n-th convergent of the continued fraction of $e^{1/s}$, he showed that for $n \ge 0$

$$p_{3n} - q_{3n}e^{1/s} = -\frac{1}{s^{n+1}}\int_0^1 \frac{x^n(x-1)^n}{n!}e^{x/s}dx\,, \tag{1}$$

$$p_{3n+1} - q_{3n+1}e^{1/s} = \frac{1}{s^{n+1}}\int_0^1 \frac{x^{n+1}(x-1)^n}{n!}e^{x/s}dx \tag{2}$$

and

$$p_{3n+2} - q_{3n+2}e^{1/s} = \frac{1}{s^{n+1}}\int_0^1 \frac{x^n(x-1)^{n+1}}{n!}e^{x/s}dx\,. \tag{3}$$

This was the direct extension of the result given by Cohn [2] concerning e. A similar expression can be seen in [3] too.

It is known that the continued fraction expansion of $e^{2/s}$ is given by

$$e^{2/s} = \left[1; \overline{\frac{(6k-5)s-1}{2}, (12k-6)s, \frac{(6k-1)s-1}{2}, 1, 1}\right]_{k=1}^{\infty}\,,$$

where $s > 1$ is odd (See [9], §32, (2)). In [6] the author gave a proof of the continued fraction expansion of $e^{2/s}$ by showing similar errors explicitly.

THEOREM 1 Let p_n/q_n be the n-th convergent of the continued fraction of $e^{2/s}$. Then, for $n \ge 0$

$$p_{5n} - q_{5n}e^{2/s} = -\left(\frac{2}{s}\right)^{3n+1}\int_0^1 \frac{x^{3n}(x-1)^{3n}}{(3n)!}e^{2x/s}dx\,, \tag{4}$$

$$p_{5n+1} - q_{5n+1}e^{2/s} = -\frac{2^{3n+1}}{s^{3n+2}}\int_0^1 \frac{x^{3n+1}(x-1)^{3n+1}}{(3n+1)!}e^{2x/s}dx\,, \tag{5}$$

$$p_{5n+2} - q_{5n+2}e^{2/s} = -\left(\frac{2}{s}\right)^{3n+3}\int_0^1 \frac{x^{3n+2}(x-1)^{3n+2}}{(3n+2)!}e^{2x/s}dx\,, \tag{6}$$

$$p_{5n+3} - q_{5n+3}e^{2/s} = \left(\frac{2}{s}\right)^{3n+3}\int_0^1 \frac{x^{3n+3}(x-1)^{3n+2}}{(3n+2)!}e^{2x/s}dx\,, \tag{7}$$

and

$$p_{5n+4} - q_{5n+4}e^{2/s} = \left(\frac{2}{s}\right)^{3n+3}\int_0^1 \frac{x^{3n+2}(x-1)^{3n+3}}{(3n+2)!}e^{2x/s}dx\,. \tag{8}$$

In this paper we shall give a rather direct and combinatorial proof of the continued fraction expansion of $e^{1/s}$, which can be applied to obtain similar results about the continued fractions of $e^{2/s}$ and some more families of the e-type continued fractions.

2. The continued fraction of $e^{1/s}$

Let p_n/q_n be the n-th convergent of

$$e^{1/s} = [1; \overline{(2k-1)s-1, 1, 1}]_{k=1}^{\infty} \quad (s \geq 2)\,.$$

Some explicit combinatorial expressions of the leaping convergents of $e^{1/s}$ are known in [5]. Namely, for $n \geq 0$ we have the following.

PROPOSITION 1.

$$p_{3n} = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} s^k\,,$$

$$p_{3n+1} = \sum_{k=0}^{n} \frac{(n+k+1)!}{k!(n-k)!} s^{k+1}\,,$$

$$p_{3n+2} = (n+1) \sum_{k=0}^{n+1} \frac{(n+k)!}{k!(n-k+1)!} s^k\,,$$

$$q_{3n} = \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k)!}{k!(n-k)!} s^k\,,$$

$$q_{3n+1} = (n+1) \sum_{k=0}^{n+1} (-1)^{n-k+1} \frac{(n+k)!}{k!(n-k+1)!} s^k\,,$$

$$q_{3n+2} = \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k+1)!}{k!(n-k)!} s^{k+1}\,.$$

Using such expressions, we can obtain (1), (2) and (3) by directly showing the following.

THEOREM 2. *For $n \geq 0$ we have*

$$\sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} s^k - e^{1/s} \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k)!}{k!(n-k)!} s^k = -\frac{1}{s^{n+1}} \int_0^1 \frac{x^n (x-1)^n}{n!} e^{x/s} dx\,, \tag{9}$$

$$\sum_{k=0}^{n} \frac{(n+k+1)!}{k!(n-k)!} s^{k+1} - e^{1/s} (n+1) \sum_{k=0}^{n+1} (-1)^{n-k+1} \frac{(n+k)!}{k!(n-k+1)!} s^k = \frac{1}{s^{n+1}} \int_0^1 \frac{x^{n+1} (x-1)^n}{n!} e^{x/s} dx\,, \tag{10}$$

$$(n+1) \sum_{k=0}^{n+1} \frac{(n+k)!}{k!(n-k+1)!} s^k - e^{1/s} \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k+1)!}{k!(n-k)!} s^{k+1} = \frac{1}{s^{n+1}} \int_0^1 \frac{x^n (x-1)^{n+1}}{n!} e^{x/s} dx\,. \tag{11}$$

We need the following lemma in order to prove Theorem 2.

LEMMA 1.

$$\int x^n e^{x/s} dx = \left(\sum_{i=0}^{n} (-1)^i s^{i+1} \frac{n!}{(n-i)!} x^{n-i}\right) e^{x/s} .$$

PROOF. Use the relation

$$\begin{aligned} I_n &= \int x^n e^{x/s} dx \\ &= s x^n e^{x/s} - s n I_{n-1} \end{aligned}$$

with $I_0 = s e^{x/s}$. ∎

First of all, we shall prove (9). By Lemma 1 we have

$$\begin{aligned} &\frac{1}{s^{n+1}} \int_0^1 \frac{x^n (x-1)^n}{n!} e^{x/s} dx \\ &= \frac{1}{s^{n+1}} \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \int_0^1 x^{n+k} e^{x/s} dx \\ &= \frac{1}{s^{n+1}} \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left[\sum_{i=0}^{n+k} (-1)^i s^{i+1} \frac{(n+k)!}{(n+k-i)!} x^{n+k-i} e^{x/s} \right]_0^1 \\ &= \frac{1}{s^{n+1}} \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{i=0}^{n-1} (-1)^i s^{i+1} \frac{(n+k)!}{(n+k-i)!} e^{1/s} \\ &\quad + \frac{1}{s^{n+1}} \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{j=0}^{k} (-1)^{n+j} s^{n+j+1} \frac{(n+k)!}{(k-j)!} e^{1/s} \\ &\quad - \frac{1}{n!} \sum_{k=0}^{n} (n+k)! \binom{n}{k} s^k . \end{aligned}$$

Here, we put $i = n + j$ with $0 \le j \le k \le n$.

The third term is equal to $-p_{3n}$. Concerning the first term, since

$$\begin{aligned} \frac{d^i}{dx^i}(x^n (x-1)^n) &= \frac{d^i}{dx^i}\left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^{n+k}\right) \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{(n+k)!}{(n+k-i)!} x^{n+k-i} , \end{aligned}$$

we have for $i = 0, 1, \ldots, n-1$

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{(n+k)!}{(n+k-i)!} = 0 \,. \tag{12}$$

Hence, the first term is nullified.

Concerning the second term, for each integer j with $0 \le j \le k \le n$ we have

$$\frac{d^{n+j}}{dx^{n+j}}\bigl(x^n(x-1)^n\bigr) = \sum_{k=j}^{n} (-1)^{n-k} \binom{n}{k} \frac{(n+k)!}{(k-j)!} x^{k-j} \,.$$

On the other hand, we also have

$$\frac{d^{n+j}}{dx^{n+j}}\bigl(x^n(x-1)^n\bigr) = \sum_{\nu=j}^{n} \binom{n+j}{\nu} (x^n)^{(n+j-\nu)} \bigl((x-1)^n\bigr)^{(\nu)} \,.$$

From

$$(x^n)^{(j)} = \frac{n!}{(n-j)!} x^{n-j} \quad \text{and} \quad \bigl((x-1)^n\bigr)^{(n)} = n!$$

we obtain

$$\begin{aligned} \frac{d^{n+j}}{dx^{n+j}}\bigl(x^n(x-1)^n\bigr)\Big|_{x=1} &= \binom{n+j}{n} \frac{n!}{(n-j)!} \cdot n! \\ &= (n+j)! \binom{n}{j} \,. \end{aligned}$$

Therefore,

$$\sum_{k=j}^{n} (-1)^{n-k} \binom{n}{k} \frac{(n+k)!}{(k-j)!} = (n+j)! \binom{n}{j} \,. \tag{13}$$

Thus, the second term is equal to

$$\begin{aligned} &\frac{1}{s^{n+1}} \frac{1}{n!} \sum_{j=0}^{n} (-1)^{n+j} s^{n+j+1} \sum_{k=j}^{n} (-1)^{n-k} \binom{n}{k} \frac{(n+k)!}{(k-j)!} \cdot e^{1/s} \\ &= \frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j} (n+j)! \binom{n}{j} s^j \cdot e^{1/s} \\ &= q_{3n} e^{1/s} \,. \end{aligned}$$

Second, we shall prove (10). By Lemma 1 we have

$$\frac{1}{s^{n+1}} \int_0^1 \frac{x^{n+1}(x-1)^n}{n!} e^{x/s} dx$$

$$
\begin{aligned}
&= \frac{1}{s^{n+1}}\frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\int_0^1 x^{n+k+1}e^{x/s}dx\\
&= \frac{1}{s^{n+1}}\frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\left[\sum_{i=0}^{n+k+1}(-1)^i s^{i+1}\frac{(n+k+1)!}{(n+k-i+1)!}x^{n+k-i+1}e^{x/s}\right]_0^1\\
&= \frac{1}{s^{n+1}}\frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sum_{i=0}^{n-1}(-1)^i s^{i+1}\frac{(n+k+1)!}{(n+k-i+1)!}e^{1/s}\\
&\quad + \frac{1}{s^{n+1}}\frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sum_{j=0}^{k+1}(-1)^{n+j}s^{n+j+1}\frac{(n+k+1)!}{(k-j+1)!}e^{1/s}\\
&\quad + \frac{1}{n!}\sum_{k=0}^{n}(n+k+1)!\binom{n}{k}s^{k+1}\,.
\end{aligned}
$$

Here, we put $i = n + j$ with $0 \le j \le k + 1 \le n + 1$.

The third term is equal to p_{3n+1}. Concerning the first term, since

$$
\begin{aligned}
\frac{d^i}{dx^i}\bigl(x^{n+1}(x-1)^n\bigr) &= \frac{d^i}{dx^i}\left(\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}x^{n+k+1}\right)\\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(n+k+1)!}{(n+k-i+1)!}x^{n+k-i+1}\,,
\end{aligned}
$$

we have for $i = 0, 1, \dots, n-1$

$$
\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(n+k+1)!}{(n+k-i+1)!} = 0\,. \tag{14}
$$

Hence, the first term is nullified.

Concerning the second term, for each integer j with $0 \le j \le k + 1 \le n + 1$ we have

$$
\frac{d^{n+j}}{dx^{n+j}}\bigl(x^{n+1}(x-1)^n\bigr) = \sum_{k=j-1}^{n}(-1)^{n-k}\binom{n}{k}\frac{(n+k+1)!}{(k-j+1)!}x^{k-j+1}\,.
$$

On the other hand, we also have

$$
\frac{d^{n+j}}{dx^{n+j}}\bigl(x^{n+1}(x-1)^n\bigr) = \sum_{\nu=j-1}^{n}\binom{n+j}{\nu}(x^{n+1})^{(n+j-\nu)}\bigl((x-1)^n\bigr)^{(\nu)}\,.
$$

From

$$
(x^{n+1})^{(j)} = \frac{(n+1)!}{(n-j+1)!}x^{n-j+1} \quad\text{and}\quad \bigl((x-1)^n\bigr)^{(n)} = n!
$$

we obtain

$$\begin{aligned}\frac{d^{n+j}}{dx^{n+j}}\left(x^{n+1}(x-1)^n\right)\bigg|_{x=1} &= \binom{n+j}{n}\frac{(n+1)!}{(n-j+1)!}\cdot n! \\ &= (n+j)!\binom{n+1}{j}.\end{aligned}$$

Therefore,

$$\sum_{k=j-1}^{n}(-1)^{n-k}\binom{n}{k}\frac{(n+k+1)!}{(k-j+1)!} = (n+j)!\binom{n+1}{j}. \tag{15}$$

Thus, the second term is equal to

$$\begin{aligned}&\frac{1}{s^{n+1}}\frac{1}{n!}\sum_{j=0}^{n+1}(-1)^{n+j}s^{n+j+1}\sum_{k=j-1}^{n}(-1)^{n-k}\binom{n}{k}\frac{(n+k+1)!}{(k-j+1)!}\cdot e^{1/s} \\ &= -\frac{1}{n!}\sum_{j=0}^{n+1}(-1)^{n-j+1}(n+j)!\binom{n+1}{j}s^j\cdot e^{1/s} \\ &= -q_{3n+1}e^{1/s}.\end{aligned}$$

Third, we shall prove (11). By Lemma 1 we have

$$\begin{aligned}&\frac{1}{s^{n+1}}\int_0^1\frac{x^n(x-1)^{n+1}}{n!}e^{x/s}dx \\ &= \frac{1}{s^{n+1}}\frac{1}{n!}\sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\int_0^1 x^{n+k}e^{x/s}dx \\ &= \frac{1}{s^{n+1}}\frac{1}{n!}\sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\left[\sum_{i=0}^{n+k}(-1)^i s^{i+1}\frac{(n+k)!}{(n+k-i)!}x^{n+k-i}e^{x/s}\right]_0^1 \\ &= \frac{1}{s^{n+1}}\frac{1}{n!}\sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\sum_{i=0}^{n}(-1)^i s^{i+1}\frac{(n+k)!}{(n+k-i)!}e^{1/s} \\ &\quad+\frac{1}{s^{n+1}}\frac{1}{n!}\sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\sum_{j=0}^{k-1}(-1)^{n+j+1}s^{n+j+2}\frac{(n+k)!}{(k-j-1)!}e^{1/s} \\ &\quad+\frac{1}{n!}\sum_{k=0}^{n+1}(n+k)!\binom{n+1}{k}s^k.\end{aligned}$$

Here, we put $i = n+j+1$ with $0 \le j \le k-1 \le n$.

The third term is equal to p_{3n+2}. Concerning the first term, since

$$\begin{aligned}\frac{d^i}{dx^i}(x^n(x-1)^{n+1}) &= \frac{d^i}{dx^i}\left(\sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}x^{n+k}\right)\\ &= \sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\frac{(n+k)!}{(n+k-i)!}x^{n+k-i}\,,\end{aligned}$$

we have for $i=0,1,\dots,n$

$$\sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\frac{(n+k)!}{(n+k-i)!}=0\,. \tag{16}$$

Hence, the first term is nullified.

Concerning the second term, for each integer j with $0\le j\le k\le n+1$ we have

$$\frac{d^{n+j}}{dx^{n+j}}\big(x^n(x-1)^{n+1}\big)=\sum_{k=j}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\frac{(n+k)!}{(k-j)!}x^{k-j}\,.$$

On the other hand, we also have

$$\frac{d^{n+j}}{dx^{n+j}}\big(x^n(x-1)^{n+1}\big)=\sum_{\nu=j}^{n+1}\binom{n+j}{\nu}(x^n)^{(n+j-\nu)}\big((x-1)^{n+1}\big)^{(\nu)}\,.$$

From

$$(x^n)^{(j-1)}=\frac{n!}{(n-j+1)!}x^{n-j+1}\quad\text{and}\quad\big((x-1)^{n+1}\big)^{(n+1)}=(n+1)!$$

we obtain

$$\begin{aligned}\frac{d^{n+j}}{dx^{n+j}}\big(x^n(x-1)^{n+1}\big)\Big|_{x=1} &= \binom{n+j}{n+1}\frac{n!}{(n-j+1)!}\cdot(n+1)!\\ &= (n+j)!\binom{n}{j-1}\,.\end{aligned}$$

Therefore,

$$\sum_{k=j+1}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\frac{(n+k)!}{(k-j-1)!}=(n+j+1)!\binom{n}{j}\,. \tag{17}$$

Thus, the second term is equal to

$$\frac{1}{s^{n+1}}\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n+j+1}s^{n+j+2}\sum_{k=j+1}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\frac{(n+k)!}{(k-j-1)!}\cdot e^{1/s}$$

$$= -\frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}(n+j+1)!\binom{n}{j}s^{j+1}\cdot e^{1/s}$$

$$= -q_{3n+2}e^{1/s}\,.$$

We conclude this section by summarizing some useful combinatorial identities.

LEMMA 2. *For $i = 0, 1, \ldots, n-1$*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(n+k)!}{(n+k-i)!} = 0\,. \tag{12}$$

For $i = 0, 1, \ldots, n-1$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(n+k+1)!}{(n+k-i+1)!} = 0\,. \tag{14}$$

For $i = 0, 1, \ldots, n$

$$\sum_{k=0}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\frac{(n+k)!}{(n+k-i)!} = 0\,. \tag{16}$$

LEMMA 3.

$$\sum_{k=j}^{n}(-1)^{n-k}\binom{n}{k}\frac{(n+k)!}{(k-j)!} = (n+j)!\binom{n}{j}\,. \tag{13}$$

$$\sum_{k=j-1}^{n}(-1)^{n-k}\binom{n}{k}\frac{(n+k+1)!}{(k-j+1)!} = (n+j)!\binom{n+1}{j}\,. \tag{15}$$

$$\sum_{k=j+1}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}\frac{(n+k)!}{(k-j-1)!} = (n+j+1)!\binom{n}{j}\,. \tag{17}$$

3. The continued fraction of $e^{2/s}$

The proof is based upon the explicit combinatorial expressions of the leaping convergents of $e^{2/s}$ in [7]. Let p_n/q_n be the n-th convergent of

$$e^{2/s} = \left[1; \overline{\frac{(6k-5)s-1}{2}, (12k-6)s, \frac{(6k-1)s-1}{2}, 1, 1}\right]_{k=1}^{\infty}\,.$$

PROPOSITION 3. *For $n = 0, 1, 2, \ldots$ we have*

$$p_{5n} = \sum_{k=0}^{3n}\frac{(3n+k)!}{k!(3n-k)!}\left(\frac{s}{2}\right)^k\,,$$

$$p_{5n+1}=\sum_{k=0}^{3n+1}\frac{(3n+k+1)!}{k!(3n-k+1)!}\frac{s^k}{2^{k+1}}\,,$$

$$p_{5n+2}=\sum_{k=0}^{3n+2}\frac{(3n+k+2)!}{k!(3n-k+2)!}\left(\frac{s}{2}\right)^k\,,$$

$$p_{5n+3}=\sum_{k=0}^{3n+2}\frac{(3n+k+3)!}{k!(3n-k+2)!}\left(\frac{s}{2}\right)^{k+1}\,,$$

$$p_{5n+4}=3(n+1)\sum_{k=0}^{3n+3}\frac{(3n+k+2)!}{k!(3n-k+3)!}\left(\frac{s}{2}\right)^k$$

and

$$q_{5n}=\sum_{k=0}^{3n}(-1)^{3n-k}\frac{(3n+k)!}{k!(3n-k)!}\left(\frac{s}{2}\right)^k\,,$$

$$q_{5n+1}=\sum_{k=0}^{3n+1}(-1)^{3n-k+1}\frac{(3n+k+1)!}{k!(3n-k+1)!}\frac{s^k}{2^{k+1}}\,,$$

$$q_{5n+2}=\sum_{k=0}^{3n+2}(-1)^{3n-k+2}\frac{(3n+k+2)!}{k!(3n-k+2)!}\left(\frac{s}{2}\right)^k\,,$$

$$q_{5n+3}=3(n+1)\sum_{k=0}^{3n+3}(-1)^{3n-k+3}\frac{(3n+k+2)!}{k!(3n-k+3)!}\left(\frac{s}{2}\right)^k\,,$$

$$q_{5n+4}=\sum_{k=0}^{3n+2}(-1)^{3n-k+2}\frac{(3n+k+3)!}{k!(3n-k+2)!}\left(\frac{s}{2}\right)^{k+1}\,.$$

By using these combinatorial expressions of leaping convergents, we can prove Theorem 1. If we replace s by $s/2$ and n by $3n$ in (9), then we get (4). If we replace s by $s/2$ and n by $3n+1$ in (9) and divide both sides by 2, then we get (5). If we replace s by $s/2$ and n by $3n+2$ in (9), then we get (6). If we replace s by $s/2$ and n by $3n+2$ in (10), then we get (7). If we replace s by $s/2$ and n by $3n+2$ in (11), then we get (8).

4. Applications to some families of the e-type continued fractions

If we know the complete expression of the continued fraction expansion of the value of the e-type function, it is possible to obtain the similar evaluation in terms of integrals.

For example, for $s\ge 2$ we have

$$\frac{e^{1/(3s)}}{3}=[0;2,\overline{1,(2k-1)s-1,5}\,]_{k=1}^{\infty}\,,$$

$$\frac{e^{1/(3s+1)}}{3} = [0; 2, \overline{1, (6k-5)s+(2k-3), 1, 1,}$$
$$\overline{27(2k-1)s+(18k-10), 1, 1, (6k-1)s+(2k-1), 5}]_{k=1}^{\infty},$$
$$\frac{e^{1/(3s+2)}}{3} = [0; 2, \overline{1, (6k-5)s+(4k-5), 1, 5,}$$
$$\overline{(6k-3)s+(4k-3), 1, 5, (6k-1)s+(4k-1), 5}]_{k=1}^{\infty}.$$

Let p_n/q_n be the n-th convergent of the continued fraction expansion of $e^{1/(3s)}/3$. By induction, we can show the following combinatorial expressions of convergents.

LEMMA 4. *For* $n = 1, 2, \dots$ *we have*

$$p_{3n} = \sum_{k=0}^{n-1} \frac{(n+k)!}{k!(n-k-1)!} \frac{(3s)^{k+1}}{3},$$
$$p_{3n+1} = \sum_{k=0}^{n} (3n+2k) \frac{(n+k-1)!}{k!(n-k)!} \frac{(3s)^k}{3},$$
$$p_{3n+2} = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} (3s)^k,$$
$$q_{3n} = n \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k-1)!}{k!(n-k)!} (3s)^k,$$
$$q_{3n+1} = \sum_{k=0}^{n} (-1)^{n-k} (2n+3k) \frac{(n+k-1)!}{k!(n-k)!} (3s)^k,$$
$$q_{3n+2} = 3 \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k)!}{k!(n-k)!} (3s)^k.$$

Now we have the following.

THEOREM 3.

$$p_{3n} - q_{3n} \frac{e^{1/(3s)}}{3} = \frac{1}{(3s)^n} \int_0^1 \frac{x^n (x-1)^{n-1}}{(n-1)!} \frac{e^{x/(3s)}}{3} dx, \tag{18}$$
$$p_{3n+1} - q_{3n+1} \frac{e^{1/(3s)}}{3} = \frac{1}{(3s)^{n+1}} \int_0^1 \frac{(3sn-2x)x^{n-1}(x-1)^n}{n!} \frac{e^{x/(3s)}}{3} dx \tag{19}$$

and

$$p_{3n+2} - q_{3n+2} \frac{e^{1/(3s)}}{3} = -\frac{3}{(3s)^{n+1}} \int_0^1 \frac{x^n (x-1)^n}{n!} \frac{e^{x/(3s)}}{3} dx. \tag{20}$$

PROOF. If n is replaced by $n-1$, s is replaced by $3s$ and both sides are divided by 3 in (10), then we obtain (18). If s is replaced by $3s$ in (9), then we obtain (20). Since

$$p_{3n+1}=2\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}\frac{(3s)^k}{3}+n\sum_{k=0}^{n}\frac{(n+k-1)!}{k!(n-k)!}\frac{(3s)^k}{3}$$

and

$$q_{3n+1}=2\sum_{k=0}^{n}(-1)^{n-k}\frac{(n+k)!}{k!(n-k)!}(3s)^k+\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{(n+k)!}{k!(n-k-1)!}(3s)^{k+1}\,,$$

by (9) where s is replaced by $3s$, and (11) where n is replaced by $n-1$ and s is replaced by $3s$, we obtain

$$\begin{aligned}&p_{3n+1}-q_{3n+1}\frac{e^{1/(3s)}}{3}\\&\quad=-\frac{2}{(3s)^{n+1}}\int_0^1\frac{x^n(x-1)^n}{n!}\frac{e^{x/(3s)}}{3}dx+\frac{1}{(3s)^n}\int_0^1\frac{x^{n-1}(x-1)^n}{(n-1)!}\frac{e^{x/(3s)}}{3}dx\,,\end{aligned}$$

which equals (19).

Let p_n/q_n be the n-th convergent of the continued fraction expansion of $e^{1/(3s+1)}/3$. By induction, we obtain the following combinatorial expressions of convergents.

LEMMA 5. *For $n=1,2,\dots$ we have*

$$p_{9n-6}=\sum_{k=0}^{3n-2}(2k-3n+2)\frac{(3n+k-3)!}{k!(3n-k-2)!}\frac{(3s+1)^k}{3}\,,$$

$$p_{9n-5}=\sum_{k=0}^{3n-2}(6n-k-4)\frac{(3n+k-3)!}{k!(3n-k-2)!}\frac{(3s+1)^k}{3}\,,$$

$$p_{9n-4}=\frac{1}{3}\sum_{k=0}^{3n-2}\frac{(3n+k-2)!}{k!(3n-k-2)!}(3s+1)^k\,,$$

$$p_{9n-3}=\sum_{k=0}^{3n-1}(3n+2k-1)\frac{(3n+k-2)!}{k!(3n-k-1)!}\frac{(3s+1)^k}{3}\,,$$

$$p_{9n-2}=\sum_{k=0}^{3n-1}(6n+k-2)\frac{(3n+k-2)!}{k!(3n-k-1)!}\frac{(3s+1)^k}{3}\,,$$

$$p_{9n-1}=\sum_{k=0}^{3n-1}\frac{(3n+k-1)!}{k!(3n-k-1)!}(3s+1)^k\,,$$

$$p_{9n}=\sum_{k=0}^{3n-1}\frac{(3n+k)!}{k!(3n-k-1)!}\frac{(3s+1)^{k+1}}{3}\,,$$

$$p_{9n+1} = \sum_{k=0}^{3n}(9n+2k)\frac{(3n+k-1)!}{k!(3n-k)!}\frac{(3s+1)^k}{3},$$

$$p_{9n+2} = \sum_{k=0}^{3n}\frac{(3n+k)!}{k!(3n-k)!}(3s+1)^k$$

and

$$q_{9n-6} = \sum_{k=0}^{3n-2}(-1)^{3n-k-2}(6n-k-4)\frac{(3n+k-3)!}{k!(3n-k-2)!}(3s+1)^k,$$

$$q_{9n-5} = \sum_{k=0}^{3n-2}(-1)^{3n-k-2}(2k-3n+2)\frac{(3n+k-3)!}{k!(3n-k-2)!}(3s+1)^k,$$

$$q_{9n-4} = \sum_{k=0}^{3n-2}(-1)^{3n-k-2}\frac{(3n+k-2)!}{k!(3n-k-2)!}(3s+1)^k,$$

$$q_{9n-3} = \sum_{k=0}^{3n-1}(-1)^{3n-k-1}(6n+k-2)\frac{(3n+k-2)!}{k!(3n-k-1)!}(3s+1)^k,$$

$$q_{9n-2} = \sum_{k=0}^{3n-1}(-1)^{3n-k-1}(3n+2k-1)\frac{(3n+k-2)!}{k!(3n-k-1)!}(3s+1)^k,$$

$$q_{9n-1} = 3\sum_{k=0}^{3n-1}(-1)^{3n-k-1}\frac{(3n+k-1)!}{k!(3n-k-1)!}(3s+1)^k,$$

$$q_{9n} = 3n\sum_{k=0}^{3n}(-1)^{3n-k}\frac{(3n+k-1)!}{k!(3n-k)!}(3s+1)^k,$$

$$q_{9n+1} = 3\sum_{k=0}^{3n}(-1)^{3n-k}(2n+k)\frac{(3n+k-1)!}{k!(3n-k)!}(3s+1)^k,$$

$$q_{9n+2} = 3\sum_{k=0}^{3n}(-1)^{3n-k}\frac{(3n+k)!}{k!(3n-k)!}(3s+1)^k.$$

In a similar way to the proof of Theorem 3, we have the following.

THEOREM 4.

$$p_{9n-6} - q_{9n-6}\frac{e^{1/(3s+1)}}{3} = \frac{1}{(3s+1)^{3n-2}}\int_0^1 \frac{(x+1)x^{3n-3}(x-1)^{3n-3}}{(3n-3)!}\frac{e^{x/(3s+1)}}{3}dx,$$

$$p_{9n-5} - q_{9n-5}\frac{e^{1/(3s+1)}}{3} = \frac{1}{(3s+1)^{3n-2}}\int_0^1 \frac{(x-2)x^{3n-3}(x-1)^{3n-3}}{(3n-3)!}\frac{e^{x/(3s+1)}}{3}dx,$$

$$p_{9n-4}-q_{9n-4}\frac{e^{1/(3s+1)}}{3}=-\frac{1}{(3s+1)^{3n-1}}\int_0^1\frac{x^{3n-2}(x-1)^{3n-2}}{(3n-2)!}\frac{e^{x/(3s+1)}}{3}dx\,,$$

$$p_{9n-3}-q_{9n-3}\frac{e^{1/(3s+1)}}{3}=\frac{1}{(3s+1)^{3n-1}}\int_0^1\frac{(3x-1)x^{3n-2}(x-1)^{3n-2}}{(3n-2)!}\frac{e^{x/(3s+1)}}{3}dx\,,$$

$$p_{9n-2}-q_{9n-2}\frac{e^{1/(3s+1)}}{3}=\frac{1}{(3s+1)^{3n-1}}\int_0^1\frac{(3x-2)x^{3n-2}(x-1)^{3n-2}}{(3n-2)!}\frac{e^{x/(3s+1)}}{3}dx\,,$$

$$p_{9n-1}-q_{9n-1}\frac{e^{1/(3s+1)}}{3}=-\frac{3}{(3s+1)^{3n}}\int_0^1\frac{x^{3n-1}(x-1)^{3n-1}}{(3n-1)!}\frac{e^{x/(3s+1)}}{3}dx\,,$$

$$p_{9n}-q_{9n}\frac{e^{1/(3s+1)}}{3}=\frac{1}{(3s+1)^{3n}}\int_0^1\frac{x^{3n}(x-1)^{3n-1}}{(3n-1)!}\frac{e^{x/(3s+1)}}{3}dx\,,$$

$$p_{9n+1}-q_{9n+1}\frac{e^{1/(3s+1)}}{3}$$
$$=-\frac{3}{(3s+1)^{3n+1}}\int_0^1\frac{(x+(3s+1)n-1)x^{3n}(x-1)^{3n-1}}{(3n)!}\frac{e^{x/(3s+1)}}{3}dx\,,$$

and

$$p_{9n+2}-q_{9n+2}\frac{e^{1/(3s+1)}}{3}=-\frac{3}{(3s+1)^{3n+1}}\int_0^1\frac{x^{3n}(x-1)^{3n}}{(3n)!}\frac{e^{x/(3s+1)}}{3}dx\,.$$

Let p_n/q_n be the n-th convergent of the continued fraction expansion of $e^{1/(3s+2)}/3$. By induction, we obtain the following combinatorial expressions of convergents.

LEMMA 6. *For $n=1,2,\ldots$ we have*

$$p_{9n-6}=\sum_{k=0}^{3n-2}\frac{(3k-6n+4)(3n+k-3)!}{k!(3n-k-2)!}\frac{(3s+2)^k}{3}\,,$$

$$p_{9n-5}=\sum_{k=0}^{3n-2}\frac{(3n-2)(3n+k-3)!}{k!(3n-k-2)!}\frac{(3s+2)^k}{3}\,,$$

$$p_{9n-3}=\sum_{k=0}^{3n-1}\frac{(3k-6n+2)(3n+k-2)!}{k!(3n-k-1)!}\frac{(3s+2)^k}{3}\,,$$

$$p_{9n-2}=\sum_{k=0}^{3n-1}\frac{(3n-1)(3n+k-2)!}{k!(3n-k-1)!}\frac{(3s+2)^k}{3}\,,$$

$$p_{9n}=\sum_{k=0}^{3n-1}\frac{(3n+k)!}{k!(3n-k-1)!}\frac{(3s+2)^{k+1}}{3}\,,$$

$$p_{9n+1} = \sum_{k=0}^{3n} \frac{(9n+2k)(3n+k-1)!}{k!(3n-k)!} \frac{(3s+2)^k}{3},$$

$$p_{3n+2} = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!}(3s+2)^k$$

and

$$q_{9n-6} = \sum_{k=0}^{3n-2} (-1)^{3n-k-2} \frac{(9n-2k-6)(3n+k-3)!}{k!(3n-k-2)!}(3s+2)^k,$$

$$q_{9n-5} = \sum_{k=0}^{3n-3} (-1)^{3n-k-3} \frac{(3n+k-2)!}{k!(3n-k-3)!}(3s+2)^{k+1},$$

$$q_{9n-3} = \sum_{k=0}^{3n-1} (-1)^{3n-k-1} \frac{(9n-2k-3)(3n+k-2)!}{k!(3n-k-1)!}(3s+2)^k,$$

$$q_{9n-2} = \sum_{k=0}^{3n-2} (-1)^{3n-k-2} \frac{(3n+k-1)!}{k!(3n-k-2)!}(3s+2)^{k+1},$$

$$q_{9n} = 3n \sum_{k=0}^{3n} (-1)^{3n-k} \frac{(3n+k-1)!}{k!(3n-k)!}(3s+2)^k,$$

$$q_{9n+1} = 3 \sum_{k=0}^{3n} (-1)^{3n-k} \frac{(2n+k)(3n+k-1)!}{k!(3n-k)!}(3s+2)^k,$$

$$q_{3n+2} = 3 \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k)!}{k!(n-k)!}(3s+2)^k.$$

Then we have the following.

THEOREM 5.

$$p_{9n-6} - q_{9n-6}\frac{e^{1/(3s+2)}}{3} = \frac{1}{(3s+2)^{3n-2}} \int_0^1 \frac{(x+2)x^{3n-3}(x-1)^{3n-3}}{(3n-3)!} \frac{e^{x/(3s+2)}}{3} dx,$$

$$p_{9n-5} - q_{9n-5}\frac{e^{1/(3s+2)}}{3} = \frac{1}{(3s+2)^{3n-2}} \int_0^1 \frac{x^{3n-3}(x-1)^{3n-2}}{(3n-3)!} \frac{e^{x/(3s+2)}}{3} dx,$$

$$p_{9n-3} - q_{9n-3}\frac{e^{1/(3s+2)}}{3} = \frac{1}{(3s+2)^{3n-1}} \int_0^1 \frac{(x+2)x^{3n-2}(x-1)^{3n-2}}{(3n-2)!} \frac{e^{x/(3s+2)}}{3} dx,$$

$$p_{9n-2} - q_{9n-2}\frac{e^{1/(3s+2)}}{3} = \frac{1}{(3s+2)^{3n-1}} \int_0^1 \frac{x^{3n-2}(x-1)^{3n-1}}{(3n-2)!} \frac{e^{x/(3s+2)}}{3} dx,$$

$$p_{9n}-q_{9n}\frac{e^{1/(3s+2)}}{3}=\frac{1}{(3s+2)^{3n}}\int_0^1\frac{x^{3n}(x-1)^{3n-1}}{(3n-1)!}\frac{e^{x/(3s+2)}}{3}dx\,,$$

$$p_{9n+1}-q_{9n+1}\frac{e^{1/(3s+2)}}{3}=\frac{1}{(3s+2)^{3n}}\int_0^1\frac{(5x-3)x^{3n-1}(x-1)^{3n-1}}{(3n-1)!}\frac{e^{x/(3s+2)}}{3}dx\,,$$

and

$$p_{3n+2}-q_{3n+2}\frac{e^{1/(3s+2)}}{3}=-\frac{3}{(3s+2)^{n+1}}\int_0^1\frac{x^n(x-1)^n}{n!}\frac{e^{x/(3s+2)}}{3}dx\,.$$

Consider the continued fraction expansions of $se^{1/(ls)}$ and $e^{1/(ls)}/s$. We mention the following results without details.

THEOREM 6. *If p_n/q_n is the n-th convergent of the continued fraction*

$$se^{1/(ls)}=[\,s;\overline{(2k-1)l-1,1,2s-1}\,]_{k=1}^{\infty}\qquad(l\ge 2, s\ge 1)\,,$$

then for $n\ge 0$

$$\begin{aligned}
&p_{3n}-se^{1/(ls)}\cdot q_{3n}\\
&\quad=s\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}(ls)^k-se^{1/(ls)}\sum_{k=0}^{n}(-1)^{n-k}\frac{(n+k)!}{k!(n-k)!}(ls)^k\\
&\quad=-\frac{1}{(ls)^{n+1}}\int_0^1\frac{x^n(x-1)^n}{n!}\cdot se^{x/(ls)}dx\,,\\
&p_{3n+1}-se^{1/(ls)}\cdot q_{3n+1}\\
&\quad=\sum_{k=0}^{n+1}\big(l(n+1)(n+k)-k\big)\frac{(n+k-1)!}{k!(n-k+1)!}\frac{(ls)^k}{l}\\
&\qquad-se^{1/(ls)}\sum_{k=0}^{n}(-1)^{n-k}\big(l(n+k+1)-1\big)\frac{(n+k)!}{k!(n-k)!}(ls)^k\\
&\quad=\frac{1}{s(ls)^{n+1}}\int_0^1\frac{(x+s-1)x^n(x-1)^n}{n!}\cdot se^{x/(ls)}dx\,,\\
&p_{3n+2}-se^{1/(ls)}\cdot q_{3n+2}\\
&\quad=(n+1)\sum_{k=0}^{n+1}\frac{(n+k)!}{k!(n-k+1)!}(ls)^k-se^{1/(ls)}\cdot l\sum_{k=0}^{n}(-1)^{n-k}\frac{(n+k+1)!}{k!(n-k)!}(ls)^k\\
&\quad=\frac{1}{s(ls)^{n+1}}\int_0^1\frac{x^n(x-1)^{n+1}}{n!}\cdot se^{x/(ls)}dx\,.
\end{aligned}$$

If p_n/q_n is the n-th convergent of the continued fraction

$$\frac{e^{1/(ls)}}{s} = [\,0; s-1, \overline{1, (2k-1)l-1, 2s-1}\,]_{k=1}^{\infty} \quad (l \ge 2, s \ge 2)\,,$$

then for $n \ge 0$

$$\begin{aligned}
&p_{3n} - \frac{e^{1/(ls)}}{s} \cdot q_{3n} \\
&\quad = l \sum_{k=0}^{n-1} \frac{(n+k)!}{k!(n-k-1)!}(ls)^k - \frac{e^{1/(ls)}}{s} \cdot n \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k-1)!}{k!(n-k)!}(ls)^k \\
&\quad = \frac{1}{(ls)^n} \int_0^1 \frac{x^n(x-1)^{n-1}}{(n-1)!} \cdot \frac{e^{x/(ls)}}{s} dx\,, \\
&p_{3n+1} - \frac{e^{1/(ls)}}{s} \cdot q_{3n+1} \\
&\quad = \sum_{k=0}^{n} \bigl(l(k-n)+1\bigr) \frac{(n+k)!}{k!(n-k)!}(ls)^k \\
&\qquad - \frac{e^{1/(ls)}}{s} \sum_{k=0}^{n} (-1)^{n-k} \bigl(s(n+k)-n\bigr) \frac{(n+k-1)!}{k!(n-k)!}(ls)^k \\
&\quad = -\frac{s}{(ls)^{n+1}} \int_0^1 \frac{(x+nl-1)x^n(x-1)^{n-1}}{n!} \cdot \frac{e^{x/(ls)}}{s} dx\,, \\
&p_{3n+2} - \frac{e^{1/(ls)}}{s} \cdot q_{3n+2} \\
&\quad = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!}(ls)^k - \frac{e^{1/(ls)}}{s} \cdot s \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k)!}{k!(n-k)!}(ls)^k \\
&\quad = -\frac{s}{(ls)^{n+1}} \int_0^1 \frac{x^n(x-1)^n}{n!} \cdot \frac{e^{x/(ls)}}{s} dx\,.
\end{aligned}$$

When $l = 1$, let p_n^*/q_n^* be the n-th convergent of continued fraction

$$se^{1/s} = [\,s+1; \overline{2s-1, 2k, 1}\,]_{k=1}^{\infty} \quad (s \ge 1)\,.$$

When $l = 1$, let p_n^*/q_n^* be the n-th convergent of continued fraction

$$\frac{e^{1/s}}{s} = [\,0; s-1, 2s, \overline{1, 2k, 2s-1}\,]_{k=1}^{\infty} \quad (s \ge 1)\,.$$

In both cases the above theorem holds for $p_n^*/q_n^* = p_{n+2}/q_{n+2}$.

5. Comments

Once we know the pattern in the continued fraction of the real number belonging to the e family, we can express its diophantine error by using integrals. In other words, if we do not know any regularity in its continued fraction expansion like

$$e^3 = [20; 11, 1, 2, 4, 3, 1, 5, 1, 2, 16, 1, 1, 16, 2, 13, 14,$$
$$4, 6, 2, 1, 1, 2, 2, 2, 3, 5, 1, 3, 1, 1, 68, \ldots],$$

we have no way for the moment.

ACKNOWLEDGEMENT. This research was supported in part by the Grant-in-Aid for Scientific Research (C) (No. 18540006), the Japan Society for the Promotion of Science.

References

[1] E. B. BURGER, *Exploring the Number Jungle*: *A journey into Diophantine Analysis*, STML 8, American Mathematical Society, 2000.

[2] H. COHN, A short proof of the simple continued fraction expansion of e, Amer. Math. Monthly **113** (2006), 57–62.

[3] C. S. DAVIS, On some simple continued fractions connected with e, J. London Math. Soc. **20** (1945), 194–198.

[4] A. YA. KHINCHIN, *Continued fractions*, Dover, New York, 1997.

[5] T. KOMATSU, Some combinatorial properties of the leaping convergents, Combinatorial Number Theory, Proceedings of the Integers Conference 2005 in Celebration of the 70th Birthday of Ronald Graham, Carrollton, Georgia, USA, October 27–30,2005, eds. by B. M. LANDMAN, M. B. NATHANSON, J. NESETRIL, R. J. NOWAKOWSKI and C. POMERANCE, Walter de Gruyter, 2007, 315–325.

[6] T. KOMATSU, A proof of the continued fraction expansion of $e^{2/s}$, Integers **7** (2007), #30.

[7] T. KOMATSU, Leaping convergents of Hurwitz continued fractions, in Diophantine Analysis and related fields (DARF 2007/2008), AIP Conf. Proc. 976, Amer. Inst. Phys., Melville, NY, 2008, 130–143.

[8] T. OSLER, A proof of the continued fraction expansion of $e^{1/M}$, Amer. Math. Monthly **113** (2006), 62–66.

[9] O. PERRON, *Die Lehre von den Kettenbrüchen*, Chelsea Publishing Company, New York, 1950.

Present Address:
DEPARTMENT OF MATHEMATICAL SCIENCES,
GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY,
HIROSAKI UNIVERSITY,
HIROSAKI, AOMORI, 036–8561 JAPAN.
e-mail: komatsu@cc.hirosaki-u.ac.jp